ON THE ORDER AND LOWER ORDER OF LAPLACE-STIELTJES TRANSFORMATIONS WITH INDEX PAIR (p, q)

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Abstract. In this paper, in order to study the precise growth of entire functions represented by Laplace-Stieltjes transformations, we have introduced the concept of (p, q)-order and lower (p, q)-order and obtained their coefficient characterizations.

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1. Introduction

Consider the Laplace-Stieltjes transformation defined by

$$G(s) = \int_0^\infty \exp(-sx)\, d\alpha(x) \tag{1.1}$$

where $\alpha(x)$ is a function of bounded variation on any finite interval $[0, X]$, $(0 < X < +\infty)$, $s = \sigma + it$, σ and t are real variables. We choose a monotonic increasing sequence of real numbers $\{\lambda_n\}$ satisfying the following conditions:

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdot \cdot \cdot < \lambda_n \uparrow +\infty, \tag{1.2}$$

$$\limsup_{n\to\infty} (\lambda_{n+1} - \lambda_n) < +\infty, \ \limsup_{n\to\infty} \frac{\ln n}{\lambda_n} = D < +\infty. \tag{1.3}$$

We put

$$K_n^* = \sup_{\substack{\lambda_n < x \le \lambda_{n+1}, \\ -\infty < t < +\infty}} \left| \int_{\lambda_n}^{x} e^{-ity} d\alpha(y) \right|.$$

In [2], Jiarong obtained the following Valiron-Knopp-Bohr formula:

Theorem A. *Suppose that the Laplace-Stieltjes transformation* (1.1) *satisfies*

$$\limsup_{n\to\infty}\ (\lambda_{n+1}-\lambda_n) < +\infty \ \ \text{and} \ \ \limsup_{n\to\infty}\frac{\ln n}{\lambda_n} < +\infty,$$

and σ_μ^G *denotes the abscissa of uniform convergence of the integral in* (1.1). *Then*

$$\limsup_{n\to\infty}\frac{\ln K_n^*}{\lambda_n} \le \sigma_\mu^G \le \limsup_{n\to\infty}\frac{\ln K_n^*}{\lambda_n} + \limsup_{n\to\infty}\frac{\ln n}{\lambda_n}. \tag{1.4}$$

Suppose that

$$\limsup_{n\to\infty}\frac{\ln K_n^*}{\lambda_n} = 0. \tag{1.5}$$

If $D = 0$, *then, by* (1.4) *and* (1.5), *it follows that* $\sigma_\mu^G = 0$ *and* $F(s)$ *is analytic in the right half plane* $\sigma > 0$.

Various authors have obtained the growth properties of the analytic function defined by (1.1). Yinying and Daochun [6] introduced a type function using the proximate orders. Similarly, Hong et al [1] also obtained some growth properties using the proximate order and type function. Kong and Yang [7] considered the Laplace-Stieltjes transformations given by (1.1) converging uniformly in the whole complex plane $Re(s) > -\infty$ and studied their growth properties. Yingying and Hong [8] have also studied the properties of Laplace-Stieltjes transformations. In 2012, Luo Xi and Kong Yinying [5] defined Laplace-Stieltjes transformations in a different manner by taking positive exponents in the integral in (1.1). Thus they defined Laplace-Stieltjes transformations as given below:

$$F(s) = \int_0^{+\infty} \exp(sy)d\alpha(y), (s = \sigma + it), \tag{1.6}$$

where $\alpha(y)$ satisfies same conditions as stated earlier. Let the sequence $\{\lambda_n\}$ satisfy both conditions stated in (1.3). We put

$$A_n^* = \sup_{\substack{\lambda_n < x \le \lambda_{n+1},\\ -\infty < t < +\infty}} \left| \int_{\lambda_n}^{x} e^{ity} d\alpha(y) \right|.$$

Then the result of Theorem A can be proved for (1.6) also. We assume that the function $F(s)$ given by (1.6) satisfies the condition $\liminf\limits_{n\to\infty} \frac{\ln (A_n^*)^{-1}}{\lambda_n} = +\infty$ i.e. $F(s)$ represents an entire function.

