IDEMPOTENT ELEMENTS OF PRE-GENERALIZED HYPERSUBSTITUTIONS OF TYPE (m, n)

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Abstract. A generalized hypersubstitution of type $\tau = (m, n)$ is a mapping σ which maps the m-ary operation symbol f and n-ary operation symbol g to the term $\sigma(f)$ and $\sigma(g)$, and may not preserved arities. Each generalized hypersubstitution can be extended to a mapping $\hat{\sigma}$ on the set of all terms of type $\tau = (m, n)$. The structure $(\mathrm{Hyp_G}(\tau); \circ_{\mathrm{G}}, \sigma_{\mathrm{id}})$ is a monoid where σ_{id} is an identity hypersubstitution. A pre-generalized hypersubstitution of type $\tau = (m, n)$, namely σ, is a generalized hypersubstitution of type $\tau = (m, n)$ where $\sigma(f)$ and $\sigma(g)$ are not variables. In this paper, we characterize idempotent pre-generalized hypersubstitutions of type $\tau = (m, n)$.

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1. Introduction

Let n be a natural number. Let $X_n = \{x_1, \ldots, x_n\}$ be an n-element set. The set X_n is called an *n-element alphabet* and its elements are called *variables*. Let $\{f_i : i \in I\}$ be the set of *operation symbols*, indexed by set I. The sets X_n and $\{f_i : i \in I\}$ have to be disjoint. To every operation symbol f_i, we assign a natural number $n_i \geq 1$, called the *arity* of f_i. As in the definition of algebra, the sequence $\tau = (n_i)_{i \in I}$ of all the arities is called the *type*. The classes of algebras are described by logical expressions. This formal language is built up by variables from an n-element set. With this notation for operation symbols and variables, we can define the terms of type τ, (see [3], [4], [5]).

An n-ary terms of type τ is defined in the following inductive way:

(i) Every variable $x_i \in X_n$ is an n-ary term.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

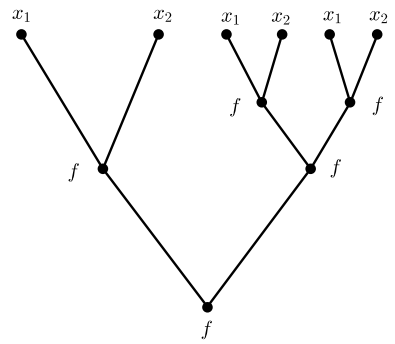
The set $W_\tau(X_n) = W_\tau(\{x_1, \ldots, x_n\})$ of all n-ary terms is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii). We denote the set of all terms of type τ by

$$W_\tau(X) := \bigcup_{m=1}^{\infty} W_\tau(X_m).$$

Denoted by var(t), op(t), ops(t) and firstop(t) the set of all variables occurring in the term t, the number of operation symbols occurring in the term t, the set of all operation symbols occurring in the term t and the first operation symbol (from the left) occurring in the term t, respectively.

Terms can be visualized as trees, where the vertices are labeled by operation symbols and the leaves are labeled by variables (see [1]). Trees have many applications in Mathematics, Computer Science, Linguistic and in other fields. For instance, the following tree corresponds to the term:

$$f(f(x_1, x_2), f(f(x_1, x_2), f(x_1, x_2))).$$



In universal algebra, identities are used to classify algebras into collections, called *varieties* and hyperidentities are used to classify varieties into collections, called *hypervarities*. The concept of hypersubstitution was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert [2]. In [8], S. Leeratanavalee and K. Denecke gave the concept of *generalized superposition* of terms $S^k : (W_\tau(X))^{k+1} \to W_\tau(X)$ by the following steps: for any term $t \in W_\tau(X)$,

(i) if $t = x_j \in X_n$, then $S^k(x_j, t_1, \ldots, t_k) := t_j$;

(ii) if $t = x \in X \setminus X_k$, then $S^k(x, t_1, \ldots, t_k) := x$;

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$ and assume that $S^k(s_j, t_1, \ldots, t_k)$ for $1 \leq j \leq n_i$ are already defined, then

$$S^k(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_k) := f_i(S^k(s_1, t_1, \ldots, t_k), \ldots, S^k(s_{n_i}, t_1, \ldots, t_k)).$$

Such operation is a tool to study generalized hypersubstitutions.

A *generalized hypersubstitution* of type τ is a mapping $\sigma:\{f_i:i \in I\}\rightarrow W_\tau(X)$ which maps operation symbols of type τ to a term of the same type, and may not preserved arity. We denoted by $\mathrm{Hyp_G}(\tau)$ the set of all generalized hypersubstitutions of type τ. The generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$ on the set of all terms of type τ inductively defined as follows:

(i) $\hat{\sigma}[x] := x \in X$;

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ for any n_i-ary operation symbol f_i and assume that $\hat{\sigma}[t_j]$ are already defined for all $1 \leq j \leq n_i$.

The concept of generalized hypersubstitutions is a generalization of the concept of hypersubstitutions.

In [8], the authors defined a binary operation $\circ_{\mathrm{G}}$ on $\mathrm{Hyp_G}(\tau)$ by $\sigma_1 \circ_{\mathrm{G}} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ is an usual composition. They have shown that the structure $\mathbf{Hyp}_{\mathrm{G}}(\tau) := (\mathrm{Hyp_G}(\tau); \circ_{\mathrm{G}}, \sigma_{\mathrm{id}})$ is a monoid where σ_{id} is an identity hypersubstitution. The authors also proved the following propositions.

