DYNAMIC BEHAVIOR OF TRAVELING WAVE SOLUTIONS FOR A CLASS FOR THE TIME-SPACE COUPLED FRACTIONAL kdV SYSTEM WITH TIME-DEPENDENT COEFFICIENTS

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Abstract. In this paper, a simplified bilinear method combined with a fractional transform has been used to obtain a new multiple soliton solutions for the Fractional coupled fractional kdV equations with variable coefficients. These systems appear in biology, engineering, mechanics, complex physics phenomena economics, signal image processing, notably control theory, groundwater problems and chemistry. Dispersion relations on the effects of the inhomogeneities of the model "due to the variable coefficients" are derived and interpreted for deterministic of the characteristic-line and velocity of each obtained soliton waves.

Keywords: simplified bilinear method; N-soliton solutions; modified Riemann-Liouville derivative; fractional coupled KdV systems.

1. Introduction

Fractional differential equations (FDEs) are the generalizations of classical differential equations with integer orders. The search of multiple soliton solutions to coupled systems of nonlinear fractional partial differential equations is of great importance, because these systems appear in biology, engineering, mechanics, complex physics phenomena economics, signal image processing, notably control theory, groundwater problems and chemistry. A variety of powerful and direct methods have been developed in this field. The analytical and approximate solutions of deterministic fractional partial differential equations attracted great attention and became a considerably interesting subject in mathematical physics. There are many methods for calculating the approximate solutions for nonlinear fractional partial differential equations such as Adomian decomposition method [1, 2], the expansion-function method [3], the variational iterations method [4] and the homotopy perturbation method [5], [6]. The exact solutions for nonlinear FDEs are still of great interest. Li and He [7] introduced complex transform for reducing FDEs into ordinary differential equations (ODEs), so that all analytical methods for advanced calculus can be easily applied to fractional calculus [8], [9]. The coupled FKdV equations are very complicated and not easy to solve by direct integration method. To the best of my knowledge, all the numerical or analytical methods that deals with these systems had found only one soliton solutions, such as in [10].

Our aim in this paper is to find N- soliton solutions and multiple singular soliton solutions for the generalized space-time coupled fractional Korteweg–de Vries (CFKdV) equations with variable coefficients of the form

$$\begin{aligned} &D_t^{\alpha}u_1 + f(t)D_x^{3\beta}u_1 + g(t)u_1D_x^{\beta}u_1 + h(t)u_2D_x^{\beta}u_2 = 0,\\ &D_t^{\alpha}u_2 + f(t)D_x^{3\beta}u_2 + k(t)u_1D_x^{\beta}u_2 = 0, \end{aligned} \tag{1.1}$$

where D_t^{α} and D_x^{β} denote the modified Riemann-Liouville derivatives with respect to the time $t > 0$ and the horizontal coordinate space x, respectively, of the the amplitude of the relevant wave models $u_1(x, t)$ and $u_2(x, t)$, and $f(t), g(t), h(t)$ and $k(t)$ are time-dependent functions, and $0 < \alpha, \beta \leq 1$ are parameters describing the order of the fractional derivatives of u_1 and u_2. The corresponding constraint conditions in this paper for (1.1), will be $f(t) = c_1k(t)$, and $k(t) - g(t) = c_2h(t)$, where c_1 and c_2 are arbitrary constants. To our knowledge, there have been no discussions on (1.1) under these conditions. We will apply a new simplified bilinear method with fractional transforms that transform the fractional partial differential equation into a classical one, then we investigate the N-soliton solutions for equataion (1.1). In this work, a new form of the simplified bilinear method is adopted. It is well-known that the Hirota bilinear form provides an efficient tool to solve nonlinear differential equations of mathematical physics [11]–[14], [15], [16].

This paper is organized as follows: some necessary definition and notation of the modified Riemann–Liouville derivative are given. In Section 2, a new simplified bilinear method based on a transformation method combined with the Hirota's bilinear sense is introduced. The N-soliton solutions and multiple singular soliton solutions for equation (1.1) are constructed in Sections 3 and 4. Section 5 is devoted for discussion and concluding remarks regards the effects of the variable coefficients and the collision behavior and propagation properties.

2. Preliminaries

There are many definitions of fractional derivative; for more details see [17]. In this paper, we will consider the modified Riemann–Liouville derivative defined by Jumarie [18], [19].