We now give

Definition 1. The maximum modulus, the maximum term and central index of $F(s)$ given by (1.6) are defined as

$$\begin{aligned} M(\sigma, F) &= \sup_{-\infty<t<+\infty} |F(\sigma+it)|, \\ M_\mu(\sigma, F) &= \sup_{\substack{0<x\leqslant+\infty,\\ -\infty<t<+\infty}} \left| \int_0^x e^{sy} d\alpha(y) \right|, s=\sigma+it, \sigma>0, \\ \mu(\sigma, F) &= \max_{1\leq n<N} \left\{ A_n^* e^{\lambda_n \sigma} \right\}, \sigma>0, \\ N(\sigma, F) &= \max\left\{ n; \mu(\sigma, F) = A_n^* e^{\lambda_n \sigma} \right\}. \end{aligned}$$

Using Lemma 1 [5], we can easily show for our case that for $\varepsilon > 0$ and σ sufficiently large,

$$\frac{1}{2}\mu(\sigma, F) \leq M_\mu(\sigma, F) \leq \mu(\sigma+D+\varepsilon, F). \tag{1.7}$$

In [7], Kong and Yang have studied the growth properties of L-S transform representing entire function using generalized functions. In this paper we have defined (p,q)-order and lower (p,q)-order of Laplace-Stieltjes transformations and obtained their coefficient characterizations. These results are more explicit and depict the relation between growth parameters and behaviour of the sequence $\{A_n^*\}$.

Following the definitions given by Juneja et al [4] for classical Dirichlet series, we define the index pair (p,q) of entire functions represented by L-S transformation and their order and lower order. We use the following notations: $\exp^{[0]}x=\ln^{[0]}x=x$; $\exp^{[m]}x = \ln^{[-m]}x = \exp(\exp^{[m-1]}x) = \ln(\ln^{[-m-1]})$, $m = \pm1, \pm2, \ldots$ $E_{[m]}(x) = \prod_{i=0}^{m} \exp^{[i]}x$; $\Lambda_{[m]}(x) = \prod_{i=0}^{m} \ln^{[i]}x$, $E_{[-m]}(x) = \frac{x}{\Lambda_{[m-1]}(x)}$, $\Lambda_{[-m]}(x) = \frac{x}{E_{[m-1]}(x)}$, $m = 0, \pm1, \ldots$

We define the (p,q)-order and lower (p,q)-order with index pair (p,q) of entire functions represented by Laplace-Stieltjes transformations $F(s)$ given by (1.6) above. Hence we put

$$\rho(\lambda) = \lim_{\sigma\to\infty} \sup(\inf) \frac{\ln^{[p]} M_\mu(\sigma, F)}{\ln^{[q]}\sigma}, \tag{1.8}$$

where p and q are integers such that $p > q \geq 0$. It can be easily seen that $0 \leq \rho(p,q) \leq \infty$ if $p > q+1$ and $1 \leq \rho(p,q) \leq \infty$ if $p = q+1$. Further, in view of (1.7) we have the equivalent formula for order ρ and lower order λ i.e.,

$$\rho(\lambda) = \lim_{\sigma\to\infty} \sup(\inf) \frac{\ln^{[p]} \mu(\sigma, F)}{\ln^{[q]}\sigma}. \tag{1.9}$$

We also put for $\alpha \geq 0$,

$$P_\alpha(x) = \begin{cases} x & \text{if} \quad q+1 < p < \infty \\ x+\alpha & \text{if} \quad p = q+1 = 2 \\ \max(1,x) & \text{if} \quad 3 \leq q+1 = p < \infty \\ \infty & \text{if} \quad p = q = \infty. \end{cases}$$

We put $P_1(x) = P(x)$. For more details about the notations, we shall refer to [4].

2. Main results

We now prove

Theorem 1. *Let* $F(s) = \int_0^\infty \exp(sy) d\alpha(y)$ *be Laplace-Stieltjes transformation and let for a pair of integers* (p,q), $p \geq 2, q \geq 0$, *the* (p,q)*-order* ρ *of* $F(s)$ *be defined by* (1.8). *Then*

$$\rho = P(L), \tag{2.1}$$

where

$$L \equiv L(p,q) = \limsup_{n\to\infty} \frac{\ln^{[p-1]}\lambda_n}{\ln^{[q]}\left\{(1/\lambda_n)\ln(A_n^*)^{-1}\right\}}. \tag{2.2}$$

