AN UNCONDITIONALLY STABLE FINITE DIFFERENCE SCHEME FOR EQUATIONS OF CONSERVATION LAW FORM

Saeid Abbasbandy[1]
Ahmad Shirzadi

Department of Mathematics
Imam Khomeini International University
Qazvin
Iran

Abstract. This study presents a numerical scheme for solving one dimensional equations of conservation law form. The Saulyev's finite difference techniques are used to compute the solution. Although the resulting difference equation do not appear explicit, a suitable use of the equation make it explicit. It is shown that this explicit scheme is unconditionally stable. A numerical example is presented to demonstrate the accuracy and efficiency of the proposed computational procedure.
Keywords: finite difference schemes; implicit methods; explicit techniques; Saulyev's technique; Lax-Wendroff formula.

1. Introduction

Finite elements [1], [2], Finite differences [3], [4] and recently meshless methods [5], [6] are known to be powerful numerical methods to solve partial differential equations with boundary conditions. In the theory of fluid flow, the equations of motions, of continuity, and of energy can be combined into one conservation equation of the form

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \tag{1.1}$$

where U and F are column vectors [7, 8]. The Lax-Wendroff method, can be used to approximate Eqn. (1.1) by an explicit difference equation of second order accuracy. Consider the following conservation equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \tag{1.2}$$

where a is a positive constant and u is a function of x and t. Grid points (x_i, t_j) are defined by $x_i = x_0 + ih, i = 1, 2, ...$ and $t_j = t_0 + jk, j = 1, 2, ...$ The notation $u_{i,j}$ is used for the finite difference approximations of $u(x_i, t_j)$. By Taylor's expansion and elimination of the t-derivatives using the differential equation(1.2) we obtain:

$$u_{i,j+1} = u_{i,j} - ka\left(\frac{\partial u}{\partial x}\right)_{i,j} + \frac{1}{2}k^2a^2\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \cdots . \tag{1.3}$$

[1]Corresponding author. Email: abbasbandy@yahoo.com, Tel.:+98(912)1305326.

The replacement of the x-derivatives by central-difference approximations gives the explicit difference equation:

$$u_{i,j+1} = u_{i,j} - \frac{a\varphi}{2}(u_{i+1,j} - u_{i-1,j}) + \frac{a^2\varphi^2}{2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}), \tag{1.4}$$

where $\varphi = \frac{k}{h}$. Scheme (1.4)is named the Lax-Wendroff explicit scheme and is stable for $0 < a\varphi \leq 1$ [3], [4]. In the next section we will present a scheme which is unconditionally stable.

2. Main results

2.1. Proposed finite difference scheme

A noticeable feature of the current explicit finite difference methods for the numerical solution is the restriction of the size of the time step due to stability requirements. For most problems these are impractical methods. This limitation is removed when the implicit finite difference schemes are used for the numerical solution of the equations. However, a disadvantage of these techniques is the extensive amount of CPU times utilized in determining the numerical solution compared to the explicit methods for the same selection of values of step-sizes k and h. Implicit finite difference schemes require the solution of a large number of simultaneous linear algebraic equations at each time step. The number of iterations require to achieve a modest accuracy may become large, particularly for large time increments and small space mesh size. So the need to develop unconditionally stable Saulyev's finite difference schemes is clear[9], [10]. The main advantage of these techniques is that they are unconditionally stable and are explicit in nature [3], [4]. If we use the Saulyev's A formula for (1.4) we get the following equation:

$$\begin{aligned}\left(1+\frac{a^2\varphi^2}{2}\right)u_{i,j+1} &= \frac{a^2\varphi^2}{2}u_{i-1,j+1} + \frac{a\varphi}{2}u_{i-1,j} + \left(1-\frac{a^2\varphi^2}{2}\right)u_{i,j} \\ &+ \frac{a\varphi}{2}(a\varphi-1)u_{i+1,j}.\end{aligned} \tag{2.1}$$

Although the above approximation does not appear explicit, because $u_{i,j+1}$ and $u_{i,j}$ are on the right-hand side, a suitable use of the equation makes it explicit. If we begin the calculation from the left to right, the only unknown is $u_{i,j+1}$.

2.2. Stability analysis

For stability analysis we consider the equation

$$\left(1+\frac{a^2\varphi^2}{2}\right)u_{i,j+1} = \frac{a^2\varphi^2}{2}u_{i-1,j+1} + \frac{a\varphi}{2}u_{i-1,j} + \left(1-\frac{a^2\varphi^2}{2}\right)u_{i,j} + \frac{a\varphi}{2}(a\varphi-1)u_{i+1,j}.$$

Substituting $u_{i,j} = z^j e^{\sqrt{-1}\lambda i h}$ and eliminating common factors from the both sides of the equation we have

$$\left(1+\frac{a^2\varphi^2}{2}\right)z = \frac{a^2\varphi^2}{2}ze^{-\sqrt{-1}\lambda h} + \frac{a\varphi}{2}e^{-\sqrt{-1}\lambda h} + \left(1-\frac{a^2\varphi^2}{2}\right) + \frac{a\varphi}{2}(a\varphi-1)e^{\sqrt{-1}\lambda h},$$

therefore,

$$\left(1+\frac{a^2\varphi^2}{2}-\frac{a^2\varphi^2}{2}\cos(\lambda h)\right)z+\sqrt{-1}\frac{a^2\varphi^2}{2}\sin(\lambda h)z$$
$$=1-\frac{a^2\varphi^2}{2}+\frac{a^2\varphi^2}{2}\cos(\lambda h)+\sqrt{-1}\left(\frac{a^2\varphi^2}{2}-a\varphi\right)\sin(\lambda h).$$