Definition 2.1 A real function $f(x)$, $x > 0$, is said to be in the space C_α; $\alpha \in \mathbb{R}$; if there exist a real number $p > \alpha$ such that $f(x) = x^p f_1(x)$, where $f_1(x)$ is continuous in $[0, \infty)$, and is said to be in the space C_α^m, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, if $f^{(m)} \in C_\alpha$.

Definition 2.2 Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous (but not necessarily differentiable) function, the Jumarie's modified Riemann-Liouville fractional derivative of order α is defined by

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x \frac{f(\tau) - f(0)}{(x-\tau)^{\alpha+1}} d\tau, & \alpha < 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x \frac{f(\tau) - f(0)}{(x-\tau)^{\alpha}} d\tau, & 0 < \alpha < 1 \\ \dfrac{d^n}{dx^n} \left(f^{(\alpha-n)}(x) \right), & n \leq \alpha \leq n+1, n \geq 1. \end{cases} \tag{2.1}$$

Some properties of Jumarie's modified Riemann-Liouville fractional derivative (2.1) are listed below [18], [19].

For $f, g \in C_\alpha, \alpha \in \mathbb{R}$,

$$D_x^0 f(x) = f(x).$$

i.e. $D_x^0 = I$ is the identity operator.

$$D_x^\alpha \left(k_1 f(x) + k_2 g(x) \right) = k_1 D_x^\alpha f(x) + k_2 D_x^\alpha g(x), \qquad k_1, k_2 \in \mathbb{C}.$$

In Jumarie's sense, the fractional derivative of a constant function is zero, which means that it has merits over the original one: For $c \in \mathbb{R}, \alpha \geq 0$,

$$D_x^\alpha c = 0.$$

If $f(x) = x^r$, then

$$D_x^\alpha f(x) = D_x^\alpha (x^r) = \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} x^{r-\alpha}.$$

Jumarie's fractional derivative of the product is as the same as the usual rule

$$D_x^\alpha \left(f(x) g(x) \right) = f(x) D_x^\alpha \left(g(x) \right) + g(x) D_x^\alpha \left(f(x) \right).$$

Finally,

$$D_x^\alpha f\left(g(x) \right) = \frac{df}{dg} D_x^\alpha g(x) = \left(\frac{dg}{dx} \right)^\alpha D_g^\alpha f\left(g(x) \right).$$

3. Multiple soliton solutions for the fractional coupled kdV system

In order to derive the N-soliton solutions of any completely integrable equation, we usually use the Hirota's direct method, which depends on a transformation for considered equation to a bilinear form. These bilinear forms are usually used to derive the auxiliary function. It is worth to mention that it is not easy to find the bilinear form for many equations, it sometimes requires to introduce a new dependent variables, or even independent variables. To avoid using the bilinear forms, Hereman et al. [20], [21], formally introduced the simplified algorithm to derive the auxiliary functions by assuming that the N-soliton solutions can be expressed as polynomials of exponential functions. The Cole-Hopf transformation method combined with the simplified Hirota's sense is a powerful method to determine multiple soliton solutions and multiple singular soliton solutions for integrable systems [22]–[30].

In this section, we apply simplified bilinear method, to construct soliton solutions of space-time coupled fractional Korteweg–de Vries (1.1):

Consider the following fractional transforms

$$\begin{aligned} T &= \frac{p_1 t^{\alpha}}{\Gamma(1+\alpha)}, \\ X &= \frac{p_2 x^{\beta}}{\Gamma(1+\beta)}, \end{aligned} \tag{3.1}$$

where p_1 and p_2 are constants. Using (3.1), we can convert fractional derivatives into the following classical derivatives:

$$\begin{aligned} \frac{\partial^{\alpha} u_j}{\partial t^{\alpha}} &= p_1 \frac{\partial u}{\partial T}, \\ \frac{\partial^{\beta} u_j}{\partial x^{\beta}} &= p_2 \frac{\partial u}{\partial X}, \; j=1,2, \end{aligned}$$

see [23]. Therefore, (1.1) becomes

$$\begin{aligned} &\frac{\partial u_1}{\partial T} + F(T)\frac{\partial^3 u_1}{\partial X^3} + G(T) u_1 \frac{\partial u_1}{\partial X} + H(T) u_2 \frac{\partial u_2}{\partial X} = 0, \\ &\frac{\partial u_2}{\partial T} + F(T)\frac{\partial^3 u_2}{\partial X^3} + K(T) u_1 \frac{\partial u_2}{\partial X} = 0, \end{aligned} \tag{3.2}$$