Proof. Let us assume that $\rho = \rho(p,q) < \infty$, then from the definition of ρ, for any $\varepsilon > 0$ and all $\sigma > \sigma_0(\varepsilon)$, we get

$$\ln M_\mu(\sigma, F) < \exp^{[p-1]}\left\{(\rho+\varepsilon)\ln^{[q]}\sigma\right\}.$$

From (1.7) and the definition of $\mu(\sigma, F)$, we have

$$\ln A_n^* + \lambda_n\sigma \ \leq\ \ln\mu(\sigma, F) \ \leq\ \ln M_\mu(\sigma, F) + o(1).$$

Hence

$$\ln A_n^* < \exp^{[p-1]}\left\{(\rho+\varepsilon)\ln^{[q]}\sigma\right\} - \lambda_n\sigma. \tag{2.3}$$

For $(p,q) \neq (2,1)$, we choose

$$\sigma = \exp^{[q-1]}\left\{\ln^{[p-2]}(\lambda_n)\right\}^{1/(\rho+\varepsilon)}.$$

For $p \geq 2$ and $q \geq 0$, from (2.3) we have

$$\ln A_n^* < \ \lambda_n - \lambda_n \exp^{[q-1]}\left\{\ln^{[p-2]}\lambda_n\right\}^{1/(\rho+\varepsilon)}.$$

Now, if $p = q+1$, then $\rho \geq 1$. Therefore, for $p = q+1 \geq 3$, the above inequality gives

$$\frac{1}{\lambda_n}\ln(A_n^*)^{-1} > \exp^{[q-1]}\left\{\ln^{[p-2]}\lambda_n\right\}^{1/(\rho+\varepsilon)} - 1,$$

or

$$\ln^{[q]}\left\{\frac{1}{\lambda_n}\ln(A_n^*)^{-1}\right\} > \frac{1}{\rho+\varepsilon}\left[\ln^{[p-1]}\lambda_n\right](1+o(1)),$$

or

$$\rho+\varepsilon > \frac{\left[\ln^{[p-1]}\lambda_n\right]}{\ln^{[q]}\left\{(1/\lambda_n)\ln(A_n^*)^{-1}\right\}}(1+o(1)).$$

By proceeding to limits, we get

$$\rho\,(p,p-1) \geq \max\,(1, L\,(p,p-1))\,. \tag{2.4}$$

From the above inequality, we also have for $p > q+1$ and $(p,q) \neq (2,1)$,

$$\rho\,(p,q) \geq L\,(p,q)\,.$$

Thus (2.1) is proved for all pairs (p,q) except for $(p,q) = (2,1)$.

Now, for $(p,q) = (2,1)$, we choose

$$\sigma = \lambda_n^{1/(\rho+\varepsilon)}.$$

After substituting this value of σ in (2.3), we get for $(p,q) = (2,1)$

$$\ln A_n^* < \lambda_n - \lambda_n(\lambda_n)^{1/(\rho+\varepsilon)},$$

or

$$\ln\left\{\frac{1}{\lambda_n}\ln\,(A_n^*)^{-1}\right\} > \frac{1}{(\rho+\varepsilon)}\ln\,(\lambda_n) + o\,(1)\,.$$

By proceeding to limits, we get

$$\limsup_{n\to\infty}\frac{\ln\lambda_n}{\ln\left\{(1/\lambda_n)\ln\,(A_n^*)^{-1}\right\}} \;\leq\; \rho\,.$$

Hence, for all index pair (p,q),

$$\rho\,(p,q) \geq L\,(p,q)\,. \tag{2.5}$$

To prove the reverse inequality, we assume that $L(p,q) < \infty$. For any $\varepsilon > 0$, there is a positive integer n_o such that for $n > n_o$, we have

$$A_n^* < \exp\left\{-\lambda_n \exp^{[q-1]}\left(\ln^{[p-2]}\lambda_n\right)^{1/L+\varepsilon}\right\}. \tag{2.6}$$

Also, for a number $\sigma_o > 1$ and $\sigma > \sigma_o$, we can find an integer S such that

$$\lambda_S < \exp^{[p-2]}\left(\ln^{[q]}2e^{\sigma}\right)^{(L+\varepsilon)} < \lambda_{S+1}.$$

