EQ-ALGEBRAS WITH PSEUDO PRE-VALUATIONS

Yongwei Yang[1]

School of Mathematics and Statistics
Anyang Normal University
Anyang 455000
China

Xiaolong Xin

School of Mathematics
Northwest University
Xi'an, 710127
China

Abstract. The concepts of (positive implicative, implicative) pseudo pre-valuations and strong pseudo pre-valuations are introduced and some related characterizations are studied. The relationships among positive implicative pseudo pre-valuations, implicative pseudo pre-valuations and pseudo pre-valuations are investigated, and conditions for a real-valued function to be a pseudo pre-valuation are also discussed. By using a congruence relation induced via a pseudo valuation, we construct a quotient structure and prove certain isomorphism theorems.

Keywords: EQ-algebra, pseudo pre-valuation, pseudo-metric space, (positive) implicative pseudo pre-valuation.

1. Introduction

Non-classical logic systems which lay logical foundation for dealing with uncertain information processing and fuzzy information, are uniquely determined by the algebraic properties of the structure of their truth values. Residuated lattices have been proposed as the most generalis algebraic counterparts of residuated systems with t-norm based semantics, where the conjunction connective is interpreted by a t-norm and the implication operator by its residuum. Some other logical algebras such as BL-algebras [10], MTL-algebras [6], lattice implication algebras [18], R_0-algebras [17] and MV-algebras [4] are considered as particular classes of residuated lattices. In a residuated lattice, the basic typical operations multiplication and implication which are closely tied by adjointness property, and the fuzzy equality (i.e., biresiduum) is derived from the operations meet and implication. Due to the algebra of truth values is no longer a residuated lattice, a new algebra for

[1]Corresponding author. E-mail: yangyw2010@126.com

the fuzzy theory which called an EQ-algebra [16] was proposed by Novák and De Baets.

EQ-algebras provide a possibility to develop fuzzy logics with the basic connective being a fuzzy equality instead of an implication. In EQ-algebras, there are three primitive operations—meet, multiplication and a fuzzy equality, and residuum and multiplication are no more closely tied by the adjunction. The implication $\to$ in EQ-algebras is defined directly from fuzzy equality $\sim$ by the formula $x \to y = (x \wedge y) \sim x$. Since the equation holds also for the biresiduum, thus each residuated lattice can be seen as an EQ-algebra but not vice versa. From these points of view, it is known that the notion of EQ-algebras generalizes that of commutative residuated lattices, so it is interesting to investigate the properties of EQ-algebras. EI-Zekey [7] introduced the notion of prelinear ordered EQ-algebras which were proved to be lattice EQ-algebras, and he also characterized the representable good EQ-algebras. As a continuation of good EQ-algebras, [8] studied the prefilters and filters of separated EQ-algebras. Ma and Hu [15] investigated the compatibility of multiplication w.r.t. the fuzzy equality in an arbitrary EQ-algebra and characterized compatible EQ-algebras by using special GLE-algebras. Since EQ-algebras are generalize form of residuated lattices, [14] extended some notions of residuated lattices to EQ-algebras, then the notions of implicative and positive implicative prefilters of EQ-algebras were proposed. Moreover, based on the fuzzy set theory it is meaningful to study the related fuzzy structures of prefilters in EQ-algebras [19, 1, 11].

Recently, Buşneag [2] defined pseudo-valuation on a Hilbert algebra, and proved that every pseudo-valuation induces a pseudo metric on a Hilbert algebra. Later he gave the notions of pseudo-valuations (valuations) on residuated lattices, and proved some theorems of extension for pseudo-valuations [3]. By using a pseudo-metric induced by a pseudo valuation on BCI-algebra [5], Ghorbani [9] introduced a congruence relation and defined the quotient algebra. Following the research of Jun et al. [12, 13], [20] investigated related characterizations of (implicative) pseudo-valuations on R_0-algebras, and showed that a pseudo-valuation on R_0-algebras is Boolean if and only if it is implicative.

In the paper, we discuss a theoretical approach of the algebraic system in EQ-algebras by using the Buşneag's model. We introduce notions of (positive implicative, implicative) pseudo pre-valuations and strong pseudo pre-valuations, then investigate some characterizations of them. A congruence relation on an EQ-algebra is constructed by using a pseudo-metric induced via a pseudo valuation. Furthermore, we construct a quotient structure related to the congruence relation and prove certain isomorphism theorems.

2. Preliminaries

In this section, we give the basic definitions and results of EQ-algebras that are useful for subsequent discussions.

An algebra $(L, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$ is called an EQ-algebra if it satisfies the following axioms: for all $x, y, s, t \in L$, (E1) $(L, \wedge, 1)$ is a $\wedge$-semilattice with top element 1,

(E2) $(L, \otimes, 1)$ is a commutative monoid and $\otimes$ is isotone with respect to $\leq$, where $x \leq y$ if and only if $x \wedge y = x$,

(E3) $x \sim x = 1$,

(E4) $((x \wedge y) \sim s) \otimes (t \sim x) \leq s \sim (t \wedge y)$,

(E5) $(x \sim y) \otimes (s \sim t) \leq (x \sim s) \otimes (y \sim t)$,

(E6) $(x \wedge y \wedge s) \sim x \leq (x \wedge y) \sim x$,

(E7) $x \otimes y \leq x \sim y$.

In what follows, L is an EQ-algebra unless otherwise specified. For any $x, y \in L$, we put $x \to y = (x \wedge y) \sim x$ and $\tilde{x} = x \sim 1$.

Definition 2.1 [16] An EQ-algebra L is called

(1) a good EQ-algebra if $\tilde{x} = x$ for any $x \in L$;

(2) a separated EQ-algebra if $x \sim y = 1$ implies $x = y$ for any $x, y \in L$;

(3) a residuated EQ-algebra if $x \otimes y \leq z$ if and only if $x \leq y \to z$ for any $x, y, z \in L$;

(4) an involutive EQ-algebra, if L contains a bottom element 0, and $\neg\neg x = x$ holds for any $x \in L$, where $\neg x = x \sim 0 = x \to 0$.

Remark 2.2 Let $(L, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$ be a residuated lattice. For any $x, y \in L$, we define $x \sim y = (x \Rightarrow y) \wedge (y \Rightarrow x)$, then $(L, \wedge, \otimes, \sim, 1)$ is a residuated EQ-algebra ([16]). In general, a residuated EQ-algebra may not be a residuated lattice ([8]), however residuated lattices are proper classes of EQ-algebras ([14]).

Proposition 2.3 [16],[8] *Let $(L, \wedge, \otimes, \sim, 1)$ be an EQ-algebra. Then the following assertions are valid: for any $x, y, z \in L$,*

(1) $x \sim y = y \sim x$, $x \sim y \leq x \to y$, $x \leq y \to x$, $x \otimes y \leq x \wedge y$;

(2) $(x \sim y) \otimes (y \sim z) \leq x \sim z$, $(x \to y) \otimes (y \to z) \leq x \to z$;

(3) $x \sim y \leq (x \sim z) \sim (y \sim z)$, $x \sim y \leq (x \wedge z) \sim (y \wedge z)$;

(4) $x \to y \leq (y \to z) \to (x \to z)$, $x \to y \leq (z \to x) \to (z \to y)$;

(5) $x \to y = x \to (x \wedge y)$, $x \to y \leq (x \wedge z) \to (y \wedge z)$;

(6) $x \leq y$ *implies* $x \to y = 1$, $x \sim y = y \to x$, $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$;

(7) $(x \to y) \otimes (y \to x) \leq x \sim y$.

Lemma 2.4 [16],[14] *Let L be a good EQ-algebra. Then we have, for any $x, y, z \in L$,*

(1) $x \leq (x \to y) \to y$, $x \leq (x \sim y) \sim y$,

(2) $x \to (y \to z) = y \to (x \to z) \leq (x \otimes y) \to z$,

(3) $x \otimes (x \to y) \leq x \wedge y$.