If we substitute

$$u_j(X,T) = e^{\theta_i(X,T)}, \; j=1,2, \quad \text{with} \quad \theta_i(X,T) = h_i X - \omega_i(T), \tag{3.3}$$

into the linear terms of equation (3.2), we get the dispersion relation as follows

$$\omega_i(T) = \int h_i^3 F(T) dT. \tag{3.4}$$

Thus, we obtain

$$\theta_i(X,T) = h_i X - \int h_i^3 F(T) dT. \tag{3.5}$$

Assuming the single soliton solutions of (3.2) as

$$u_j(X,T) = R_j (\ln a(X,T))_{XX}, \tag{3.6}$$

where $a(X,T)$ is the auxiliary function given by

$$a(X,T) = 1 + e^{\theta_1(X,T)} = 1 + e^{h_1 X - \int h_1^3 F(T) dT}. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.2), then solving for R_1, R_2 and R_3, we find two sets of solutions as

$$R_1 = \frac{12F(T)}{K(T)}, \quad R_2 = \frac{\pm 12F(T)}{K(T)}\sqrt{\frac{K(T)-G(T)}{H(T)}}. \tag{3.8}$$

To obtain a numerical value of R_j, we set the constraints $F(T) = c_1 K(T)$, and $K(T) - G(T) = c_2 H(T)$, where c_1 and c_2 are arbitrary constants

Now, substituting (3.8) into (3.6), we get the following sets of single soliton solutions for (3.2)

$$\begin{aligned} u_1(X,T) &= \frac{12F(T)}{K(T)} h_1^2 \frac{e^{\theta_1(X,T)}}{\left(1+e^{\theta_1(X,T)}\right)^2} = \frac{3F(T)}{K(T)} h_1^2 \sec h^2\left(\frac{\theta_1(X,T)}{2}\right), \\ u_2(X,T) &= \frac{\pm 12F(T)}{K(T)}\sqrt{\frac{K(T)-G(T)}{H(T)}} h_1^2 \frac{e^{\theta_1(X,T)}}{\left(1+e^{\theta_1(X,T)}\right)^2} \\ &= \frac{\pm 3(F(T))}{K(T)}\sqrt{\frac{K(T)-G(T)}{H(T)}} h_1^2 \sec h^2\left(\frac{\theta_1(X,T)}{2}\right), \end{aligned} \tag{3.9}$$

where

$$\theta_1(X,T) = h_1 X - \int h_1^3 F(T) dT.$$

Using the fractional transforms in (3.1), we obtain the following sets of single soliton solutions $u_j(x,t)$ for (1.1)

$$\begin{aligned} u_1(x,t) &= \frac{12f(t)}{k(t)} h_1^2 \frac{e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}}}{\left(1 + e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}}\right)^2} \\ &= \frac{3f(t)}{k(t)} h_1^2 \sec h^2\left(\frac{\theta_1(x,t)}{2}\right), \end{aligned} \tag{3.10}$$

$$\begin{aligned} u_2(x,t) &= \frac{\pm 12f(t)}{k(t)}\sqrt{\frac{k(t)-g(t)}{h(t)}} h_1^2 \frac{e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}}}{\left(1 + e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}}\right)^2} \\ &= \frac{\pm 12f(t)}{k(t)}\sqrt{\frac{k(t)-g(t)}{h(t)}} h_1^2 \sec h\left(\frac{\theta_1(x,t)}{2}\right), \end{aligned} \tag{3.11}$$

where

$$\theta_1(x,t) = \frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}, \tag{3.12}$$

with the constraints $f(t) = c_1 k(t)$, and $k(t) - g(t) = c_2 h(t)$, where c_1 and c_2 are arbitrary constants.

To obtain the two-soliton solutions, we let

$$a(X,T) = 1 + e^{\theta_1(X,T)} + e^{\theta_2(X,T)} + b_{12} e^{\theta_1(X,T)+\theta_2(X,T)}\}, \tag{3.13}$$

where θ_1 and θ_2 are defined in (3.5). Using (3.13) in (3.6) and substituting the results in (3.2), we obtained the phase shift as follows

$$b_{12} = \frac{(h_1 - h_2)^2}{(h_1 + h_2)^2}. \tag{3.14}$$

Substituting (3.14), (3.13) and (3.8) into (3.6), we obtain sets of two solitons solutions for (3.2), then by using (3.1) and

$$\theta_i(x,t) = \frac{h_i x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_i^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}, \quad i = 1, 2,$$

we can find two solitons solutions for (1.1).