From the definition, we have $M_\mu\,(\sigma,F) \leq \sum\limits_{n=0}^{\infty} A_n^* \exp\,(\sigma\lambda_n)\,.$ Then

$$\begin{aligned} M_\mu\,(\sigma,F) \leq O(1) &+ \sum_{n=n_o+1}^{S} A_n^* \exp\,(\sigma\lambda_n) \\ &+ \sum_{n=S+1}^{\infty} \exp\left\{\sigma\lambda_n - \lambda_n \exp^{[q-1]}\left(\ln^{[p-2]}\lambda_n\right)^{1/(L+\varepsilon)}\right\}. \end{aligned} \tag{2.7}$$

Hence, using (2.6) we obtain

$$(2.8)\quad \begin{aligned}\sum_{n=n_o+1}^{S} A_n^* \exp(\sigma\lambda_n) &< \exp(\sigma\lambda_S) \sum_{n=n_o+1}^{S} \exp\left\{-\lambda_n \exp^{[q-1]}\left(\ln^{[p-2]}\lambda_n\right)^{1/L+\varepsilon}\right\} \\ &< \exp(\sigma\lambda_S) \sum_{n=n_o+1}^{\infty} \exp\left\{-\lambda_{n_o} \exp^{[q-1]}\left(\ln^{[p-2]}\lambda_n\right)^{1/L+\varepsilon}\right\} \\ &< B\exp(\sigma\lambda_S)\end{aligned}$$

since the series on right hand side is convergent. Similarly, we have

$$(2.9)\quad \begin{aligned}&\sum_{n=S+1}^{\infty} \exp\left\{\sigma\lambda_n - \lambda_n \exp^{[q-1]}\left(\ln^{[p-2]}\lambda_S\right)^{1/(L+\varepsilon)}\right\} \\ &\leq \sum_{n=S+1}^{\infty} \exp\{\sigma\lambda_n - \lambda_n \ln(2e^{\sigma})\} \ \leq \sum_{n=0}^{\infty} 2^{-\lambda_n} = C < \infty.\end{aligned}$$

Combining the estimates obtained in (2.8) and (2.9), we get

$$M_\mu(\sigma, F) \leq A + B\exp\left\{\sigma \exp^{[p-2]}\left(\ln^{[q]} 2e^{\sigma}\right)^{(L+\varepsilon)}\right\} + C$$

or

$$\ln^{[2]} M_\mu(\sigma, F) \leq \exp^{[p-3]}\left(\ln^{[q]} 2e^{\sigma}\right)^{(L+\varepsilon)} + o(1)$$

or

$$\begin{aligned}\ln^{[p]} M_\mu(\sigma, F) &\leq \ln\left(\ln^{[q]} 2e^{\sigma}\right)^{(L+\varepsilon)} (1 + o(1)) \\ &= \left\{(L+\varepsilon)\ln^{[q]} \ln(2e^{\sigma})\right\} (1+o(1)) \\ &\simeq (L+\varepsilon)\ln^{[q]}(\sigma).\end{aligned}$$

Proceeding to limits, we obtain

$$(2.10)\quad \lim_{\sigma\to\infty}\sup \frac{\ln^{[p]} M_\mu(\sigma, F)}{\ln^{[q]}\sigma} = \rho(p,q) \leq L(p,q).$$

Thus the above inequality is true for all pairs of (p, q). From (2.5) and (2.10), we get (2.1) and the proof of the Theorem 1 is complete. ∎

Next, we prove

Lemma 1. *Let* $F(s) = \int_0^\infty \exp(sy) \mathrm{d}\alpha(y)$ *be Laplace-Stieltjes transformation having* (p, q)*-order* ρ *and lower* (p, q)*-order* λ*. Then*

$$(2.11)\quad \lim_{\sigma\to\infty}\sup(\inf) \frac{\ln^{[p-1]}\lambda_{N(\sigma,F)}}{\ln^{[q]}\sigma} = \rho(\lambda).$$

Proof. From a result of Jingjing [3, p.7], we have

$$\ln \mu(\sigma, F) = \ln \mu(\sigma', F) + \int_{\sigma'}^{\sigma} \lambda_{N(x,F)}\, dx, \quad \text{where } 0 < \sigma' < \sigma < \infty.$$