Consequently,

$$\left(1+\frac{a^2\varphi^2}{2}-\frac{a^2\varphi^2}{2}\cos(\lambda h)\right)^2+\frac{a^4\varphi^4}{4}\sin^2(\lambda h))|z|^2$$
$$=\left(1-\frac{a^2\varphi^2}{2}+\frac{a^2\varphi^2}{2}\cos(\lambda h)\right)^2+a^2\varphi^2\left(\frac{a\varphi}{2}-1\right)^2\sin^2(\lambda h),$$

so we have

$$|z|^2=\frac{\left(1-\frac{a^2\varphi^2}{2}+\frac{a^2\varphi^2}{2}\cos(\lambda h)\right)^2+a^2\varphi^2\left(\frac{a\varphi}{2}-1\right)^2\sin^2(\lambda h)}{\left(1+\frac{a^2\varphi^2}{2}-\frac{a^2\varphi^2}{2}\cos(\lambda h)\right)^2+\frac{a^4\varphi^4}{4}\sin^2(\lambda h)}.$$

For stability we should have$|z|^2\leq 1$ and, therefore,

$$\left(1-\frac{a^2\varphi^2}{2}+\frac{a^2\varphi^2}{2}\cos(\lambda h)\right)^2+a^2\varphi^2\left(\frac{a\varphi}{2}-1\right)^2\sin^2(\lambda h)$$
$$\leq(1+\frac{a^2\varphi^2}{2}-\frac{a^2\varphi^2}{2}\cos(\lambda h))^2+\frac{a^4\varphi^4}{4}\sin^2(\lambda h),$$

simplifying the above non-equality we have $(1-a\varphi)(1+\cos(\lambda h))\leq 2$, therefore,

$$(1-a\varphi)\leq\frac{2}{(1+\cos(\lambda h))},$$

since $1\leq\cos(\lambda h)\leq 1$, if $1-a\varphi\leq 1$ or $a\varphi\geq 0$ then the above non-equality is hold. Because $a\geq 0$ and $\varphi\geq 0$, the relation is hold and, therefore, scheme (2.1) is unconditionally stable.

3. Test example

For a test example, we consider the following equation

$$\frac{\partial U}{\partial t}+\frac{\partial U}{\partial x}=0,\quad 0<x<\infty,\quad t>0, \tag{3.1}$$

with the boundary condition

$$U(0,t)=2t,\quad t>0, \tag{3.2}$$

and initial conditions

$$\begin{aligned}U(x,0)&=x(x-2),\quad 0\leq x\leq 2,\\ U(x,0)&=2(x-2),\quad 2\leq x.\end{aligned} \tag{3.3}$$

The solution obtained via the proposed finite difference scheme are presented in some mesh points(x=1 and $t = 0.5\ i,\ i = 1, 2, ..., 8$) and are compared with the exact solution:

t	0.5	1	1.5	2	2.5	3	3.5	4
Exact	-0.75	0	1	2	3	4	5	6
approximate	-0.556	0.185	1.074	2.028	3.010	4.004	5.001	6.000

4. Conclusions

This article has outlined an approach for the study of a particular class of hyperbolic partial differential equations. The Saulyev's first kind explicit finite difference technique was applied. The Proposed finite difference method has been proved to be unconditionally stable. The new algorithm outlined here, was tested on a problem and was seen to produce good results that suggest convergence to the exact solution when h goes to zero. The new scheme discussed in this report, had the advantage of being stable and explicit. The results reveal that the method is remarkably effective.

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References

[1] Braess, D., *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*, Cambridge University Press; 3 edition, 2007.

[2] Johnson, C., *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Dover Publications, 2009.

[3] Lapidus, L., Pinder, G.F., *Numerical solution of partial differential equations in science and engineering*, Wiley, New York, 1982.

[4] Smith, G.D., *Numerical solution of partial differential equations*, Oxford University Press, 1985.

[5] Abbasbandy, S., Shirzadi, A., *A meshless method for two-dimensional diffusion equation with an integral condition*, Eng. Anal. Bound. Elem. 34 (12) (2010) 1031–1037.

[6] Abbasbandy, S., Shirzadi, A., *MLPG method for two-dimensional diffusion equation with Neumann's and non-classical boundary conditions*, Appl. Numer. Math. 61 (2011) 170–180.

[7] Li,X.-G., Yu, X.J., Chen, G.-N., *The third-order relaxation schemes for hyperbolic conservation laws*, J. Computat. Appl. Math., 138 (2002), 93-108.

[8] Qiu, J., Dumbser, M., C.-W., *The discontinuous Galerkin method with LaxWendroff type time discretizations*, Comput.Methods Appl.Mech.Engrg., 194 (2005), 4528-4543.

[9] Dehghan, M., *Identifying a control function in two-dimensional parabolic inverse problems*, Appl. Math. Comput., 143(2003), 375-391.

[10] Dehghan, M., *The one-dimensional heat equation subject to a boundary integral specification*, Chaos, Solitons and Fractals, 32 (2007), 661-675.

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