Lemma 2.5 [16], [8], [1] *Let L be an EQ-algebra. Then the following conditions are equivalent: $x, y, z \in L$,*

(1) *L is residuated;*

(2) *L is good and $x \to y \leq (x \otimes z) \to (y \otimes z)$;*

(3) *L is separated and $(x \otimes y) \to z = x \to (y \to z)$.*

In what follows, we recall some types of prefilters in EQ-algebras.

Definition 2.6 [8, 14, 19] *Let F be a nonempty subset of L. Then F is called:*

(1) *a prefilter of L if it satisfies for any $x, y \in L$,* (F1) $1 \in F$; (F2) $x \in F$, $x \to y \in F$ *imply* $y \in F$; *if a prefilter F satisfies* (F3) $x \to y \in F$ *implies* $(x \otimes z) \to (y \otimes z) \in F$ *for any $x, y, z \in L$, then F is called a filter of L.*

(2) *a positive implicative prefilter of L if F is a prefilter of L and it satisfies* (F4) $x \to (y \to z) \in F$ *and* $x \to y \in F$ *imply* $x \to z \in F$ *for any $x, y, z \in L$.*

(3) *an implicative prefilter of L if it satisfies* (F1) *and* (F5) $z \to ((x \to y) \to x) \in F$ *and $z \in F$ imply $x \in F$ for any $x, y, z \in L$.*

Definition 2.7 [8] Let L, L' be EQ-algebras. A function $f : L \to L'$ satisfying $f(1) = 1'$ (where 1 and $1'$ are the top elements of L and L', respectively) is called a homomorphism if $f(a \boxdot b) = f(a) \boxdot' f(b)$, where $\boxdot \in \{\wedge, \otimes, \sim\}$ in L and $\boxdot' \in \{\wedge', \otimes', \sim'\}$ in L'. The order is clearly stable under homomorphism because it is defined by using meet.

3. Pseudo pre-valuations on EQ-algebras

In the section, we introduce the notion of pseudo pre-valuations, and give some characterizations of a pseudo pre-valuation on EQ-algebras. By discussing the concept of pseudo-metrics induced by pseudo pre-valuations, we obtain that the binary operations on EQ-algebras are uniformly continuous.

Definition 3.1 Let $\varphi : L \to R$ be a real-valued function, where R is the set of all real numbers. Then φ is called a pseudo pre-valuation on L if it satisfies the following conditions: for any $x, y \in L$,

(1) $\varphi(1) = 0$,

(2) $\varphi(y) \leq \varphi(x) + \varphi(x \to y)$.

If a real-valued function $\varphi : L \to R$ satisfies conditions (1) and (3) $\varphi(y) + \varphi(y \to x) = \varphi(x) + \varphi(x \to y)$, then φ is said to be a strong pseudo pre-valuation on L.

Let φ be a pseudo pre-valuation on L. φ is said to be a pseudo valuation on L if $\varphi((x \otimes z) \to (y \otimes z)) \leq \varphi(x \to y)$ for any $x, y, z \in L$. A strong pseudo pre-valuation φ is called a strong pseudo valuation if φ is a pseudo valuation. A pseudo pre-valuation φ is called a pre-valuation if $\varphi(x) = 0$ implies $x = 1$.

From the definitions of pseudo pre-valuations and strong pseudo pre-valuations, it is easy to see that a strong pseudo pre-valuation is pseudo pre-valuation, however the converse is not true in general.

Example 3.2 Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

One can check that $(L, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. Define $\varphi : L \to R$ by $\varphi(0)$=9, $\varphi(a) = 5$, $\varphi(b) = 2$, $\varphi(1) = 0$. Routine calculation shows that φ is a pseudo pre-valuation but φ is not a strong pseudo pre-valuation since $\varphi(a) + \varphi(a \to b) = 5 \neq 7 = \varphi(b) + \varphi(b \to a)$.

The following example shows that strong pseudo pre-valuations on EQ-algebras exist.

Example 3.3 Let $L = \{0, a, b, c, d, 1\}$ be a chain with Cayley tables as follows:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	a	a	b
c	0	0	a	0	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

$\sim$	0	a	b	c	d	1
0	1	0	0	0	0	0
a	0	1	d	d	d	d
b	0	d	1	d	d	d
c	0	d	d	1	d	d
d	0	d	d	d	1	1
1	0	d	d	d	1	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	d	1	1	1	1
c	0	d	d	1	1	1
d	0	d	d	d	1	1
1	0	d	d	d	1	1

Then $(L, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. Define a function $\varphi : L \to R$ as follows: $\varphi(0) = 3$, $\varphi(a) = \varphi(b) = \varphi(c) = \varphi(d) = \varphi(1) = 0$. One can check that φ is a strong pseudo pre-valuation on L.

Proposition 3.4 *Let φ be a real-valued function on L. If φ is a pseudo pre-valuation on L, then the following properties hold: for any $x, y \in L$,*

(1) $x \leq y$ *implies that* $\varphi(y) \leq \varphi(x)$*;*

(2) $0 \leq \varphi(x)$*;*

(3) $\varphi(x \to y) \leq \varphi(y)$, $\varphi(x) = \varphi(1 \to x)$.

Proof. (1) Let $x, y \in L$ such that $x \leq y$, then $x \to y = 1$, we obtain that

$$\varphi(y) \leq \varphi(x) + \varphi(x \to y) = \varphi(x) + \varphi(1) = \varphi(x) + 0 = \varphi(x).$$

(2) For any $x \in L$, we have

$$0 = \varphi(1) \leq \varphi(x \to 1) + \varphi(x) = \varphi(x).$$

(3) Since $y \leq x \to y$, thus $\varphi(x \to y) \leq \varphi(y)$. Obviously, $\varphi(1 \to x) \leq \varphi(x)$. Notice φ is a pseudo pre-valuation on L, we have $\varphi(x) \leq \varphi(1) + \varphi(1 \to x) = \varphi(1 \to x)$, thus $\varphi(1 \to x) = \varphi(x)$ ■

Proposition 3.5 *Let L be an EQ-algebra with a bottom element 0 and φ a real-valued function on L. If φ is a strong pseudo pre-valuation on L, then we have: for any $x, y \in L$,*

(1) $x \leq y$ *implies that* $\varphi(x \sim y) = \varphi(y \to x) = \varphi(x) - \varphi(y)$,

(2) $\varphi(x) + \varphi(\neg x) = \varphi(0)$,

(3) $\varphi(\neg\neg x) = \varphi(x) = \varphi(\tilde{x})$.

Proof. (1) Assume that $x \leq y$, then $x \to y = 1$, and so $\varphi(x \to y) = 0$, therefore $\varphi(y \to x) = \varphi(x) - \varphi(y) + \varphi(x \to y) = \varphi(x) - \varphi(y)$. Notice that $y \to x = x \sim y$, thus (1) holds.

(2) Since $\neg x = x \sim 0$, then $\varphi(\neg x) = \varphi(0) - \varphi(x)$, and thus $\varphi(x) + \varphi(\neg x) = \varphi(0)$.