Proposition 1.1 ([8]). *For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\widehat{\ } = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

Proposition 1.2 ([8]). *The monoid $\boldsymbol{Hyp}(\tau) := (Hyp(\tau); \circ_h, \sigma_{id})$ of all arity preserving hypersubstitutions of type τ forms a submonoid of $\boldsymbol{Hyp}_G(\tau)$.*

2. Pre-generalized hypersubstitutions of type (m, n)

In [2], the authors studied M-hyperidentities and M-solid varieties based on submonoid M of the monoid $\mathbf{Hyp}(\tau)$. They defined a number of natural such monoid based on various properties of hypersubstitutions.

According to [10], the authors studied pre-generalized hypersubstitutions of type $(2, 2)$. In practice, the type of these two operation symbols that we consider need not to be $(2, 2)$. Thus, we extend this idea to consider operation symbols of type (m, n). This section we provide the definition of pre-generalized hypersubstitutions of type (m, n). We also recall the definitions of projection generalized hypersubstitutions and weak projection generalized hypersubstitutions of type (m, n).

Definition 2.1. Let f and g be operation symbols of type (m, n). We denote the generalized hypersubstitution σ with $\sigma(f) = t_1$ and $\sigma(g) = t_2$ by σ_{t_1,t_2}.

(i) A generalized hypersubstitution σ of type (m, n) is called a *projection generalized hypersubstitution* if the term $\sigma(f)$ and $\sigma(g)$ are variables. We denote the set of all projection generalized hypersubstitutions of type (m, n) by $P_{\mathrm{G}}(m, n)$.

(ii) A generalized hypersubstitution σ of type (m, n) is called a *weak projection generalized hypersubstitution* if the term $\sigma(f)$ or $\sigma(g)$ is variable. We denote the set of all weak projection generalized hypersubstitutions of type (m, n) by $WP_{\mathrm{G}}(m, n)$.

(iii) A generalized hypersubstitution σ of type (m, n) is called a *pre-generalized hypersubstitution* if the terms $\sigma(f)$ and $\sigma(g)$ are not variables. We denote the set of all pre-generalized hypersubstitutions of type (m, n) by $Pre_{\mathrm{G}}(m, n)$. That is, $Pre_{\mathrm{G}}(m, n) := \mathrm{Hyp}_{\mathrm{G}}(m, n) \setminus WP_{\mathrm{G}}(m, n)$.

In [7], S. Leeratanavalee showed that for any type τ, the set $P_{\mathrm{G}}(\tau) \cup \{\sigma_{\mathrm{id}}\}$ and $Pre_{\mathrm{G}}(\tau)$ are submonoids of $\mathbf{Hyp}_{\mathrm{G}}(\tau)$. It is easy to see that $WP_{\mathrm{G}}(\tau) \cup \{\sigma_{\mathrm{id}}\}$ is a submonoid of $\mathbf{Hyp}_{\mathrm{G}}(\tau)$, and $P_{\mathrm{G}}(\tau) \cup \{\sigma_{\mathrm{id}}\}$ forms a submonoid of $WP_{\mathrm{G}}(\tau) \cup \{\sigma_{\mathrm{id}}\}$, (see [7]).

In 2007, S. Leeratanavalee [6] characterized idempotent elements of weak projection generalized hypersubstitutions of type $(2, 2)$ which was published in 2011. Later, in 2015, N. Lekkoksung and P. Jampachon [9] characterized idempotent elements of weak projection generalized hypersubstitutions of type (m, n).

Another submonoid of the monoid of all generalized hypersubstitutions of type τ is the monoid of all pre-generalized hypersubstitutions of type τ. In 2007, W. Puninagool and S. Leeratanavalee [10] characterized idempotent elements of pre-generalized hypersubstitutions of type $(2, 2)$. In this paper, we characterize idempotent pre-generalized hypersubstitutions of type (m, n).

3. Main results

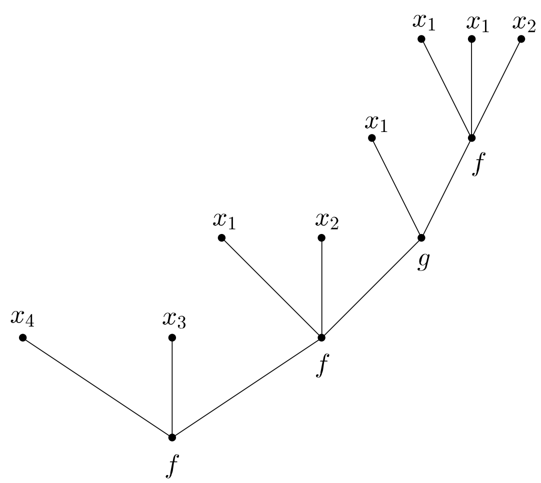
For any semigroup S, an element e of S is called an *idempotent element* of S if $ee = e$. In this section, we present a sufficient and necessary condition for elements of $Pre_{\mathrm{G}}(m, n)$ to be an idempotent element.

Let F be a variable over the two-elements alphabet $\{f, g\}$. Let $t = F(t_1, \ldots, t_j)$ where F has arity $j \in \{m, n\}$ and $i \leq \min\{m, n\}$, we define $M^i(t)$ by:

(i) if $t_i \in X$, then $M^i(t) = t_i$;

(ii) if $t_i = F'(s_1 \ldots, s_k)$ where F' has arity $k \in \{m, n\}$ and assume that $M^i(s_i)$ are already defined, then $M^i(t) = M^i(s_i)$.

For example, let f and g be operation symbols of type 3 and 2, respectively, and $t = f(x_4, x_3, f(x_1, x_2, g(x_1, f(x_1, x_1, x_2))))$. This term t can be visualized by a tree:



Then

$M^1(t) = x_4$ $M^2(t) = x_3$ $M^3(t)$ does not define.
$M^2(g(x_1, f(x_1, x_1, x_2)) = x_1$ $M^3(f(x_1, x_1, x_2)) = x_2$.