Since $\lambda_{N(\sigma,F)}$ is nondecreasing, therefore we have

$$\ln \mu(\sigma, F) \;\leq\; \ln \mu(\sigma', F) + \lambda_{N(\sigma,F)}(\sigma - \sigma')$$

or

$$(1 - o(1))\; \ln \ln \mu(\sigma, F) \;\leq\; (1 + o(1))\; \ln \lambda_{N(\sigma,F)},$$

or

$$\ln^{[p]} \mu(\sigma, F) \;\leq\; \ln^{[p-1]} \lambda_{N(\sigma,F)}.$$

Hence,

$$\limsup_{\sigma\to\infty}(\inf)\; \frac{\ln^{[p]} \mu(\sigma, F)}{\ln^{[q]} \sigma} \;\leq\; \limsup_{\sigma\to\infty}(\inf)\; \frac{\ln^{[p-1]} \lambda_{N(\sigma,F)}}{\ln^{[q]} \sigma}.$$

Conversely, we have

$$\ln \mu(2\sigma, F) = \ln \mu(\sigma', F) + \int_{\sigma'}^{2\sigma} \lambda_{N(x,F)}\, dx \geq O(1) + \int_{\sigma}^{2\sigma} \lambda_{N(x,F)}\, dx \geq \sigma\, \lambda_{N(\sigma,F)}.$$

Following as above, we obtain

$$\limsup_{\sigma\to\infty}(\inf)\; \frac{\ln^{[p]} \mu(\sigma, F)}{\ln^{[q]} \sigma} \;\geq\; \limsup_{\sigma\to\infty}(\inf)\; \frac{\ln^{[p-1]} \lambda_{N(\sigma,F)}}{\ln^{[q]} \sigma}.$$

Combining the two inequalities obtained above, we get (2.11). ∎

Now, we obtain some lower bounds for the lower (p, q)-order λ.

Lemma 2. *Let $F(s) = \int_0^\infty \exp(sy) d\alpha(y)$ be Laplace-Stieltjes transformation having lower (p, q)-order λ and let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of natural numbers. Then*

$$\lambda \geq P_\alpha(\Im) \tag{2.12}$$

where

$$\alpha \equiv \alpha(\{n_k\}) = \lim_{k\to\infty} \inf \frac{\ln \lambda_{n_{k-1}}}{\ln \lambda_{n_k}},$$

and

$$\Im \equiv \Im(\{n_k\}, p, q) = \lim_{k\to\infty} \inf \frac{\ln^{[p-1]} \lambda_{n_{k-1}}}{\ln^{[q]} \left\{ 1/\lambda_{n_k} \ln \left(A^*_{n_k} \right)^{-1} \right\}}. \tag{2.13}$$

Proof. We assume that $0 < \Im < \infty$. Then, for a given ε, $\Im > \varepsilon > 0$, we have from (2.13)

$$(A^*_{n_k}) > \exp\left[-\lambda_{n_k}\exp^{[q]}\left\{\ln^{[p-1]}\lambda_{n_{k-1}}\Big/(\Im-\varepsilon)\right\}\right]. \tag{2.14}$$

Since addition of an exponential polynomial does not affect the growth of $F(s)$, we can assume that (2.14) holds for all k. Now for the pair $(p,q)\neq(2,1)$, we choose a sequence

$$\sigma_k = 2+\exp^{[q]}\left\{\ln^{[p-1]}\lambda_{n_{k-1}}\Big/(\Im-\varepsilon)\right\}, k=1,2,.... \tag{2.15}$$

Then $\{\sigma_k\}$ is monotonic increasing and $\sigma_k\to\infty$ as $k\to\infty$. If $\sigma_k\le\sigma<\sigma_{k+1}$, then for $(p,q)\neq(2,1)$, from (2.14) we get

$$\ln M_\mu(\sigma,F)\ge \ln A^*_{n_k}+\lambda_{n_k}\sigma_k > \quad \sigma_k\lambda_{n_k}-\lambda_{n_k}\exp^{[q]}\left\{\ln^{[p-1]}\lambda_{n_{k-1}}\Big/(\Im-\varepsilon)\right\}.$$

Now, substituting for σ_k, from (2.15), we get

$$\begin{aligned}\ln M_\mu(\sigma,F) \ge 2\lambda_{n_k} &= 2\exp^{[p-1]}\left\{(\Im-\varepsilon)\ln^{[q]}(\sigma_{k+1}-2)\right\}\\ &\ge 2\exp^{[p-1]}\left\{(\Im-\varepsilon)\ln^{[q]}(\sigma-2)\right\}.\end{aligned}$$