(3) $\varphi(\neg\neg x) = \varphi(x)$ is immediately from (2). Consider that $\tilde{x} = x \sim 1$, we get that $\varphi(\tilde{x}) = \varphi(x) - \varphi(1) = \varphi(x)$, thus (3) is valid. ■

For any real-valued function φ on L, we consider the following set

$$\varphi_* := \{x \in L | \varphi(x) = 0\}.$$

Theorem 3.6 *Let φ be a pseudo pre-valuation on L. Then the set φ_* is a prefilter of L which is called the prefilter induced by the pseudo pre-valuation φ. If L is a residuated EQ-algebra, then φ_* is a filter.*

Proof. From $\varphi(1) = 0$, it follows that $1 \in \varphi_*$. Suppose that $x, y \in L$ such that $x, x \to y \in \varphi_*$, then we get $\varphi(x) = 0$ and $\varphi(x \to y) = 0$. Since $\varphi(y) \leq \varphi(x) + \varphi(x \to y) = 0$, therefore $\varphi(y) = 0$, and so $y \in \varphi_*$. Thus φ_* is a prefilter of L. Now let L be a residuated EQ-algebra and $x \to y \in \varphi_*$, then we get that $x \to y \leq (x \otimes z) \to (y \otimes z)$ for any $x, y, z \in L$ by Lemma 2.5, and so $\varphi((x \otimes z) \to (y \otimes z)) = 0$. Consequently, $(x \otimes z) \to (y \otimes z) \in \varphi_*$, and therefore φ_* is a filter. ∎

The following example shows that the converse of Theorem 3.6 may not be true, that is, there exist a EQ-algebra L and a real-valued function $\varphi : L \to R$ such that φ_* is a prefilter of L but φ is not a pseudo pre-valuation on L.

Example 3.7 Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	1	1

It is easy to see that $(L, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. Define a real-valued function $\varphi : L \to R$ by $\varphi(0) = 2$, $\varphi(a) = -1$, $\varphi(b) = \varphi(1) = 0$. Then $\varphi_* = \{b, 1\}$ is a prefilter of L, but φ is not a pseudo pre-valuation since

$$\varphi(b) = 0 \nleq \varphi(a) + \varphi(a \to b) = -1.$$

Proposition 3.8 *Let F be a prefilter of L and t a positive element of R. Define*

$$\varphi^F(x) = \begin{cases} 0, & x \in F, \\ t, & x \notin F, \end{cases}$$

then φ^F is a pseudo pre-valuation on L which is called the pseudo pre-valuation induced by the prefilter F. Moreover, $(\varphi^F)_* = F$.

Proof. It is obvious that φ^F is a pseudo pre-valuation on L.

$$(\varphi^F)_* = \{x \in L | \varphi^F(x) = 0\} = \{x \in L | x \in F\} = F.$$ ∎

In the following, we provide some conditions under which a real-valued function on L becomes to a pseudo pre-valuation.

Theorem 3.9 *Let φ be a real-valued function on L with $\varphi(1) = 0$. Then φ is a pseudo pre-valuation if and only if $x \leq y \to z$ implies $\varphi(z) \leq \varphi(x) + \varphi(y)$ for any $x, y, z \in L$.*

Proof. Assume that φ is a pseudo pre-valuation and $x \leq y \to z$. It follows that $\varphi(y \to z) \leq \varphi(x)$, and so $\varphi(z) \leq \varphi(y) + \varphi(y \to z) \leq \varphi(x) + \varphi(y)$.

Conversely, since $x \to y \leq x \to y$, thus $\varphi(y) \leq \varphi(x \to y) + \varphi(x)$, and so φ is a pseudo pre-valuation. ■

An interesting application of Theorem 3.9 is to some proofs of the following important results.

Proposition 3.10 *If φ is a pseudo pre-valuation on L, then for any $x, y, z, s, t \in L$,*

(1) $\varphi(x \wedge y) \leq \varphi(x) + \varphi(y)$;

(2) $\varphi(x \to z) \leq \varphi(x \to y) + \varphi(y \to z)$;

(3) $\varphi(x \sim z) \leq \varphi(x \sim y) + \varphi(y \sim z)$;

(4) $\varphi((x \wedge s) \sim (y \wedge t)) \leq \varphi(x \wedge y) + \varphi(s \sim t)$;

(5) $\varphi((x \to s) \sim (y \to t)) \leq \varphi(x \sim y) + \varphi(s \sim t)$,

Proof. (1) For any $x, y \in L$, we have $x \leq y \to x = y \to (x \wedge y)$ by Proposition 2.3. Using Theorem 3.9, we can get $\varphi(x \wedge y) \leq \varphi(x) + \varphi(y)$.

(2) Notice that $x \to y \leq (y \to z) \to (x \to z)$, we have $\varphi(x \to z) \leq \varphi((y \to z) \to (x \to z)) + \varphi(y \to z) \leq \varphi(x \to y) + \varphi(y \to z)$.

(3) According to Proposition 2.3, we obtain that $x \sim y \leq (y \sim z) \sim (x \sim z) \leq (y \sim z) \to (x \sim z)$, therefore $\varphi(x \sim z) \leq \varphi(x \sim y) + \varphi(y \sim z)$ by Theorem 3.9.

(4) For any $x, y, s, t \in L$, we have $x \sim y \leq (x \wedge s) \sim (y \wedge s)$ and $s \sim t \leq (y \wedge s) \sim (y \wedge t)$ by Proposition 2.3. From (3) and Proposition 3.4, it follows that $\varphi(x \wedge y) + \varphi(s \sim t) \geq \varphi((x \wedge s) \sim (y \wedge s)) + \varphi((y \wedge s) \sim (y \wedge t)) \geq \varphi((x \wedge s) \sim (y \wedge t))$.

(5) It follows immediately by the definition of $\to$ and items (3) and (4). ■

Let L be a residuated EQ-algebra. Since $x \otimes y \leq z$ if and only if $x \leq y \to z$ for any $x, y, z \in L$, then the following proposition is a consequence of Theorem 3.9.

Proposition 3.11 *Let L be a residuated EQ-algebra and φ a real-valued function on L with $\varphi(1) = 0$. Then φ is a pseudo pre-valuation if and only if $x \otimes y \leq z$ implies $\varphi(z) \leq \varphi(x) + \varphi(y)$ for any $x, y, z \in L$.*

Proposition 3.12 *Let L be a residuated EQ-algebra and φ a pseudo pre-valuation. Then for any $x, y \in L$, we have:*

(1) $\varphi(x \otimes y) \leq \varphi(x) + \varphi(y)$;

(2) $\varphi(x \wedge y) \leq \varphi(x) + \varphi(y)$;

(3) $\varphi((x \to y) \to z) \leq \varphi(x \to (y \to z))$.

Proof. (1) Since $x \otimes y \leq x \otimes y$, then $\varphi(x \otimes y) \leq \varphi(x) + \varphi(y)$ by Proposition 3.11.

(2) Notice that L is a residuated EQ-algebra, we get $x \otimes (x \to y) \leq x \wedge y$ by Lemma 2.4 and Lemma 2.5. From Proposition 3.4 and Proposition 3.11, it follows that $\varphi(x \wedge y) \leq \varphi(x) + \varphi(x \to y) \leq \varphi(x) + \varphi(y)$.

(3) Since $x \otimes y \leq x \sim y \leq x \to y$, then $(x \to y) \to z \leq (x \otimes y) \to z = x \to (y \to z)$ by Lemma 2.5. Thus $\varphi(x \to (y \to z)) \leq \varphi((x \to y) \to z)$. ∎

Proposition 3.13 *Let L be a good EQ-algebra and φ a real-valued function on L with $\varphi(1) = 0$. If φ satisfies the condition: $\varphi(x \to (x \to z)) \leq \varphi(x \to y)$ $+\, \varphi(x \to (y \to z))$ for any $x, y, z \in L$, then φ is a pseudo pre-valuation.*

Proof. Notice that L is a good EQ-algebra, we obtain that $1 \to x = 1 \sim x = x \sim 1 = x$ for any $x \in L$. For any $x, y \in L$, we get $\varphi(x) + \varphi(x \to y) = \varphi(1 \to x) + \varphi(1 \to (x \to y)) \geq \varphi(1 \to (1 \to y)) = \varphi(y)$. Hence φ is a pseudo pre-valuation. ∎

Proposition 3.14 *Let φ be a real-valued function on L. Then the following statements are equivalent: for any $x, y \in L$,*

(1) *φ is a strong pseudo pre-valuation,*

(2) *$x \leq y$ implies that $\varphi(y \to x) = \varphi(x) - \varphi(y)$,*

(3) *$\varphi(y \to x) = \varphi(x \wedge y) - \varphi(y)$.*

Proof. (1) $\Rightarrow$ (2) The proof follows immediately from Proposition 3.5.