Proposition 3.1 ([9]). *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m, n). Then the following statements are equivalent:*

(i) *σ_{t_1,t_2} is idempotent;*

(ii) *$\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$.*

To characterize idempotent elements of $Pre_G(m, n)$, we separate our consideration into three cases:

Case 1. $\text{op}(t_1) = 1$ and $\text{op}(t_2) = 1$;
Case 2. $\text{op}(t_1) = 1$, $\text{op}(t_2) > 1$ and $\text{op}(t_1) > 1$, $\text{op}(t_2) = 1$;
Case 3. $\text{op}(t_1) > 1$ and $\text{op}(t_2) > 1$.

Case 1. op$(t_1) = 1$ and op$(t_2) = 1$

Assume that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$, in case of $\text{op}(t_1) = 1 = \text{op}(t_2)$, we consider only three cases:

1. $\text{firstop}(t_1) = f$ and $\text{firstop}(t_2) = f$;
2. $\text{firstop}(t_1) = g$ and $\text{firstop}(t_2) = g$;
3. $\text{firstop}(t_1) = f$ and $\text{firstop}(t_2) = g$

since for the case $\text{firstop}(t_1) = g$ and $\text{firstop}(t_2) = f$ is impossible. Indeed, for $s_1, \ldots, s_m \in X$,

$$t_1 = \hat{\sigma}_{t_1,t_2}[t_1] = S^m(\sigma_{t_1,t_2}(g), s_1, \ldots, s_m) = S^m(t_2, s_1, \ldots, s_m).$$

This implies $\text{firstop}(t_1) = f$, a contradiction.

Proposition 3.2. *Let* $t_1 = f(s_1, \ldots, s_m)$ *and* $t_2 = f(s'_1, \ldots, s'_m)$ *where* $s_1, \ldots, s_m, s'_1, \ldots, s'_m \in X$. *Then the following statements are equivalent:*

(1) σ_{t_1,t_2} *is idempotent;*

(2) *the following conditions holds:*

(i) *if* $x_j \in var(t_1)$ *where* $1 \leq j \leq m$, *then* $s_j = x_j$;

(ii) *if* $s_i = x_j$ *where* $1 \leq i, j \leq m$, *then* $s'_i = s'_j$;

(iii) *if* $s_j = x \in X \setminus X_m$ *where* $1 \leq j \leq m$, *then* $s'_j = x$.

Proof. (1) $\Rightarrow$ (2) : Since σ_{t_1,t_2} is idempotent, $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. That is, $S^m(t_1, s_1, \ldots, s_m) = t_1$ and $S^m(t_1, s'_1, \ldots, s'_m) = t_1$.

(i) Assume that $x_j \in \text{var}(t_1)$ where $1 \leq j \leq m$. We suppose that $s_j = x_i$ where $x_i \in X_m$ and $i \neq j$. Thus, we have to replace x_j in the term t_1 by x_i. This implies that $S^m(t_1, s_1, \ldots, s_m) \neq t_1$, a contradiction. Hence, $s_j = x_j$.

(ii) Assume that $s_j = x_i$ where $1 \leq i, j \leq m$. Suppose that $s'_j \neq s'_i$. From $S^m(t_1, s'_1, \ldots, s'_m)$, we have to replace s_j in the term t_1 by s'_i. Since $s'_j \neq s'_i$, $S^m(t_1, s'_1, \ldots, s'_m) \neq t_2$, this is a contradiction. Thus, $s'_j = s'_i$.

(iii) Assume that $s_j = x \in X \setminus X_m$ where $1 \leq j \leq m$. Suppose that $s'_j \neq x$. From $S^m(t_1, s'_1, \ldots, s'_m)$, we have to replace s_j in the term t_1 by x. Since $s'_j \neq x$, $S^m(t_1, s'_1, \ldots, s'_m) \neq t_2$, this is a contradiction. Thus, $s'_j = x$.

(2) $\Rightarrow$ (1) : We consider these conditions:

(i) If $x_j \in \text{var}(t_1)$ where $1 \leq j \leq m$, then we replace x_j in the term t_1 by x_j.

(ii) If $s_i = x_j$ where $1 \leq i, j \leq m$, then we replace x_j in the term t_1 by s'_i.

(iii) If $s_j = x \in X \setminus X_m$ where $1 \leq j \leq m$, then we replace x in the term t_1 by x.

This shows that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Therefore, σ_{t_1,t_2} is idempotent. ∎

Similarly, we obtain the following proposition.

Proposition 3.3. *Let* $t_1 = g(s_1, \ldots, s_n)$ *and* $t_2 = g(s'_1, \ldots, s'_n)$ *where* $s_1, \ldots, s_n, s'_1, \ldots, s'_n \in X$. *Then the following statements are equivalents:*

(1) σ_{t_1,t_2} *is idempotent;*

(2) *the following conditions hold:*

(i) *if* $x_j \in var(t_2)$ *where* $1 \leq j \leq n$, *then* $s'_j = x_j$;

(ii) *if* $s'_i = x_j$ *where* $1 \leq i, j \leq n$, *then* $s_i = s_j$;

(iii) *if* $s'_j = x \in X \setminus X_n$ *where* $1 \leq j \leq n$, *then* $s_j = x$.

For the last possibility, in case of $\mathrm{op}(t_1) = 1 = \mathrm{op}(t_2)$, we obtain the following proposition.