For sufficiently large value of σ, we have

$$\ln^{[p]}M_\mu(\sigma,F)\ge(\Im-\varepsilon)\ln^{[q]}\sigma+o(1).$$

For $p=q+1\ge 3$, we have $\lambda\ge 1$, so we have, by the above inequality,

$$\lim_{\sigma\to\infty}\inf\frac{\ln^{[p]}M_\mu(\sigma,F)}{\ln^{[q]}\sigma}\ge\max(1,\Im). \tag{2.16}$$

For $p>q+1$, the same inequality implies

$$\lim_{\sigma\to\infty}\inf\frac{\ln^{[p]}M_\mu(\sigma,F)}{\ln^{[q]}\sigma}\ge\Im. \tag{2.17}$$

Now, for $(p,q)=(2,1)$, as $1\le\Im<\infty$, we choose

$$\sigma_k = 2\left(\lambda_{n_{k-1}}\right)^{1/(\Im-\varepsilon)}, k=1,2,... \tag{2.18}$$

If $\sigma_k\le\sigma<\sigma_{k+1}$, after proceeding as in the previous case, we get

$$\ln M_\mu(\sigma,F)\ge\lambda_{n_k}\left(\lambda_{n_{k-1}}\right)^{1/(\Im-\varepsilon)},$$

which gives

$$\frac{\ln\ln M_\mu(\sigma,F)}{\ln\sigma}\ge\frac{\ln\lambda_{n_k}}{\ln\sigma_{k+1}}+\frac{1}{(\Im-\varepsilon)}\,\frac{\ln\lambda_{n_{k-1}}}{\ln\sigma_{k+1}}.$$

After substituting the value of σ_{k+1}, from (2.18) and proceeding to limits as $k \to \infty$, we get

$$\lambda \geq \Im + \chi. \tag{2.19}$$

After combining (2.16), (2.17) and (2.19) we get the desired result for $0 < \Im < \infty$. If $\Im = 0$ (2.12) is trivially true and if $\Im = \infty$, then, repeating the above arguments with an arbitrarily large number in place of $(\Im - \varepsilon)$, we get the desired result. This completes the proof of the Lemma 2. ∎

In the next result, we characterize the lower $(p, q)-$ order in terms of the ratio $\left\{A^*_{n-1}/A^*_n\right\}$. We prove

Lemma 3. *Let $F(s) = \int_0^\infty \exp(sy) \mathrm{d}\alpha(y)$ be Laplace-Stieltjes transformation having lower (p, q)-order λ and let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of natural numbers, then*

$$\lambda \geq C_\chi(\Im^*) \tag{2.20}$$

where

$$\Im^* \equiv \Im^*(\{n_k\}, p, q) = \liminf_{k\to\infty} \frac{\ln^{[p-1]} \lambda_{n_{k-1}}}{\ln^{[q]} \left\{ \left(\ln A^*_{n_{k-1}} \middle/ A^*_{n_k} \right) \middle/ \left(\lambda_{n_k} - \lambda_{n_{k-1}} \right) \right\}}, \tag{2.21}$$

and χ is as defined in Lemma 2.

Proof. If $\Im^*$ is zero then (2.20) is trivially true, therefore it is sufficient to consider the case when $0 < \Im^* \leq \infty$. For any ε such that $\Im^* > \varepsilon > 0$ and for all $k > k_1 = k_1(\varepsilon)$, from (2.21), we have

$$\left(A^*_{n_{k-1}} \middle/ A^*_{n_k} \right) < \exp\left[\left(\lambda_{n_k} - \lambda_{n_{k-1}} \right) \exp^{[q]} \left\{ \ln^{[p-1]} \lambda_{n_{k-1}} \middle/ (\Im^* - \varepsilon) \right\} \right].$$

Writing the above inequality for $m = k_1 + 1, ..., k$ and multiplying side by side, we get

$$A^*_{n_k} > A^*_{n_{k_1}} \prod_{m=k_1+1}^{k} \exp\left[- \left(\lambda_{n_m} - \lambda_{n_{m-1}} \right) \exp^{[q]} \left\{ \ln^{[p-1]} \lambda_{n_{m-1}} \middle/ (\Im^* - \varepsilon) \right\} \right]. \tag{2.22}$$