(2) $\Rightarrow$ (3) Observe that $x \wedge y \leq y$, we get that $\varphi(y \to x) = \varphi(y \to (x \wedge y)) = \varphi(x \wedge y) - \varphi(y)$.

(3) $\Rightarrow$ (1) Suppose that (3) holds, then $\varphi(1) = \varphi(x \to x) = \varphi(x \wedge x) - \varphi(x) = 0$. For any $x, y \in L$, we have $\varphi(x) + \varphi(x \to y) = \varphi(x) + \varphi(x \wedge y) - \varphi(x) = \varphi(x \wedge y) = \varphi(y) + \varphi(x \wedge y) - \varphi(y) = \varphi(y) + \varphi(y \to x)$, thus φ is a strong pseudo pre-valuation. ∎

Let (M, d) be an ordered pair, where M is a nonempty set and $d : M \times M \to R$ is a positive function. If d satisfies the following conditions: for any $x, y, z \in M$,

(1) $d(x, x) = 0$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, z) \leq d(x, y) + d(y, z)$,

then (M, d) is called a pseudo-metric space. Moreover, if $d(x, y) = 0$ implies $x = y$, then (M, d) is called a metric space.

Theorem 3.15 *Let φ be a pseudo pre-valuation on L. Define a real-valued function on $d_\varphi : L \times L \to R$ by $d_\varphi(x, y) = \varphi(x \to y) + \varphi(y \to x)$ for any $x, y \in L$, then (M, d_φ) is a pseudo-metric space, where d_φ is called the pseudo-metric induced by the pseudo pre-valuation φ.*

Proof. It is obvious that $d_\varphi(x,y) \geq 0$, $d_\varphi(x,x) = 0$ and $d_\varphi(x,y) = d_\varphi(y,x)$ for any $x,y \in L$. According to Proposition 3.10, we have $d_\varphi(x,y) + d_\varphi(y,z) = (\varphi(x \to y) + \varphi(y \to x)) + (\varphi(y \to z) + \varphi(z \to y)) = (\varphi(x \to y) + \varphi(y \to z)) + (\varphi(z \to y) + \varphi(y \to x)) \geq \varphi(x \to z) + \varphi(z \to x) = d_\varphi(x,z)$. Hence (M, d_φ) is a pseudo-metric space. ∎

Proposition 3.16 *Let φ be a pseudo pre-valuation on L and d_φ the pseudo-metric induced by φ. Then the following inequalities hold: for any $x,y,z,\omega,\nu \in L$,*

(1) $\max\{d_\varphi(x \to z, y \to z), d_\varphi(z \to x, z \to y)\} \leq d_\varphi(x,y)$,

(2) $d_\varphi(x \to y, \omega \to \nu) \leq d_\varphi(x \to y, \omega \to y) + d_\varphi(\omega \to y, \omega \to \nu)$,

(3) $d_\varphi(x \wedge z, y \wedge z) \leq d_\varphi(x,y)$,

(4) *if φ is a pseudo valuation, then* $d_\varphi(x \otimes z, y \otimes z) \leq d_\varphi(x,y)$,

(5) $d_\varphi(x \sim z, y \sim z) \leq d_\varphi(x,y)$.

Proof. (1) For any $x,y,z \in L$, we have $y \to x \leq (x \to z) \to (y \to z)$ and $x \to y \leq (y \to z) \to (x \to z)$, thus $\varphi(y \to x) \geq \varphi((x \to z) \to (y \to z))$ and $\varphi(x \to y) \geq \varphi((y \to z) \to (x \to z))$. And so $d_\varphi(x,y) = \varphi(y \to x) + \varphi(x \to y) \geq \varphi((x \to z) \to (y \to z)) + \varphi((y \to z) \to (x \to z)) = d_\varphi(x \to z, y \to z)$. Analogously, $d_\varphi(x,y) \geq d_\varphi(z \to x, z \to y)$. Hence $\max\{d_\varphi(x \to z, y \to z), d_\varphi(z \to x,\ z \to y)\} \leq d_\varphi(x,y)$.

(2) It is trivial since d_φ is the pseudo-metric induced by φ.

(3) For any $x,y,z \in L$, we obtain $d_\varphi(x\wedge z, y\wedge z) = \varphi((x\wedge z) \to (y\wedge z)) + \varphi((y\wedge z) \to (x \wedge z))$. Notice that $x \to y \leq (x \wedge z) \to (y \wedge z)$ and $y \to x \leq (y \wedge z) \to (x \wedge z)$, we get $\varphi(x \to y) \geq \varphi((x \wedge z) \to (y \wedge z))$ and $\varphi(y \to x) \geq \varphi((y \wedge z) \to (x \wedge z))$, and so $d_\varphi(x,y) = \varphi(x \to y) + \varphi(y \to x) \geq \varphi((x \wedge z) \to (y \wedge z)) + \varphi((y \wedge z) \to (x \wedge z)) = d_\varphi(x \wedge z, y \wedge z)$.

(4) Since φ is a pseudo valuation, then we have $\varphi((x \otimes z) \to (y \otimes z)) \leq \varphi(x \to y)$ and $\varphi((y \otimes z) \to (x \otimes z)) \leq \varphi(y \to x)$ for any $x,y,z \in L$. And thus $d_\varphi(x,y) = \varphi(x \to y) + \varphi(y \to x) \geq \varphi((x\otimes z) \to (y\otimes z)) + \varphi((y\otimes z) \to (x\otimes z)) = d_\varphi(x\otimes z, y\otimes z)$.

(5) is similar to the proof of (3). ∎

Proposition 3.17 *Let d_φ be the pseudo-metric induced by a pseudo pre-valuation φ. Then $(L \times L, d^*_\varphi)$ is a pseudo-metric space, where*

$$d^*_\varphi((x,y),(\omega,\nu)) = \max\{d_\varphi(x,\omega), d_\varphi(y,\nu)\},$$

for any $(x,y),(\omega,\nu) \in L \times L$.

Proof. Consider that d_φ is a pseudo-metric on L, we have $d^*_\varphi((x,y),(x,y)) = \max\{d_\varphi(x,x), d_\varphi(y,y)\} = 0$ and $d^*_\varphi((x,y),(\omega,\nu)) = \max\{d_\varphi(x,\omega), d_\varphi(y,\nu)\} = \max\{d_\varphi(\omega,x), d_\varphi(\nu,y)\} = d^*_\varphi((\omega,\nu),(x,y))$. Now, let $(x,y),(a,b),(\omega,\nu) \in L \times L$, we get

$$\begin{aligned}
&d^*_\varphi((x,y),(a,b)) + d^*_\varphi((a,b),(\omega,\nu))\\
&\qquad = \max\{d_\varphi(x,a), d_\varphi(y,b)\} + \max\{d_\varphi(a,\omega), d_\varphi(b,\nu)\}\\
&\qquad \geq \max\{d_\varphi(x,a) + d_\varphi(a,\omega), d_\varphi(y,b) + d_\varphi(b,\nu)\}\\
&\qquad \geq \max\{d_\varphi(x,\omega), d_\varphi(y,\nu)\}\\
&\qquad = d^*_\varphi((x,y),(\omega,\nu)).
\end{aligned}$$

Consequently, $(L \times L, d^*_\varphi)$ is a pseudo-metric space. ∎

Theorem 3.18 *Let φ be a pre-valuation on L and d_φ the pseudo-metric induced by φ. Then the operations $\wedge, \otimes, \sim, \rightarrow: L \times L \rightarrow L$ are uniformly continuous.*

Proof. Here we only prove $\wedge : L\times L \rightarrow L$ is uniformly continuous, other cases can be proved in a similar way. For any $x, y, \omega, \nu \in L$ and $\varepsilon > 0$, if $d^*_\varphi((x,y),(\omega,\nu)) < \frac{\varepsilon}{2}$, then $d_\varphi(x,\omega) < \frac{\varepsilon}{2}$ and $d_\varphi(y,\nu) < \frac{\varepsilon}{2}$. According to Proposition 3.16, we get $d_\varphi(x\wedge y, \omega\wedge\nu) \leq d_\varphi(x\wedge y, \omega\wedge y) + d_\varphi(\omega\wedge y, \omega\wedge\nu) \leq d_\varphi(x,\omega) + d_\varphi(y,\nu) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence the operation $\wedge : L \times L \rightarrow L$ is uniformly continuous. ∎

4. Quotient structures induced by pseudo valuations

In the section, we will construct a quotient EQ-algebra related to a pseudo valuation, and investigate some isomorphism theorems.