Proposition 3.4. *Let $t_1 = f(s_1, \ldots, s_m)$ and $t_2 = g(s'_1, \ldots, s'_n)$ where $s_1, \ldots, s_m$, $s'_1, \ldots, s'_n \in X$. Then the following statements are equivalents:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *the following conditions holds:*

  (i) *if $x_j \in var(t_1)$ where $1 \leq j \leq m$, then $s_j = x_j$;*

  (ii) *if $x_k \in var(t_2)$ where $1 \leq k \leq n$, then $s_k = x_k$.*

Proof. (1) $\Rightarrow$ (2) : Since σ_{t_1,t_2} is idempotent,

$$t_1 = \hat{\sigma}_{t_1,t_2}[t_1] = S^m(\sigma_{t_1,t_2}(f), s_1, \ldots, s_m) = S^m(t_1, s_1, \ldots, s_m) \tag{3.1}$$

$$t_2 = \hat{\sigma}_{t_1,t_2}[t_2] = S^n(\sigma_{t_1,t_2}(g), s'_1, \ldots, s'_n) = S^n(t_2, s'_1, \ldots, s'_n). \tag{3.2}$$

(i) Assume that $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$. We suppose that $s_j = x_i$ where $x_i \in X$ and $i \neq j$. Then, in the right-hand-side of (3.1), we have to replace x_j in the term t_1 by x_i. This implies $S^m(t_1, s_1, \ldots, s_m) \neq t_1$, which is a contradiction. Thus, $s_j = x_j$.

(ii) Similarly to (i).

(2) $\Rightarrow$ (1) : We consider the following conditions:

(i) If $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$, then we replace x_j in the term t_1 by x_j.

(ii) If $x_k \in \mathrm{var}(t_2)$ where $1 \leq j \leq n$, then we replace x_k in the term t_2 by x_k.

This shows that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Therefore, σ_{t_1,t_2} is idempotent. ∎

Case 2. op$(t_1) = 1$, op$(t_2) > 1$ and op$(t_2) = 1$, op$(t_1) > 1$

At first, we consider in case that $\mathrm{op}(t_1) = 1$ and $\mathrm{op}(t_2) > 1$. We need the following lemmas before proving main results.

Lemma 3.5. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m, n). If op(t_1)=1, op$(t_2) > 1$ and σ_{t_1,t_2} is idempotent, then firstop$(t_1) = f$.*

Proof. Suppose that $\mathrm{firstop}(t_1) = g$. That is, $t_1 = g(s_1, \ldots, s_n)$ where $s_1, \ldots, s_n \in W_{(m,n)}(X)$. Then

$$t_1 = \hat{\sigma}_{t_1,t_2}[t_1] = S^n(\sigma_{t_1,t_2}(g), s_1, \ldots, s_n) = S^n(t_2, s_1, \ldots, s_n) \neq t_1$$

since $\mathrm{op}(t_2) > 1$. This is a contradiction, so $\mathrm{firstop}(t_1) = f$. ∎

Lemma 3.6. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1)$=1, $op(t_2) > 1$ and $t_1 = f(s_1,\ldots,s_m)$ where $s_1,\ldots,s_m \in X$. Then the following conditions are equivalent:*

(1) $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$;

(2) *firstop*$(t_1) = f$ *and if* $x_j \in var(t_1)$ *where* $1 \leq j \leq m$, *then* $s_j = x_j$.

Proof. (1) $\Rightarrow$ (2) : By Lemma 3.5, we have firstop$(t_1) = f$. Assume that $x_j \in \text{var}(t_1)$ where $1 \leq j \leq m$. We suppose that $s_j = x_i$ where $x_i \in X_m$ and $i \neq j$. Thus, we have to replace x_j in the term t_1 by x_i. This implies that $S^m(t_1, s_1,\ldots,s_m) \neq t_1$, a contradiction. Hence, $s_j = x_j$.

(2) $\Rightarrow$ (1) : Since op$(t_1) = 1$ and firstop$(t_1) = f$, $t_1 = f(s_1,\ldots,s_n)$ where $s_1,\ldots,s_n \in X$. Consider

$$\hat{\sigma}_{t_1,t_2}[t_1] = S^m(\sigma_{t_1,t_2}(f), s_1,\ldots,s_n) = S^m(t_1, s_1,\ldots,s_n),$$

if $x_j \in \text{var}(t_1)$ where $1 \leq j \leq m$, then we replace x_j in the term t_1 by x_j. In case that $x \in \text{var}(t_1)$ where $x \in X \setminus X_m$, then we replace x in the term t_1 by x. Therefore, $S^m(t_1, s_1,\ldots,s_n) = t_1$, so $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. ∎

By the above lemma, in the case that op$(t_1) = 1$, we have to consider the following two cases:

- firstop$(t_2) = g$;
- firstop$(t_2) = f$.

Then we obtain the following proposition.

Proposition 3.7. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1)$=1, $op(t_2) > 1$, t_1=$f(s'_1,\ldots,s'_m)$ and t_2=$g(s_1,\ldots,s_n)$ where $s'_1,\ldots,s'_m \in X$, $s_1,\ldots,s_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:*

(1) σ_{t_1,t_2} *is idempotent;*

(2) *the following conditions hold*

  (i) *if* $x_j \in var(t_1)$ *where* $1 \leq j \leq m$, *then* $s'_j = x_j$;

  (ii) *if* $x_j \in var(t_2)$ *where* $1 \leq j \leq n$, *then* $s_j = x_j$.

Proof. (1) $\Rightarrow$ (2) : Since $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$, (i) is satisfied by Lemma 3.6.