Let the sequence $\{\sigma_k\}$ be chosen as in equations (2.15) and (2.18) for the two cases and let $\sigma_k \leq \sigma < \sigma_{k+1}$. Then for all $\sigma > \sigma_o = \sigma_o(\varepsilon)$, we have by (2.22)

$$\begin{aligned}
\ln M_\mu(\sigma, F) &\geq \ln A^*_{n_k} + \lambda_{n_k}\sigma_k \\
&> \ln A^*_{n_{k_1}} - \sum_{m=k_1+1}^{k} \left(\lambda_{n_m} - \lambda_{n_{m-1}} \right) \exp^{[q]} \left\{ 1/(\Im^* - \varepsilon) \ln^{[p-1]} \lambda_{n_{m-1}} \right\} + \lambda_{n_k}\sigma_k \\
&= \ln A^*_{n_{k_1}} - \lambda_{n_k} \exp^{[q]} \left\{ 1/(\Im^* - \varepsilon) \ln^{[p-1]} \lambda_{n_{m-1}} \right\} \\
&+ \sum_{m=k_1}^{k-1} \lambda_{n_m} \left[\exp^{[q]} \left\{ 1/(\Im^* - \varepsilon) \ln^{[p-1]} \lambda_{n_m} \right\} - \exp^{[q]} \left\{ 1/(\Im^* - \varepsilon) \ln^{[p-1]} \lambda_{n_{m-1}} \right\} \right] \\
&> \ln A^*_{n_{k_1}} - \lambda_{n_k} \exp^{[q]} \left\{ 1/(\Im^* - \varepsilon) \ln^{[p-1]} \lambda_{n_{k-1}} \right\} + \lambda_{n_k}\sigma_k.
\end{aligned}$$

Now, choosing the sequence $\{\sigma_k\}$ for the two cases and proceeding as in Lemma 2 we get the desired result (2.20). ■

In the next result, we obtain the reverse estimates for the lower (p,q) order. We have

Lemma 4. *Let* $F(s) = \int_0^\infty \exp(sy)d\alpha(y)$ *be Laplace-Stieltjes transformation having lower* (p,q)*-order* λ *such that* $\theta(n) \equiv \frac{\ln(A_n^*/A_{n+1}^*)}{\lambda_{n+1}-\lambda_n}$ *forms a nondecreasing function of* n *for* $n > n_o$*. Then*

$$\lambda \leq C(\Im_o) \tag{2.23}$$

and

$$\lambda \leq C(\Im_o^*) \tag{2.24}$$

where

$$\Im_o \equiv \Im_o(p,q) = \liminf_{n\to\infty} \frac{\ln^{[p-1]}\lambda_{n-1}}{\ln^{[q]}\left\{1/\lambda_n \ln(A_n^*)^{-1}\right\}}, \tag{2.25}$$

and

$$\Im_o^* \equiv \Im_o^*(p,q) = \liminf_{n\to\infty} \frac{\ln^{[p-1]}\lambda_{n-1}}{\ln^{[q]}\left\{1/(\lambda_n - \lambda_{n-1})\ln\left(A_{n-1}^*/A_n^*\right)\right\}}.$$ ■

Proof. We know that, if $\mu(\sigma,F) = A_n^* e^{\lambda_n\sigma}$ is the maximum term for a given σ, then

$$A_{n-1}^* e^{\lambda_{n-1}\sigma} \leq A_n^* e^{\lambda_n\sigma} > A_{n+1}^* e^{\lambda_{n+1}\sigma}$$

or

$$\theta(n-1) \leq \sigma < \theta(n).$$

Hence, for infinitely many values of n, $\theta(n) > \theta(n-1)$ and $\theta(n) \to \infty$ as $n \to \infty$. When $\theta(n) > \theta(n-1)$, we have $\mu(\sigma,F) = \max_{n\in N}\left\{A_n^* e^{\lambda_n\sigma}\right\}$, $\sigma > 0$ and $N(\sigma,F) = n$ for $\theta(n-1) \leq \sigma < \theta(n)$.