Lemma 4.1 *Let φ be a pseudo valuation on L. A relation $\approx_\varphi$ on L is defined as follows: for any $x, y \in L$,*

$$x \approx_\varphi y \text{ if and only if } \varphi(x \rightarrow y) = \varphi(y \rightarrow x) = 0,$$

Then $\approx_\varphi$ is a congruence relation on L. We denote the quotient set $L/\varphi = \{[x]_\varphi \mid x \in L\}$ induced via φ, where $[x]_\varphi$ is the equivalence class of x with respect to $\approx_\varphi$.

Proof. The proof follows immediately from Proposition 3.16. ∎

Let φ be a pre-valuation on L. It is easy to verify that $(L/\varphi, \wedge, \otimes, \sim, [1]_\varphi)$ is an EQ-algebra which is called a quotient EQ-algebra induced by the pseudo valuation φ, where the operation $\wedge$ on L/φ is defined by $[x]_\varphi \wedge [y]_\varphi = [x \wedge y]_\varphi$, and similarly for other operations. The partial order on L/F is $[x]_\varphi \leq [y]_\varphi$ if and only if $[x]_\varphi = [x]_\varphi \wedge [y]_\varphi$.

Lemma 4.2 *Let φ be a pre-valuation on L. Then $[x]_\varphi \leq [y]_\varphi$ if and only if $\varphi(x \rightarrow y) = 0$.*

Proof. Using $x \rightarrow y = x \rightarrow (x \wedge y)$, it follows that $[x]_\varphi \leq [y]_\varphi$ if and only if $[x]_\varphi = [x]_\varphi \wedge [y]_\varphi$ if and only if $\varphi(x \rightarrow y) = 0$ by Lemma 4.1. ∎

Proposition 4.3 *Let L be a residuated EQ-algebra with a bottom element* 0 *and* φ *a strong pseudo valuation on* L. *Then* L/φ *is an involutive EQ-algebra.*

Proof. Obviously, L/φ is an EQ-algebra, moreover L/φ is a separated EQ-algebra. Indeed, let $[x]_\varphi \sim [y]_\varphi$=$[1]_\varphi$, then $[x \sim y]_\varphi$=$[1]_\varphi$, and so $\max\{\varphi(x \to y), \varphi(x \to y)\} \leq \varphi(x \sim y) = \varphi(1 \to (x \sim y)) = 0$. Thus $\varphi(x \to y) = \varphi(x \to y) = 0$, and so $[x]_\varphi = [y]_\varphi$, consequently L/φ is a separated EQ-algebra. Next, we will show that $\neg\neg[x]_\varphi = [x]_\varphi$ for any $x \in L$.

Observe that $x \leq \neg\neg x$, we get $\varphi(\neg\neg x \to x) = \varphi(x) - \varphi(\neg\neg x)$. Due to the fact that $\varphi(\neg\neg x) + \varphi(\neg\neg x \to x) = \varphi(x) + \varphi(x \to \neg\neg x)$ and $\varphi(x) = \varphi(\neg\neg x)$, it follows that $\varphi(\neg\neg x \to x) = \varphi(x \to \neg\neg x) = 0$. Thus $\neg\neg[x]_\varphi = [\neg\neg x]_\varphi = [x]_\varphi$, and so L/φ is an involutive EQ-algebra. ∎

As an immediate consequence of the above proposition together with Lemma 4.2, we record here the following result.

Proposition 4.4 *Let L be a residuated EQ-algebra with a bottom element* 0, *and* φ *a strong pseudo valuation on* L. *Then in the involutive EQ-algebra* L/φ, *we have:*

(1) $[x]_\varphi \leq [y]_\varphi$ *if and only if* $\varphi(x \to y) = 0$ *if and only if* $\varphi(x) = \varphi(x \wedge y)$;

(2) $[x]_\varphi = [y]_\varphi$ *if and only if* $\varphi(x \to y) = \varphi(y \to x) = 0$ *if and only if* $\varphi(x) = \varphi(y) = \varphi(x \wedge y)$.

Moreover, the mapping $\hat{\varphi} : L/\varphi \to R$ *defined by* $\hat{\varphi}([x]_\varphi) = \varphi(x)$ *is a strong pseudo valuation on* L/φ.

Proposition 4.5 *Let* φ *be a pseudo valuation on* L *and* J *be a prefilter of* L *such that* $\varphi_* \subseteq J$. *Denote* $J/\varphi = \{[x]_\varphi | x \in J\}$, *then we have:*

(1) $x \in J$ *if and only if* $[x]_\varphi \in J/\varphi$ *for any* $x \in L$;

(2) J/φ *is a prefiter of* L/φ.

Proof. (1) Suppose that $[x]_\varphi \in J/\varphi$, then there exists $y \in J$ such that $[x]_\varphi = [y]_\varphi$. Using Lemma 4.2, we get $\varphi(y \to x) = 0$, and so $y \to x \in \varphi_* \subseteq J$. Since J is a prefilter of L, then $x \in J$. The converse is obviously.

(2) Obviously, $[1]_\varphi \subseteq J/\varphi$. Suppose that $[x]_\varphi, [x]_\varphi \to [y]_\varphi \in J/\varphi$, then we get $x, x \to y \in J$. Hence $y \in J$, and so $[y]_\varphi \in J/\varphi$. Consequently, J/φ is a prefiter of L/φ. ∎

Lemma 4.6 *Let* L, L' *be EQ-algebras,* φ *be a pseudo pre-valuation on* L *and* $f : L \to L'$ *be an epimorphism. Then* $f(\varphi) : L \to R$ *is a pseudo pre-valuation on* L', *where* $f(\varphi)$ *is defined by* $f(\varphi)(y') = \inf\{\varphi(y) | f(y) = y', y \in L\}$ *for any* $y' \in L'$. *Moreover,* $f(\varphi_*) \subseteq (f(\varphi))_*$.

Proof. Due to the fact that φ is a pseudo pre-valuation on L and f is an epimorphism from L to L', it follows that $f(\varphi)(1') = \inf\{\varphi(x)|f(x) = 1', x \in L\} = \varphi(1) = 0$. For any $x', y' \in L'$, $f(\varphi)(x') + f(\varphi)(x' \to y') = \inf\{\varphi(x)|f(x) = x', x \in L\} + \inf\{\varphi(z)|f(z) = x' \to y', z \in L\} \geq \inf\{\varphi(x) + \varphi(x \to y)|f(x) = x', f(y) = y', x, y \in L\} \geq \inf\{\varphi(y)|f(y) = y', y \in L\} = f(\varphi)(y')$, thus $f(\varphi)$ is a pseudo pre-valuation on L'. Now let $x \in \varphi_*$, then we get $\varphi(x) = 0$. Since $f(\varphi)(f(x)) = \inf\{\varphi(t)|f(t) = f(x), t \in L\} \leq \varphi(x) = 0$, therefore $f(\varphi)(f(x)) = 0$, that is $x \in (f(\varphi))_*$, and so $f(\varphi_*) \subseteq (f(\varphi))_*$. ∎

Lemma 4.7 *Let L, L' be EQ-algebras, φ a pseudo valuation on L' and $f : L \to L'$ a homomorphism. Then $f^{-1}(\varphi) : L' \to R$ is a pseudo pre-valuation on L, where $f(\varphi)$ is defined by $f^{-1}(\varphi)(x) = \varphi(f(x))$ for any $x \in L$. Moreover, $f^{-1}(\varphi_*) = (f^{-1}(\varphi))_*$.*

Proof. The proof is straightforward. ∎

Filters are important tools to study logical algebras, and closely related to congruence relations with which we can associate quotient algebras.