This shows that (1.1) does not show any resonant phenomenon [15] because the phase shift term b_{12} in (3.14) cannot be 0 or ∞ for $|h_1| \neq |h_2|$. It is well known that a two-soliton solution can degenerate into a resonant trial under the conditions

$$b_{12} = 0 \quad \text{or} \quad (b_{12})^{-1} = 0, \quad \text{for} \quad |h_1| \neq |h_1|.$$

The three-soliton solution is determined by

$$\begin{aligned} a(X,T) = 1 &+ e^{\theta_1(X,T)} + e^{\theta_2(X,T)} + e^{\theta_3(X,T)} + b_{12} e^{\theta_1(X,T)+\theta_2(X,T)} \\ &+ b_{13} e^{\theta_1(X,T)+\theta_3(X,T)} + b_{23} e^{\theta_2(X,T)+\theta_3(X,T)} \\ &+ b_{123} e^{\theta_1(X,T)+\theta_2(X,T)+\theta_3(X,T)}, \end{aligned} \tag{3.15}$$

where

$$b_{ij} = \frac{(h_i - h_j)^2}{(h_i + h_j)^2}, \qquad 1 \leq i < j \leq 3. \tag{3.16}$$

Proceeding as before we find

$$b_{123} = b_{12} b_{13} b_{23}. \tag{3.17}$$

Since (3.17) holds, we can find three-soliton solutions for transformed coupled mkdv system (3.2), and with the fractional transformation (3.1) and using

$$\theta_i(x,t) = \frac{h_i x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_i^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}, i = 1, 2, 3,$$

we can obtain three solitons solutions for (1.1), which implies that N-soliton solutions exist for any order N [11], [12].

4. Multiple singular soliton solutions for the fractional coupled kdV system

In order to obtain a single singular soliton solution, we substitute

$$u_j(X,T) = e^{h_i X - \omega_i(T)}, \tag{4.1}$$

into the linear part of (3.2), concluding the dispersion relation (3.4), then, as a result we get

$$w_i(t) = \int h_i^3 F(T) dT. \tag{4.2}$$

The singular single soliton solution of (3.2) is assumed to be

$$u_j(X,T) = R_j \left(\ln(a(X,T))\right)_{XX}, \quad j = 1,2, \tag{4.3}$$

where $a(X,T)$ is given by

$$a(X,T) = 1 - e^{\theta_1(X,T)} = 1 - e^{h_1 X - \int h_1^3 F(T) dT}. \tag{4.4}$$

When we substitute (4.4) into (3.2), and solving for R_j, we found

$$R_1 = \frac{12F(T)}{K(T)}, R_2 = \frac{\pm 12F(T)}{K(T)} \sqrt{\frac{K(T)-G(T)}{H(T)}}. \tag{4.5}$$

Putting these values of R_j, we get the following sets of singular soliton solutions for (3.2)

$$u_1(X,T) = \frac{12F(T)}{K(T)} h_1^2 \frac{e^{\theta_1(X,T)}}{\left(1-e^{\theta_1(X,T)}\right)^2} = \frac{3F(T)}{K(T)} h_1^2 \csc h^2 \left(\frac{\theta_1(X,T)}{2}\right), \tag{4.6}$$

$$\begin{aligned} u_2(X,T) &= \frac{\pm 12F(T)}{K(T)} \sqrt{\frac{K(T)-G(T)}{H(T)}} h_1^2 \frac{e^{\theta_1(X,T)}}{\left(1-e^{\theta_1(X,T)}\right)^2} \\ &= \frac{\pm 3(F(T))}{K(T)} \sqrt{\frac{K(T)-G(T)}{H(T)}} h_1^2 \csc h^2 \left(\frac{\theta_1(X,T)}{2}\right), \end{aligned} \tag{4.7}$$

where

$$\theta_1(X,T) = h_1 X - \int h_1^3 F(T) dT.$$

Using the fractional transforms in (3.1), we obtain the following sets of singular soliton solutions $u_j(x,t)$ for (1.1)