Assume that $x_j \in \text{var}(t_2)$ where $1 \leq j \leq n$. We suppose that $s_j \neq x_j$ where $1 \leq j \leq n$. Then $s_j \in W_{(m,n)}(X) \setminus X$ or $s_j = x_i$ where $i \neq j$. If $s_j \in W_{(m,n)}(X) \setminus X$, then $\text{op}(t_2) < \text{op}(S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1],\ldots,\hat{\sigma}_{t_1,t_2}[s_n]))$. If $s_j = x_i$ where $i \neq j$, then we have to replace x_j in the term t_2 by x_i. It follows that $S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1],\ldots,\hat{\sigma}_{t_1,t_2}[s_n]) \neq t_2$, which contradicts to the idempotency of σ_{t_1,t_2}. Thus, $s_j = x_j$, that is, the condition (ii) is satisfied.

(2) $\Rightarrow$ (1) : By Lemma 3.6, we have $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. We show that $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1],\ldots,\hat{\sigma}_{t_1,t_2}[s_n])$, we have to replace s_j in the term t_2

by x_j since the condition (ii) is satisfied. Thus, $S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) = t_2$, so $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Hence, σ_{t_1,t_2} is idempotent. ∎

To prove the case firstop$(t_2) = f$, we need the following lemmas.

Lemma 3.8. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type (m, n), op$(t_1) = 1$, op$(t_2) > 1$ and $t_2 = f(s_1, \ldots, s_m)$ where $s_1, \ldots, s_m \in W_{(m,n)}(X)$. If $x_j \in$ var(t_1) where $1 \leq j \leq m$, then firstop$(s_j) = f$ or $s_j \in X$.*

Proof. Assume that $x_j \in \text{var}(t_1)$. Suppose that $s_j = g(s'_1, \ldots, s'_n)$ where $s'_1, \ldots, s'_n \in W_{(m,n)}(X)$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$,

$$t_2 = S^m(t_1, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]). \tag{3.3}$$

Now, $\hat{\sigma}_{t_1,t_2}[s_j] = S^n(t_2, \hat{\sigma}_{t_1,t_2}[s'_1], \ldots, \hat{\sigma}_{t_1,t_2}[s'_n])$, we replace $\hat{\sigma}_{t_1,t_2}[s_j]$ in (3.3), yield,

$$\begin{aligned} t_2 = S^m(t_1, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_{j-1}], S^n(t_2, \hat{\sigma}_{t_1,t_2}[s'_1], \ldots, \hat{\sigma}_{t_1,t_2}[s'_n]),\\ \hat{\sigma}_{t_1,t_2}[s_{j+1}], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]). \end{aligned} \tag{3.4}$$

For simplicity, we write the right-hand-side of (3.4) by A. Since $x_j \in \text{var}(t_1)$, op$(t_2) <$ op(A). This is a contradiction. Thus, firstop$(s_j) = f$. ∎

Lemma 3.9. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type (m, n), op$(t_1) = 1$, op$(t_2) > 1$, $t_1 = f(x_{j_1}, \ldots, x_{j_m})$ and $t_2 = f(\overline{s}_1, \ldots, \overline{s}_m)$ where $x_{j_1}, \ldots, x_{j_m} \in X$, $\overline{s}_1, \ldots, \overline{s}_m \in W_{(m,n)}(X)$. Then ops$(t_2) = \{f\}$.*

Proof. Suppose that $g \in \text{ops}(t_2)$. Without losing of generality, let firstop$(\overline{s}_k) = g$ for some $1 \leq k \leq m$. We write $\overline{s}_k = g(s'_1, \ldots, s'_n)$ where $s'_1, \ldots, s'_n \in W_{(m,n)}(X)$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$,

$$t_2 = S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\overline{s}_m]).$$

Thus, there are two possibilities to consider.

If $x_{j_k} \in X \setminus X_m$ where $1 \leq k \leq m$, then $\hat{\sigma}_{t_1,t_2}[\overline{s}_k]$ must be x_{j_k}, otherwise $t_2 \neq S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\overline{s}_m])$.

If $x_{j_k} \in X_m$ where $1 \leq k \leq m$, then by Lemma 3.8, firstop$(\overline{s}_k) = f$ or $\overline{s}_k \in X$. Therefore, we have ops$(t_2) = \{f\}$. ∎

Lemma 3.10. Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m, n), op(t_1)=1, op$(t_2) > 1$, $t_1 = f(x_{j_1}, \ldots, x_{j_m})$ and $t_2 = f(\overline{s}_1, \ldots, \overline{s}_m)$ where $x_{j_1}, \ldots, x_{j_m} \in X$, $\overline{s}_1, \ldots, \overline{s}_n \in W_{(m,n)}(X)$. If σ_{t_1,t_2} is idempotent, then $\hat{\sigma}_{t_1,t_2}[\overline{s}_k] = \overline{s}_k$ for all $k \in \{1, \ldots, m\}$.

Proof. Let $i \in \{1, \ldots, m\}$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$,

$$t_2 = f(\overline{s}_1, \ldots, \overline{s}_m) = S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\overline{s}_m]). \tag{3.5}$$

Thus, we replace x_{j_i} in the term t_1 by x_{j_i} if $x_{j_i} \in X \setminus X_m$. If $x_{j_i} \in X_m$, then we replace x_{j_i} in the term t_1 by $\hat{\sigma}_{t_1,t_2}[\overline{s}_{j_i}]$. By (3.5), we obtain $\hat{\sigma}_{t_1,t_2}[\overline{s}_{j_i}] = \overline{s}_i$. Then

$$\begin{aligned}\hat{\sigma}_{t_1,t_2}[\overline{s}_i] &= (\hat{\sigma}_{t_1,t_2} \circ \hat{\sigma}_{t_1,t_2})[\overline{s}_{j_i}] \\ &= (\sigma_{t_1,t_2} \circ_G \sigma_{t_1,t_2})\widehat{\ }[\overline{s}_{j_i}] \\ &= \hat{\sigma}_{t_1,t_2}[\overline{s}_{j_i}].\end{aligned}$$