Let F be a filter in EQ-algebra L. Define a relation $\equiv$ on L as: $x \equiv y$ if and only if $x \sim y \in F$, then $\equiv$ is a congruence relation on L. Let L/F denote the quotient algebra induced by F, and $[x]^F$ denote the equivalence class of x with respect to $\equiv$, then the quotient algebra L/F is a separated EQ-algebra [16].

Proposition 4.8 *Let φ be a pseudo valuation on L. Then $L/\varphi \simeq L/\varphi_*$.*

Proof. Define a function $\Phi : L/\varphi \to L/\varphi_*$ by $\Phi([x]_\varphi) = [x]^{\varphi_*}$. We only need to prove that Φ is an isomorphism. Suppose that $[x]_\varphi, [y]_\varphi \in L/\varphi$. It is not difficult to prove that $[x]_\varphi = [y]_\varphi$ if and only if $\varphi(x \to y) = \varphi(y \to x) = 0$ if and only $[x]^{\varphi_*} = [y]^{\varphi_*}$, which implies that Φ is an one-to-one function. Obviously, Φ is subjective. Now we prove that Φ is a homomorphism. Indeed, $\Phi([1]_\varphi) = [1]^{\varphi_*}$ and $\Phi([x]_\varphi \wedge [y]_\varphi) = [x]^{\varphi_*} \wedge [y]^{\varphi_*}$; similarly for $\otimes$ and $\sim$. ∎

For the purpose of investigating homomorphism theorems of EQ-algebras based on pseudo valuations, we introduce the following notion.

Definition 4.9 Let L, L' be EQ-algebras, φ a pseudo valuation on L and $f : L \to L'$ an epimorphism. φ is called an invariant pseudo valuation with respect to f if $f(x_1) = f(x_2)$ implies $\varphi(x_1) = \varphi(x_2)$, for any $x_1, x_2 \in L$.

Proposition 4.10 *Let L, L' be EQ-algebras, $f : L \to L'$ an epimorphism and φ an invariant pseudo valuation with respect to f. Then $L/\varphi \simeq L'/f(\varphi)$.*

Proof. It is known that L/φ and $L'/f(\varphi)$ are EQ-algebras. Define a function $\psi : L/\varphi \to L'/f(\varphi)$ by $\psi([x]_\varphi) = [f(x)]_{f(\varphi)}$. (1) Suppose that $[x]_\varphi, [y]_\varphi \in L/\varphi$ such that $[x]_\varphi = [y]_\varphi$, then $\varphi(x \to y) = \varphi(y \to x) = 0$. Since f is an epimorphism, then $f(\varphi)(f(x) \to f(y)) = \inf\{\varphi(z)|f(z) = f(x) \to f(y) = f(x \to y), z \in L\} \leq \varphi(x \to y) = 0$, that is $f(\varphi)(f(x) \to f(y)) = 0$. Similarly, $f(\varphi)(f(y) \to f(x)) = 0$, and so $[x]_{f(\varphi)} = [y]_{f(\varphi)}$. Consequently ψ is well defined. (2) Now we show that ψ is a homomorphism. Due to the fact that f is a homomorphism, it follows that $\psi([1]_\varphi) = [f(1)]_{f(\varphi)} = [1']_{f(\varphi)}$; $\psi([x \boxdot y]_\varphi) = [f(x \boxdot y)]_{f(\varphi)} = [f(x)]_{f(\varphi)} \boxdot' [f(y)]_{f(\varphi)}$, where $\boxdot \in \{\wedge, \otimes, \sim\}$ in L and $\boxdot' \in \{\wedge', \otimes', \sim'\}$ in L'. (3) It is obvious that φ is subjective. (4) Suppose that $\psi([x]_\varphi) = \psi([y]_\varphi)$, that is, $[f(x)]_{f(\varphi)} = [f(y)]_{f(\varphi)}$, then $f(\varphi)(f(x) \to f(y)) = f(\varphi)(f(x \to y)) = f(\varphi)(f(y \to x)) = f(\varphi)(f(y) \to f(x)) = 0$. Moreover, $f(\varphi)(f(x \to y)) = \inf\{\varphi(z) | f(z) = f(x \to y), z \in L\} = \varphi(x \to y) = 0$, similarly for $\varphi(y \to x) = 0$, thus $[x]_\varphi = [y]_\varphi$, and so ψ is an one-to-one function. Therefore $L/\varphi \simeq L'/f(\varphi)$. ∎

Proposition 4.11 *Let L, L' be EQ-algebras, φ a pseudo valuation on L' and $f : L \to L'$ an epimorphism. Then $L/f^{-1}(\varphi) \simeq L'/\varphi$.*

Proof. Let $x_1, x_2 \in L$ such that $f(x_1) = f(x_2)$. From $f^{-1}(\varphi)(x_1) = \varphi(f(x_1)) = \varphi(f(x_2)) = f^{-1}(\varphi)(x_2)$, it follows that $f^{-1}(\varphi)$ is an invariant pseudo valuation with respect to f. Notice that f is a subjective function, it is not difficult to prove that $f(f^{-1}(\varphi)) = \varphi$. According to Proposition 4.10, we have $L/f^{-1}(\varphi) \simeq L'/\varphi$. ∎

5. Some special pseudo pre-valuations on EQ-algebras

In the section, we introduce the concepts of implicative pseudo pre-valuations and positive implicative pseudo pre-valuations, and investigate their relationships.

5.1. Positive implicative pseudo pre-valuations

Definition 5.1 *rm Let $\varphi : L \to R$ be a pseudo valuation. If φ satisfies: $\varphi(x \to z) \leq \varphi(x \to (y \to z)) + \varphi(x \to y)$ for any $x, y, z \in L$, then φ is called a positive implicative pseudo pre-valuation on L.*

The following example shows that positive implicative pseudo pre-valuations exist.

Example 5.2 *rm Let L be the EQ-algebra defined in Example 3.7. Define a real-valued function $\varphi : L \to R$ by $\varphi(0) = 2$, $\varphi(a) = 1$, $\varphi(b) = \varphi(1) = 0$, then φ is a positive implicative pseudo pre-valuation on L.*

It is obvious that every positive implicative pseudo pre-valuation on an EQ-algebra is a pseudo pre-valuation, while the converse is not true in general. In fact, let φ be a pseudo valuation on L in Example 3.2, however φ is not a positive implicative pseudo pre-valuation since $\varphi(a \to 0) = 5 > \varphi(a \to (a \to 0)) + \varphi(a \to a) = 0$.

In the following, we provide some characterizations of positive implicative pseudo pre-valuations.

Theorem 5.3 *Let $\varphi : L \to R$ be a pseudo pre-valuation on L. Then the following conditions are equivalent:*

(1) *φ is a positive implicative pseudo pre-valuation;*

(2) *$\varphi((x \wedge (x \to y)) \to y) = 0$ for any $x, y \in L$;*

(3) *$\varphi(x \to y) = \varphi(x \to (x \to y))$ for any $x, y \in L$.*

Proof. (1) $\Rightarrow$ (2) Observe that $(x \wedge (x \to y)) \to x = 1$ and $(x \wedge (x \to y)) \to (x \to y) = 1$, we get that $\varphi((x \wedge (x \to y)) \to y) \leq \varphi((x \wedge (x \to y)) \to x) + \varphi((x \wedge (x \to y)) \to (x \to y)) = 0$ by hypothesis, and so $\varphi((x \wedge (x \to y)) \to y) = 0$.