$$\begin{aligned} u_1(x,t) &= \frac{12f(t)}{k(t)} h_1^2 \frac{e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}}}{\left(1 - e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}}\right)^2} \\ &= \frac{3f(t)}{k(t)} h_1^2 \csc h^2 \left(\frac{\theta_1(x,t)}{2}\right), \end{aligned} \tag{4.8}$$

$$\begin{aligned}(4.9)\qquad u_2(x,t) &= \frac{\pm 12 f(t)}{k(t)}\sqrt{\frac{k(t)-g(t)}{h(t)}}h_1^2\frac{e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)}-\frac{\alpha h_1^3\int f(t)t^{\alpha-1}dt}{\Gamma(1+\alpha)}}}{\left(1-e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)}-\frac{\alpha h_1^3\int f(t)t^{\alpha-1}dt}{\Gamma(1+\alpha)}}\right)^2}\\ &= \frac{\pm 12 f(t)}{k(t)}\sqrt{\frac{k(t)-g(t)}{h(t)}}h_1^2\csc h(\frac{\theta_1(x,t)}{2}),\end{aligned}$$

where

$$(4.10)\qquad \theta_1(x,t)=\frac{h_1 x^\beta}{\Gamma(1+\beta)}-\frac{\alpha h_1^3\int f(t)t^{\alpha-1}dt}{\Gamma(1+\alpha)}.$$

Similarly we can find the constraints $f(t)=c_1k(t)$, and $k(t)-g(t)=c_2h(t)$, to obtain a numerical value of R_j where c_1 and c_2 are arbitrary constants.

The two singular soliton solutions are obtained by setting

$$(4.11)\qquad a(X,T)=1-e^{\theta_1(X,T)}-e^{\theta_2(X,T)}+b_{12}e^{\theta_1(X,T)+\theta_2(X,T)},$$

where $\theta_1(X,T)$ and $\theta_2(X,T)$ are defined in (3.5). Now, by substituting (4.11) into (4.3), then in (3.2), to obtain the phase shift b_{12} as in (3.16). In general,

$$(4.12)\qquad b_{ij}=\frac{(h_i-h_j)^2}{(h_i+h_j)^2},\quad 1\le i<j\le 3.$$

Substituting (4.12), (4.11) and (4.5) into (4.3), a sets of two solitons solutions for (3.2) are obtained and then, by using (3.1), we can find two solitons solutions for (1.1).

For the singular three-soliton solutions we use

$$(4.13)\qquad \begin{aligned}a(X,T)=1&-e^{\theta_1(X,T)}-e^{\theta_2(X,T)}-e^{\theta_3(X,T)}+b_{12}e^{\theta_1(X,T)+\theta_2(X,T)}\\ &+b_{13}e^{\theta_1(X,T)+\theta_3(X,T)}+b_{23}e^{\theta_2(X,T)+\theta_3(X,T)}\\ &-b_{123}e^{\theta_1(X,T)+\theta_2(X,T)+\theta_3(X,T)},\end{aligned}$$

Proceeding as before, we can obtain the singular three-soliton solutions for (3.2), then by using (3.1) we can find three singular solitons solutions for (1.1).

5. Discussion and concluding remarks

In this section we will discuss the effects of the inhomogeneities, namely, variable coefficients. We will see that the dispersion relation plays a crucial role in the analysis, since it can be used to obtain the characteristic line and velocity v for each soliton.

We obtain sets of single soliton solutions (3.10) and (3.11) with distinct soliton amplitudes *amp* for $u_1(x,t)$ and $u_2(x,t)$ which can be expressed as

$$amp\ of\ u_j(x,t) = \begin{cases} \left|\dfrac{3f(t)}{k(t)}h_1^2\right|, & j=1, \\ \left|\dfrac{3f(t)}{k(t)}\sqrt{\dfrac{k(t)-g(t)}{h(t)}}h_1^2\right|, & j=2. \end{cases}$$

With the characteristic-line method [31], [32], the characteristic wedge for each solitary wave for $u_j(x,t)$ defined by

$$x^\beta = \frac{\alpha\Gamma(1+\beta)}{\Gamma(1+\alpha)}\int h_i^2 f(t)t^{\alpha-1}dt.$$