It follows that $\hat{\sigma}_{t_1,t_2}[\overline{s}_i] = \hat{\sigma}_{t_1,t_2}[\overline{s}_{j_i}] = \overline{s}_i$. ■

Lemma 3.11. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type (m,n), $op(t_1) = 1$, $op(t_2) > 1$, $t_1 = f(s_1,\dots,s_m)$ and $t_2 = f(\overline{s}_1,\dots,\overline{s}_m)$ where $s_1,\dots,s_m \in X$, $\overline{s}_1,\dots,\overline{s}_n \in W_{(m,n)}(X)$. If $s_j = x \in X \setminus X_m$ where $1 \le j \le m$, then $\overline{s}_j = x$ whenever $\overline{s}_j \in X$ and $M^j(\overline{s}_i) = x$ whenever $\overline{s_i} \notin X$ where $1 \le i \le m$.*

Proof. Assume that $s_j = x \in X \setminus X_m$ where $1 \le j \le m$.

- $\overline{s}_j \in X$: Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$,

$$t_2 = S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1],\dots,\hat{\sigma}_{t_1,t_2}[\overline{s}_m]).$$

  Thus, we have to replace $s_j = x$ in the term t_1 by x, otherwise $t_2 \neq S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1],\dots,\hat{\sigma}_{t_1,t_2}[\overline{s}_m])$. That is, $\hat{\sigma}_{t_1,t_2}[\overline{s}_j] = x$, so $\overline{s}_1 = x$.

- $\overline{s}_i \notin X$ where $1 \le i \le m$: We suppose that $M^j(\overline{s}_i) \neq x$. Now, $\overline{s}_i = f(s'_1,\dots,s'_m)$ where $s'_1,\dots,s'_m \in W_{(m,n)}(X)$. By Lemma 3.10, $\hat{\sigma}_{t_1,t_2}[\overline{s}_i] = \overline{s}_i$. Then

$$\overline{s}_i = f(s'_1,\dots,s'_m) = S^m(t_1, \hat{\sigma}_{t_1,t_2}[s'_1],\dots,\hat{\sigma}_{t_1,t_2}[s'_m]).$$

  Since $s_j \in X \setminus X_m$ we replace s_j in the term t_1 by x. Thus, $\overline{s}_i \neq S^m(t_1, \hat{\sigma}_{t_1,t_2}[s'_1],\dots,\hat{\sigma}_{t_1,t_2}[s'_m])$ since $M^j(\overline{s}_i) \neq x$, this is a contradiction. Therefore $M^j(\overline{s}_i) = x$. ■

Proposition 3.12. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1) = 1$, $op(t_2) > 1$, $t_1 = f(s_1,\dots,s_m)$ and $t_2 = f(\overline{s}_1,\dots,\overline{s}_m)$ where $s_1,\dots,s_n \in X$, $\overline{s}_1,\dots,\overline{s}_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *if $x_j \in var(t_1)$ where $1 \le j \le m$, then $s_j = x_j$ and the following conditions hold:*

  (i) *if $s_j = x \in X \setminus X_m$ where $1 \le j \le m$, then $\overline{s}_j = x$ and for each $\overline{s}_i \notin X$ where $1 \le i \le m$, $M^j(\overline{s}_i) = x$;*

  (ii) *if $s_j = s_l$ where $1 \le j,l \le m$, then $\overline{s}_j = \overline{s}_l$;*

  (iii) *For each subterm of t_2 which is not a variable $f(s'_1,\dots,s'_m)$ where $s'_1,\dots,s'_m \in W_{(m,n)}(X)$, if $s_j = s_l$ where $1 \le j,l \le m$, then $s'_j = s'_l$.*

Proof. (1) $\Rightarrow$ (2) : We now know that, by Lemma 3.9, ops$(t_2) = \{f\}$. By Lemma 3.6, $x_j \in$ var(t_1) implies $s_j = x_j$ where $1 \leq j \leq m$.

(i) This is clear by Lemma 3.11.

(ii) Assume that $s_j = s_l$. Then there is $x_k \in X_m$ such that $s_j = x_k = s_l$. Since σ_{t_1,t_2} is idempotent,

$$t_2 = f(\overline{s}_1, \ldots, \overline{s}_n) = S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\overline{s}_m]).$$

Thus, we have to replace s_j and s_l in the term t_1 by $\overline{s}_k$. That is, $\overline{s}_j = \overline{s}_k = \overline{s}_l$.

(iii) Similarly to (ii).

(2) $\Rightarrow$ (1) : It is clear that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. We have to show that $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, indeed,

$$f(\overline{s}_1, \ldots, \overline{s}_m) = S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\overline{s}_m]).$$

That is, we must replace s_i in the term t_1 by $\hat{\sigma}_{t_1,t_2}[\overline{s}_i]$ for each $i \in \{1, \ldots, m\}$.

- If $s_i = x_k$ where $1 \leq i \leq m$ and $x_k \in X_m$, then we replace s_i and s_k in the term t_1 by $\overline{s}_k = \hat{\sigma}_{t_1,t_2}[\overline{s}_k]$.
- If $s_i = x \in X \setminus X_m$ where $1 \leq i \leq m$, then we consider t_2. Since op$(t_2) > 1$, $\overline{s}_j \notin X$ for some $1 \leq j \leq m$. Thus, by assumption, $M^i(\overline{s}_j) = x$. And $\overline{s}_k = x$ if $\overline{s}_k \in X$ where $1 \leq k \leq m$. It follows that we have to replace $\hat{\sigma}_{t_1,t_2}[\overline{s}_j] = \overline{s}_j$ and $\hat{\sigma}_{t_1,t_2}[\overline{s}_k] = \overline{s}_k$. Simply calculate we have $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$.