(2) $\Rightarrow$ (3) From $x \to y \leq x \to (x \to y)$, we get that $\varphi(x \to (x \to y)) \leq \varphi(x \to y)$ by Proposition 3.4. For the inverse inequality, consider that $x \to (x \to y) = x \to (x \wedge (x \to y))$ and $\varphi((x \wedge (x \to y)) \to y) = 0$, we get that $\varphi(x \to (x \to y)) = \varphi(x \to (x \wedge (x \to y))) + \varphi((x \wedge (x \to y)) \to y) \geq \varphi(x \to y)$, consequently (3) is valid.

(3) $\Rightarrow$ (1) Suppose that (3) hold, together with $x \to (y \to z) \leq ((y \to z) \to (x \to z)) \to (x \to (x \to z))$ and $x \to y \leq (y \to z) \to (x \to z)$, we obtain that $\varphi(x \to (y \to z)) + \varphi(x \to y) \geq \varphi(((y \to z) \to (x \to z)) \to (x \to (x \to z))) + \varphi((y \to z) \to (x \to z)) \geq \varphi(x \to (x \to z)) = \varphi(x \to z)$. Hence φ is a positive implicative pseudo pre-valuation. ∎

In the following, we will investigate some characterizations of positive implicative pseudo pre-valuations in some special types of EQ-algebras.

Theorem 5.4 *Let L be a residuated EQ-algebra and $\varphi : L \to R$ a pseudo pre-valuation on L. Then the following conditions are equivalent: for any $x, y, z \in L$,*

(1) *φ is a positive implicative pseudo pre-valuation;*

(2) *$\varphi((x \otimes y) \to z) = \varphi((x \wedge y) \to z)$;*

(3) *$\varphi((x \wedge y) \to (x \otimes y)) = 0$;*

(4) *$\varphi((x \wedge (x \to y)) \to (x \otimes y)) = 0$;*

(5) *$\varphi(x \to (x \otimes x)) = 0$.*

Proof. (1) $\Rightarrow$ (2) $\varphi((x \wedge y) \to z) \geq \varphi((x \otimes y) \to z)$ follows from $(x \wedge y) \to z \leq (x \otimes y) \to z$. On the other hand, since L is a residuated EQ-algebra, then $(x \otimes y) \to z \leq x \to (y \to z) \leq (x \wedge y) \to (y \wedge (y \to z))$, therefore $\varphi((x \otimes y) \to z) \geq \varphi((x \wedge y) \to (y \wedge (y \to z)))$. According to Theorem 5.3, we get that $\varphi((y \wedge (y \to z)) \to z) = 0$, and thus $\varphi((x \otimes y) \to z) \geq \varphi((x \wedge y) \to (y \wedge (y \to z))) + \varphi((y \wedge (y \to z)) \to z) \geq \varphi((x \wedge y) \to z)$. Consequently $\varphi((x \otimes y) \to z)) = \varphi((x \wedge y) \to z)$.

(2) $\Rightarrow$ (3) For any $x, y \in L$, we obtain that $\varphi((x \wedge y) \to (x \otimes y)) = \varphi((x \otimes y) \to (x \otimes y)) = 0$.

(3) $\Rightarrow$ (4) Since $x \otimes (x \to y) \leq x \wedge y$, then $(x \wedge (x \to y)) \to (x \otimes (xvy)) \leq (x \wedge (x \to y)) \to (x \wedge y)$. Combination with the hypothesis, we obtain that $0 = \varphi((x \wedge (x \to y)) \to (x \otimes (x \to y))) \geq \varphi((x \wedge (x \to y)) \to (x \wedge y))$.

Hence $\varphi((x \wedge (x \to y)) \to (x \wedge y)) = 0$. According to Proposition 3.5, we have $\varphi((x\wedge(x \to y)) \to (x\otimes y)) \leq \varphi((x\wedge(x \to y)) \to (x\wedge y))+\varphi((x\wedge y) \to (x\otimes y)) = 0$. Hence $\varphi((x \wedge (x \to y)) \to (x \otimes y)) = 0$.

(4) $\Rightarrow$ (5) Taking $y = x$, it is easy to obtain (5) from (4).

(5) $\Rightarrow$ (1) It follows immediately from Proposition 3.5 and hypothesis that $\varphi(x \to z) \leq \varphi(x \to (x \otimes x)) + \varphi((x \otimes x) \to z) = \varphi((x \otimes x) \to z) \leq \varphi((x \otimes x) \to (y \otimes (y \to z))) \leq \varphi((x \otimes x) \to (x \otimes y)) + \varphi((x \otimes y) \to (y \otimes (y \to z)))$. Obverse that L is a residuated EQ-algebra, we have $\varphi((x \otimes x) \to (x \otimes y)) \leq \varphi(x \to y)$ and $\varphi((x \otimes y) \to (y \otimes (y \to z))) \leq \varphi(x \to (y \to z))$ by Lemma 2.5. Thus $\varphi(x \to z) \leq \varphi(x \to (y \to z)) + \varphi(x \to y)$, and so φ is a positive implicative pseudo pre-valuation. ■

Theorem 5.5 *Let L be a good EQ-algebra and φ a pseudo pre-valuation on L. Then φ is a positive implicative pseudo pre-valuation if and only if $\varphi((x \to y) \to (x \to z)) \leq \varphi(x \to (y \to z))$ for any $x, y, z \in L$.*

Proof. Since L is a good EQ-algebra, then $\varphi((x \to y) \to (x \to z)) = \varphi(x \to ((x \to y) \to z)) \leq \varphi((x \to y) \to z)$ by Lemma 2.4 and Proposition 3.4.

Conversely, we have $\varphi(x\to(y\to z))+\varphi(x\to y) \geq \varphi((x\to y)\to(x\to z))+\varphi(x\to y) \geq \varphi(x \to z)$ for any $x, y, z \in L$, thus φ is a positive implicative pseudo pre-valuation. ■

5.2. Implicative pseudo pre-valuations

We now proceed to investigate particular classes of pseudo pre-valuations. For this purpose, we introduce the concept of implicative pseudo pre-valuations as follows.

Definition 5.6 Let $\varphi : L \to R$ be a real-valued function. If φ satisfies the following conditions: for any $x, y, z \in L$,

(1) $\varphi(1) = 0$,

(2) $\varphi(x) \leq \varphi(z \to ((x \to y) \to x)) + \varphi(z)$,

then φ is called an implicative pseudo pre-valuation on L.

Example 5.7 Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	b	a
b	0	b	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Routine calculation shows that $(L, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. Define a real-valued function $\varphi : L \to R$ by $\varphi(0) = 3$, $\varphi(a) = \varphi(b) = \varphi(1) = 0$. Then φ is an implicative pseudo pre-valuation.