The velocity v of each solitary wave for $u_j(x,t)$, $j=1,2$,

$$v_i = \frac{1}{\beta}\left(\frac{\alpha\Gamma(1+\beta)}{\Gamma(1+\alpha)}\int h_i^2 f(t)t^{\alpha-1}dt\right)^{\frac{1}{\beta}-1}\frac{\alpha\Gamma(1+\beta)h_i^2 f(t)t^{\alpha-1}}{\Gamma(1+\alpha)}.$$

The analysis is illustrated by studying couplings of the kdV equations with time-fractional derivatives (FkdV)

$$\begin{aligned} &D_t^\alpha u_1 + f(t)\frac{\partial^3 u_1}{\partial x^3} + g(t)u_1\frac{\partial u_1}{\partial x} + h(t)u_2\frac{\partial u_2}{\partial x} = 0, \\ &D_t^\alpha u_2 + f(t)\frac{\partial^3 u_2}{\partial x^3} + k(t)u_1\frac{\partial u_2}{\partial x} = 0, \end{aligned} \tag{5.1}$$

From (3.10) and (3.11) we obtain the singular soliton solution for (5.1) where

$$\theta_i = h_i x - \frac{\alpha}{\Gamma(1+\alpha)}\int h_i^3 f(t)t^{\alpha-1}dt. \tag{5.2}$$

The characteristic face can be derived from relation (5.2) for each solitary wave by

$$x = \frac{\alpha}{\Gamma(1+\alpha)}\int h_i^2 f(t)t^{\alpha-1}dt. \tag{5.3}$$

Then the velocity of the wave is

$$v_x = \frac{\alpha}{\Gamma(1+\alpha)}h_i^2 t^{\alpha-1}f(t). \tag{5.4}$$

The soliton amplitude of $u_1(x,t)$ is dependent of the variable coefficients $f(t)$ and $k(t)$ and independent of the variable coefficients $g(t)$ and $h(t)$, but the soliton amplitude of $u_2(x,t)$ is dependent of the variable coefficients $f(t)$, $g(t)$, $h(t)$ and $k(t)$. The propagation velocity of the solitary wave is dependent just of the coefficient functions $f(t)$.

Moreover, we see that from (5.4) the soliton travels along the positive x-axis direction when the following inequality holds

$$\frac{\alpha}{\Gamma(1+\alpha)} h_i^2 t^{\alpha-1} f(t) > 0. \tag{5.5}$$

Fig. 1 shows that the solitonic amplitude of $u_1(x,t)$ increases with an increase in the ratio $f(t)/k(t)$.

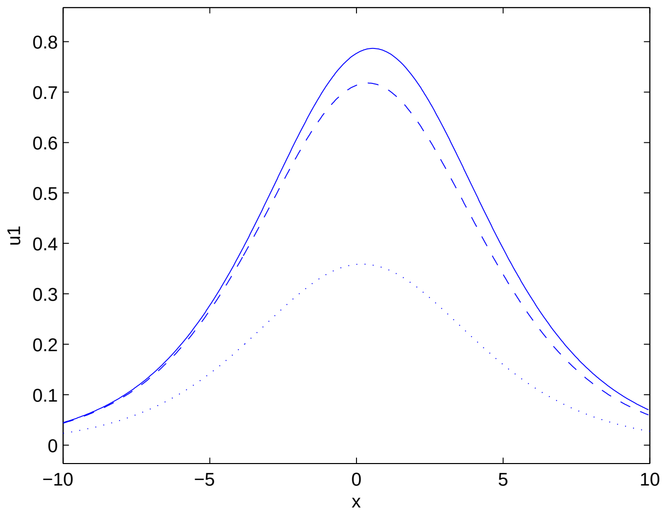

Figure 1: Evolution plots of the one-soliton solution given by expression (3.10) with $h_1 = 0.4$, $t = 2$, $\alpha = 0.8$, $f(t) = \lambda t^{2.4}$, $g(t) = t^{2.4}$, $\lambda = 1.1$ (solid line), $\lambda = 0.8$ (dashed line line), $\lambda = 0.4$ (dot line).

However, expression (5.4) indicates that the propagation velocity of the solitary wave is influenced by the coefficient functions $f(t)$, while it is independent of the coefficient function $g(t), g(t)$ and $k(t)$.