Therefore, σ_{t_1,t_2} is idempotent. ∎

On the other hand we obtain the following results.

Lemma 3.13. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n). If op$(t_1) > 1$, op$(t_2) = 1$ and σ_{t_1,t_2} is idempotent, then firstop$(t_2) = g$.*

Lemma 3.14. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), op$(t_1)>1$, op$(t_2) = 1$ and $t_2 = g(s_1, \ldots, s_n)$ where $s_1, \ldots, s_n \in X$. Then the following conditions are equivalent:*

(1) $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$*;*

(2) *firstop$(t_2) = g$ and if $x_j \in$ var(t_1) where $1 \leq j \leq n$, then $s_j = x_j$.*

Proposition 3.15. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), op$(t_1) > 1$, op$(t_2) = 1$, $t_1 = f(s'_1, \ldots, s'_m)$ and $t_2 = g(s_1, \ldots, s_n)$ where $s'_1, \ldots, s'_m \in W_{(m,n)}(X)$, $s_1, \ldots, s_n \in X$. Then the following conditions are equivalent:*

(1) σ_{t_1,t_2} *is idempotent;*

(2) *the following conditions hold*

  (i) *if $x_j \in$ var(t_1) where $1 \leq j \leq m$, then $s'_j = x_j$;*

  (ii) *if $x_j \in$ var(t_2) where $1 \leq j \leq n$, then $s_j = x_j$.*

Lemma 3.16. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) = 1$ and $t_1 = g(s_1, \ldots, s_n)$ where $s_1, \ldots, s_n \in W_{(m,n)}(X)$. If $x_j \in var(t_2)$ where $1 \leq j \leq n$, then $firstop(s_j) = g$ or $s_j \in X$.*

Lemma 3.17. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) = 1$, $t_1 = g(s_1, \ldots, s_n)$ and $t_2 = g(x_{j_1}, \ldots, x_{j_n})$ where $x_{j_1}, \ldots, x_{j_n} \in X$, $s_1, \ldots, s_n \in W_{(m,n)}(X)$. Then $ops(t_1) = \{g\}$.*

Lemma 3.18. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1)>1$, $op(t_2) = 1$, $t_1 = g(s_1, \ldots, s_n)$ and $t_2 = g(x_{j_1}, \ldots, x_{j_n})$ where $x_{j_1}, \ldots, x_{j_n} \in X$, $s_1, \ldots, s_n \in W_{(m,n)}(X)$. If σ_{t_1,t_2} is idempotent, then $\hat{\sigma}_{t_1,t_2}[s_k] = s_k$ for all $k \in \{1, \ldots, n\}$.*

Lemma 3.19. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) = 1$, $t_1 = g(s_1, \ldots, s_n)$ and $t_2 = g(\overline{s}_1, \ldots, \overline{s}_n)$ where $\overline{s}_1, \ldots, \overline{s}_n \in X$, $s_1, \ldots, s_n \in W_{(m,n)}(X)$. If $\overline{s}_j = x \in X \setminus X_n$ where $1 \leq j \leq n$, then $s_j = x$ whenever $s_j \in X$ and $M^j(s_i) = x$ whenever $s_i \notin X$ where $1 \leq i \leq n$.*

Proposition 3.20. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) = 1$, $t_1 = g(s_1, \ldots, s_n)$ and $t_2 = g(\overline{s}_1, \ldots, \overline{s}_n)$ where $\overline{s}_1, \ldots, \overline{s}_n \in X$, $s_1, \ldots, s_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *if $x_j \in var(t_2)$ where $1 \leq j \leq n$, then $s_j = x_j$ and the following conditions hold:*

  (i) *if $\overline{s}_j = x \in X \setminus X_n$ where $1 \leq j \leq n$, then $s_j = x$ and for each $s_i \notin X$ where $1 \leq i \leq n$, $M^j(s_i) = x$;*

  (ii) *if $\overline{s}_j = \overline{s}_l$ where $1 \leq j, l \leq n$, then $s_j = s_l$;*

  (iii) *For each subterm of t_1 which is not a variable $g(s'_1, \ldots, s'_n)$ where $s'_1, \ldots, s'_n \in W_{(m,n)}(X)$, if $\overline{s}_j = \overline{s}_l$ where $1 \leq j, l \leq n$, then $s'_j = s'_l$.*

Case 3. op$(t_1) > 1$ and op$(t_2) > 1$.

Assume that $\mathrm{op}(t_1) > 1$ and $\mathrm{op}(t_2) > 1$. In this case, if σ_{t_1,t_2} is idempotent, then the case $\mathrm{firstop}(t_1) = g$ and $\mathrm{firstop}(t_2) = f$ is impossible. Therefore, we consider only the following cases:

1. $\mathrm{firstop}(t_1) = f$ and $\mathrm{firstop}(t_2) = f$;
2. $\mathrm{firstop}(t_1) = g$ and $\mathrm{firstop}(t_2) = g$ and
3. $\mathrm{firstop}(t_1) = f$ and $\mathrm{firstop}(t_2) = g$.