Proposition 5.8 *Every implicative pseudo pre-valuation is a pseudo pre-valuation, moreover every implicative pseudo pre-valuation is a positive implicative pseudo pre-valuation.*

Proof. From $y \leq 1 \to y$, it follows that $x \to y \leq x \to (1 \to y)$. If $x \leq y$, then $1 = x \to (1 \to y) = x \to ((y \to y) \to y)$, and so $\varphi(y) \leq \varphi(x \to ((y \to y) \to y)) + \varphi(x) = \varphi(x)$, that is, $x \leq y$ implies that $\varphi(y) \leq \varphi(x)$. Now let $x, y \in L$, since $\varphi(x) + \varphi(x \to y) \geq \varphi(x) + \varphi(x \to (1 \to y)) = \varphi(x) + \varphi(x \to ((y \to 1) \to y)) \geq \varphi(y)$, thus φ is a pseudo pre-valuation. Since φ is an implicative pseudo pre-valuation on L, then $\varphi(x \to y) \leq \varphi(1 \to (((x \to y) \to y) \to (x \to y))) + \varphi(1) = \varphi(((x \to y) \to y) \to (x \to y))$ for any $x, y \in L$. On one hand, due to the fact that $x \to (x \to y) \leq ((x \to y) \to y) \to (x \to y)$, we have $\varphi(((x \to y) \to y) \to (x \to y)) \leq \varphi(x \to (x \to y))$, and so $\varphi(x \to y) \leq \varphi(x \to (x \to y))$. On the other hand, using Proposition 3.4, we get that $\varphi(x \to (x \to y)) \leq \varphi(x \to y)$, thus $\varphi(x \to (x \to y)) = \varphi(x \to y)$. By Theorem 5.3, φ is a positive implicative pseudo pre-valuation. ∎

A positive implicative pseudo pre-valuation is not an implicative pseudo pre-valuation in general. In fact, Let L be the EQ-algebra defined in Example 3.7 and $\varphi : L \to R$ a positive implicative pseudo pre-valuation defined in Example 5.2, while φ is not an implicative pseudo pre-valuation, since $\varphi(a) = 1 > \varphi(1 \to ((a \to 0) \to a)) + \varphi(1) = 0$.

The above discussing also displays that a pseudo pre-valuation is not an implicative pseudo pre-valuation in general. Nextly, we give some conditions for a pseudo pre-valuation to be an implicative pseudo pre-valuation.

Theorem 5.9 *Let $\varphi : L \to R$ be a real-valued function. If φ is a pseudo pre-valuation on L, then φ is an implicative pseudo pre-valuation if and only if $\varphi((x \to y) \to x) = \varphi(x)$ for any $x, y \in L$.*

Proof. Assume that φ is an implicative pseudo pre-valuation, then we get that $\varphi((x \to y) \to x) = \varphi(1 \to ((x \to y) \to x)) + \varphi(1) \geq \varphi(x)$ for any $x, y \in L$. To obtain the reverse inequality, notice that $x \leq (x \to y) \to x$, we have $\varphi(x) \geq \varphi((x \to y) \to x)$, and thus $\varphi((x \to y) \to x) = \varphi(x)$.

Conversely, suppose that $\varphi((x{\to}y){\to}x) = \varphi(x)$ for any $x, y \in L$. Consider that φ is a pseudo pre-valuation on L, we get that $\varphi(z{\to}((x{\to}y){\to}x)) + \varphi(z) \geq \varphi((x{\to}y){\to}x)$. From $\varphi((x{\to}y){\to}x) = \varphi(x)$, it follows that $\varphi(z{\to}((x{\to}y){\to}x)) + \varphi(z) \geq \varphi(x)$. Consequently, φ is an implicative pseudo pre-valuation. ∎

An interesting application of Theorem 5.9 is to a proof of the following important result.

Theorem 5.10 *Let L be an EQ-algebra with a bottom element 0 and φ a pseudo pre-valuation on L, then φ is an implicative pseudo pre-valuation if and only if $\varphi(\neg x \to x) = \varphi(x)$ for any $x \in L$.*

Proof. Suppose that φ is an implicative pseudo pre-valuation, then we have $\varphi(x) = \varphi((x \to 0) \to x) = \varphi(\neg x \to x)$ for any $x \in L$ by Theorem 5.9.

Conversely, due to the fact that $(x\to y)\to x \leq \neg x\to x$, we have $\varphi((x\to y)\to x) \geq \varphi(\neg x\to x) = \varphi(x)$. As for the reverse inequality, observe that $x \leq (x\to y)\to x$, we get $\varphi(x) \geq \varphi((x\to y)\to x)$, and thus $\varphi(x) = \varphi((x\to y)\to x)$. Therefore φ is an implicative pseudo pre-valuation. ∎

Let φ be a pre-valuation on L. If $\varphi(x \to (y \to z)) = \varphi(y \to (x \to z))$ for any $x, y, z \in L$, then we say that φ has the weak exchange principle.

Theorem 5.11 *Let L be an EQ-algebra with a bottom element 0, and φ a pseudo pre-valuation on L with the weak exchange principle. Then the following statements are equivalent:*

(1) *φ is an implicative pseudo pre-valuation;*

(2) *$\varphi(x \to (\neg z \to y)) + \varphi(y \to z) \geq \varphi(x \to z)$ for any $x, y, z \in L$;*

(3) *$\varphi(x \to (\neg z \to z)) = \varphi(x \to z)$ for any $x, z \in L$.*

Proof. (1) $\Rightarrow$ (2) From $y \to z \leq (x \to y) \to (x \to z)$, we get that $\varphi(x \to (\neg z \to y)) + \varphi(y \to z) \geq \varphi(\neg z \to (x \to y)) + \varphi((x \to y) \to (x \to z)) \geq \varphi(\neg z \to (x \to z))$ as φ has the weak exchange principle. Notice that $\neg z \to (x \to z) \leq \neg(x \to z) \to (x \to z)$, together with Theorem 5.10, we obtain that $\varphi(\neg(x \to z) \to (x \to z)) = \varphi(x \to z) \leq \varphi(\neg z \to (x \to z))$. Therefore $\varphi(x \to (\neg z \to y)) + \varphi(y \to z) \geq \varphi(x \to z)$.

(2) $\Rightarrow$ (3) By hypothesis, we have $\varphi(x \to z) \leq \varphi(x \to (\neg z \to z)) + \varphi(z \to z) = \varphi(x \to (\neg z \to z))$ for any $x, z \in L$. To obtain the reverse inequality, obverse that $x \to z \leq x \to (\neg z \to z)$, we get $\varphi(x \to (\neg z \to z)) \leq \varphi(x \to z)$. Thus $\varphi(x \to (\neg z \to z)) = \varphi(x \to z)$.

(3) $\Rightarrow$ (1) Using the hypothesis, together with Proposition 3.4, we have $\varphi(x) = \varphi(1 \to x) = \varphi(1 \to (\neg x \to x)) = \varphi(\neg x \to x)$. By Theorem 5.10, φ is an implicative pseudo pre-valuation. ∎

Next, we further find the conditions under which a positive implicative pseudo pre-valuation is equivalent to an implicative pseudo pre-valuation.

Theorem 5.12 *Let L be an EQ-algebra with a bottom element 0, and φ a positive implicative pseudo pre-valuation on L. Then φ is an implicative pseudo pre-valuation if and only if $\varphi(x) \leq \varphi(\neg\neg x)$ for any $x \in L$.*

Proof. Assume that φ is an implicative pseudo pre-valuation, from $\neg\neg x = \neg x \to 0 \leq \neg x \to x$, it follows that $\varphi(\neg x \to x) \leq \varphi(\neg\neg x)$. According to Theorem 5.10, we get $\varphi(x) = \varphi(\neg x \to x) \leq \varphi(\neg\neg x)$.

Conversely, since $\neg x \to x \leq (x \to 0) \to (\neg x \to 0) = \neg x \to (\neg x \to 0)$, then $\varphi(\neg x \to (\neg x \to 0)) \leq \varphi(\neg x \to x)$. It follows immediately from Theorem 5.3 and Proposition 3.4 that $\varphi(\neg\neg x) = \varphi(\neg x \to 0) \leq \varphi(\neg x \to x) \leq \varphi(x)$. Using hypothesis, we obtain that $\varphi(\neg x \to x) = \varphi(x)$. By Theorem 5.10, φ is an implicative pseudo pre-valuation. ∎

As a consequence of Theorem 5.12 together with Lemma 2.4, we have the following result.

Corollary 5.13 *Let L be a good EQ-algebra with a bottom element 0, and φ a positive implicative pseudo pre-valuation on L. Then φ is an implicative pseudo pre-valuation if and only if $\varphi(x) = \varphi(\neg\neg x)$ for any $x \in L$.*

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