In Fig. 2, we choose $\alpha = 0.7, h_1 = 0.6, h_2 = 1, f(t) = 2t^{2.3}$ and $k(t) = t^{2.3}$, then the characteristic curve (5.3) given by

$$x - \frac{7h_i^2}{30\Gamma(1.7)} t^3 + \eta = 0.$$

Then the soliton reveals the cubical type propagation trajectory with the unalterable amplitude but continuously changeable velocity.

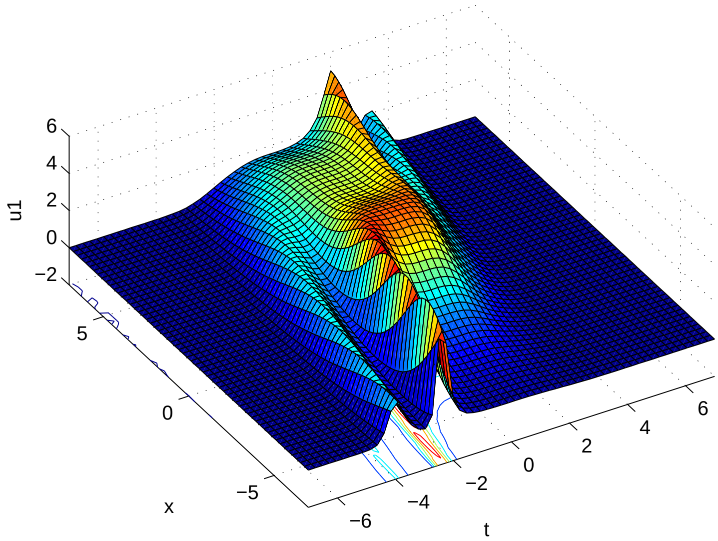

Figure 2: The profile figure for two soliton solution of $u_1(x,t)$ with $\alpha = 0.7$, $f(t) = 2t^{2.3}$, $k(t) = t^{2.3}$, $h_1 = 0.6$, $h_2 = 1$.

In Fig. 3, we choose $\alpha = 0.9, h_1 = 0, 9, h_2 = 1, f(t) = t^{0.1}\sin 8t + t^{1.1}$ and $k(t) = 2t^{0.1}\sin 8t + 2t^{1.1}$.

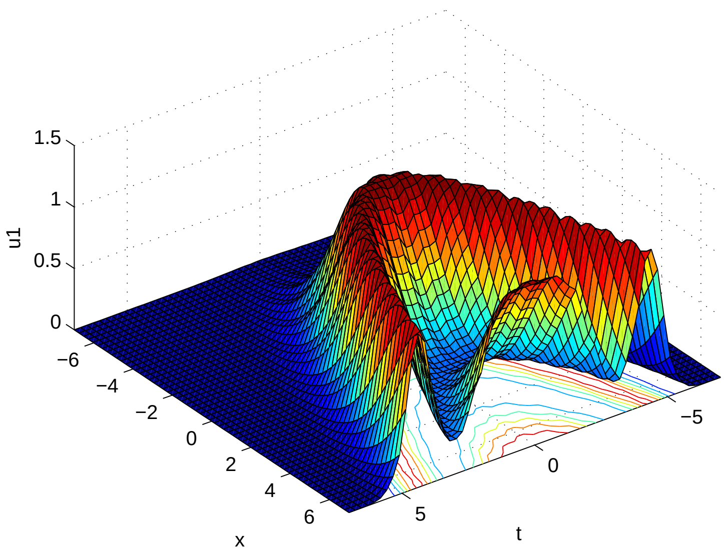

Figure 3: The profile figure for two soliton solution of $u_1(x,t)$ with $\alpha = 0.9$, $f(t) = t^{0.1}\sin 8t + t^{1.1}$, $k(t) = 2t^{0.1}\sin 8t + 2t^{1.1}$, $h_1 = 0.9$, $h_2 = 1$.

Then, in the characteristic curve (5.3), given by

$$x+\frac{9h_i^2}{10\Gamma(1.9)}\left[\frac{\cos 8t}{8}-\frac{t^2}{2}\right]+\eta=0,$$

we note that when t approaches zero, i.e., $\cos(8t)\gg t^2$, the trajectory is snake-like type with periodic oscillation. Otherwise, when t is far from the origin, the trajectory is parabolic-like type one. From which we conclude that, besides the periodic oscillation of the solitons in the local region, the large scale propagation trajectories for such a structure are the parabolic-typed curves. Likewise, if the variable coefficients are taken as the other forms, the corresponding characteristic curves will present different characters.

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