Now, let

$$\phi = \liminf_{\sigma\to\infty} \frac{\ln^{[p-1]}\lambda_{N(\sigma,F)}}{\ln^{[q]}\sigma}.$$

First, we assume that $\phi > 0$. For any ε such that $\phi > \varepsilon > 0$, and for all $\sigma > \sigma_o = \sigma_o(\varepsilon)$, we have

$$\lambda_{N(\sigma,F)} > \exp^{[p-1]}\left\{(\phi - \varepsilon)\ln^{[q]}\sigma\right\}.$$

Now, let $A_{n_1}^* \exp(\lambda_{n_1}s)$ and $A_{n_2}^* \exp(\lambda_{n_2}s)$, $(n_1 > n_o, \theta(n_1 - 1) > \sigma_o)$, be two consecutive maximum terms of $F(s)$ so that $n_1 \leq n_2 - 1$. Let $n_1 < n \leq n_2$. Since $A_{n_1}^* \exp(\lambda_{n_1}s)$ is the maximum term we have $N(\sigma,F) = n_1$ for $\theta(n_1-1) \leq \sigma < \theta(n_1)$. For σ in this interval,

$$\lambda_{n_1} > \exp^{[p-1]}\left\{(\phi - \varepsilon)\ln^{[q]}\sigma\right\}.$$

Since $\theta(n_1) = \theta(n_1+1) = ... = \theta(n-1)$, we have

$$\lambda_{n-1} \geq \lambda_{n_1} > \exp^{[p-1]}\left\{(\phi - \varepsilon)\ln^{[q]}(\theta(n-1) - \delta)\right\} \tag{2.26}$$

where $\delta = \max\{1, [\theta(n_1) - \theta(n_1 - 1)]/2\}$. Since $\theta(n)$ is nondecreasing, we have

$$\ln\left|A^*_{n_o}/A^*_{n_{o+1}}\right| + ... + \ln\left|A^*_{n-1}/A^*_n\right| = \ln\left|A^*_{n_o}/A^*_n\right| = \sum_{k=n_0}^{n-1}(\lambda_{k+1} - \lambda_k)\theta(k)$$
$$\leq (\lambda_n - \lambda_{n_o})\theta(n-1), (n > n_o),$$

and hence

$$\ln^{[q]}\left(\frac{1}{\lambda_n}\ln(A^*_n)^{-1}\right) \leq \ln^{[q]}\theta(n-1) + o(1). \tag{2.27}$$

From (2.26), we have

$$\ln^{[p-1]}\lambda_{n-1} \geq \left\{(\phi - \varepsilon)\ln^{[q]}(\theta(n-1) - \delta)\right\}. \tag{2.28}$$

Combining (2.27) and (2.28), we get

$$\frac{\ln^{[q]}\left(\frac{1}{\lambda_n}\ln(A^*_n)^{-1}\right)}{\ln^{[p-1]}\lambda_{n-1}} \leq \frac{1}{(\phi - \varepsilon)} + o(1).$$

By passing to limits, the above inequality leads to

$$\phi \leq \Im_o. \tag{2.29}$$

When $\phi = 0$, the above inequality is evidently true. It follows from (2.29) that $\lambda = P(\phi) \leq P(\Im_o)$ [using Lemma 1]. This proves (2.23). To prove (2.24), we find from (2.28) that for sufficiently large n, and for any ε such that $\phi > \varepsilon > 0$,

$$\frac{\ln^{[p-1]}\lambda_{n-1}}{\ln^{[q]}(\theta(n-1) - \delta)} \geq \phi - \varepsilon.$$

After substituting the value of $\theta(n-1)$ and passing to limits, we get

$$\phi \leq \Im^*_o.$$

The inequality is obvious if $\phi = 0$.

So, from the last inequality, we get $\lambda = C(\phi) \leq C(\Im^*_o)$ [Lemma 1] and this completes the proof of Lemma 4. ■

Combining the results of Lemmas 2, 3 and 4, we obtain the following characterization of lower (p, q) order.

Theorem 2. *Let* $F(s) = \int_0^\infty \exp(sy) d\alpha(y)$ *be Laplace-Stieltjes transformation having lower* (p, q)*-order* λ *such that* $\theta(n) \equiv \frac{\ln(A_n^*/A_{n+1}^*)}{\lambda_{n+1}-\lambda_n}$ *forms a nondecreasing function of* n *for* $n > n_o$ *and* (1.3) *holds. Then, for* $(p, q) \neq (2, 1)$,

$$\lambda = C(\Im_o) = C(\Im_0^*), \tag{2.30}$$

where $\Im_o$ *and* $\Im_0^*$ *are defined as in Lemma* 3.

Further, (2.30) *holds for* $(p, q) = (2, 1)$ *also if* $\ln \lambda_n \simeq \ln \lambda_{n+1}$, *as* $\mathrm{n} \to \infty$.

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