Proposition 3.21. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) > 1$, $t_1 = f(s_1, \ldots, s_m)$ and $t_2 = f(s'_1, \ldots, s'_m)$ where $s_1, \ldots, s_m, s'_1, \ldots, s'_m \in W_{(m,n)}(X)$. Then the following conditions are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *if* $x_j \in var(t_1)$ *where* $1 \leq j \leq m$, *then* $s_j = x_j$ *and* $t_2 = S^m(t_1, k_1^1, \ldots, k_m^1)$
where $k_j^1 \in X$ *or* $k_j^1 = S^m(t_1, k_1^2, \ldots, k_m^2)$
where $k_j^2 \in X$ *or* $k_j^2 = S^m(t_1, k_1^3, \ldots, k_m^3)$
$\vdots$
where $k_j^{l-1} \in X$ *or* $k_j^{l-1} = S^m(t_1, k_1^l, \ldots, k_m^l)$
where $k_j^l \in X$ *for some* $l \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2) : Assume that $x_j \in \text{var}(t_1)$ where $1 \leq j \leq m$. Suppose that $s_j \neq x_j$. If $s_j \in W_{(m,n)}(X) \setminus X$, then

$$\text{op}(t_1) < \text{op}(S^m(\sigma_{t_1,t_2}(f), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m])).$$

Otherwise, we consider,

$$t_1 = S^m(t_1 \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]).$$

Since $x_j \in \text{var}(t_1)$, we replace x_j in the term t_1 by $\hat{\sigma}_{t_1,t_2}[s_j] \neq x_j$. This implies

$$S^m(t_1 \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]) \neq t_1,$$

this is a contradiction.

Hence, $s_j = x_j$.
Now, $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$,

$$t_2 = S^m(t_1 \hat{\sigma}_{t_1,t_2}[s'_1], \ldots, \hat{\sigma}_{t_1,t_2}[s'_m]).$$

Since $x_j \in \text{var}(t_1)$, we have to replace s'_j in the term t_1 by $\hat{\sigma}_{t_1,t_2}[s'_j]$. That is, $\hat{\sigma}_{t_1,t_2}[s'_j] = s'_j$. Hence, s'_j is a variable or $\text{firstop}(s'_j) = f$.

In case that $s'_j \notin X$ and $\text{firstop}(s'_j) = g$ is impossible.

Indeed, let $s'_j = g(s''_1, \ldots, s''_n)$ where $s''_1, \ldots, s''_n \in W_{(m,n)}(X)$.

Then $\hat{\sigma}_{t_1,t_2}[s'_j] = S^n(t_2, \hat{\sigma}_{t_1,t_2}[s''_1], \ldots, \hat{\sigma}_{t_1,t_2}[s''_n])$, so $\text{firstop}(\hat{\sigma}_{t_1,t_2}[s'_j]) = f$ which contradicts to $\hat{\sigma}_{t_1,t_2}[s'_j] = s'_j$.

Therefore, $s'_j = f(s''_1, \ldots, s''_m)$ where $s''_1, \ldots, s''_m \in W_{(m,n)}(X)$.

Now, we obtain

$$t_2 = S^m(t_1, \hat{\sigma}_{t_1,t_2}[s'_1], \ldots, \hat{\sigma}_{t_1,t_2}[s'_{j-1}],$$
$$S^m(t_1, \hat{\sigma}_{t_1,t_2}[s''_1], \ldots, \hat{\sigma}_{t_1,t_2}[s''_m]), \hat{\sigma}_{t_1,t_2}[s'_{j+1}], \ldots, \hat{\sigma}_{t_1,t_2}[s'_m]).$$

Similarly, $s''_j \in X$ or $\text{firstop}(s''_j) = f$. We do this step and this procedure will stop after finitely many steps. Therefore,

$$t_2 = S^m(t_1, k_1^1, \ldots, k_m^1)$$

where $k_j^1 \in X$ or $k_j^1 = S^m(t_1, k_1^2, \ldots, k_m^2)$
where $k_j^2 \in X$ or $k_j^2 = S^m(t_1, k_1^3, \ldots, k_m^3)$

$\vdots$
where $k_j^{l-1} \in X$ or $k_j^{l-1} = S^m(t_1, k_1^l, \dots, k_m^l)$
where $k_j^l \in X$ for some $l \in \mathbb{N}$.

(2) $\Rightarrow$ (1) : Consider

$$\hat{\sigma}_{t_1,t_2}[t_1] = S^m(t_1, \hat{\sigma}_{t_1,t_2}[s_1], \dots, \hat{\sigma}_{t_1,t_2}[s_m]).$$

Then we replace x_j in the term t_1 by $\hat{\sigma}_{t_1,t_2}[s_j]$. Since $s_j = x_j$, $\hat{\sigma}_{t_1,t_2}[s_j] = x_j$. Thus, $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. Next we show that $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Suppose that the construction of a term t_2 terminates at the step $l \in \mathbb{N}$. This means that

$$k_j^{l-1} = S^m(t_1, k_1^l, \dots, k_{j-1}^l, x_j, k_{j+1}^l, \dots, k_m^l),$$

so $\hat{\sigma}_{t_1,t_2}[k_j^{l-1}] = k_j^{l-1}$. Then

$$k_j^{l-2} = S^m(t_1, k_1^{l-1}, \dots, k_j^{l-1}, \dots, k_m^{l-1}).$$

We replace x_j in the term t_1 by k_j^{l-1}, and so $\hat{\sigma}_{t_1,t_2}[k_j^{l-2}] = k_j^{l-2}$. We do this steps, thus, $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. ■

Similarly, we can prove the following proposition.

Proposition 3.22. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) > 1$, $t_1 = g(s_1, \dots, s_n)$ and $t_2 = g(s'_1, \dots, s'_n)$ where $s_1, \dots, s_n, s'_1, \dots, s'_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *if $x_j \in var(t_2)$ where $1 \leq j \leq n$, then $s_j = x_j$ and $t_1 = S^n(t_2, k_1^1, \dots, k_n^1)$*
where $k_j^1 \in X$ or $k_j^1 = S^m(t_1, k_1^2, \dots, k_m^2)$
where $k_j^2 \in X$ or $k_j^2 = S^m(t_1, k_1^3, \dots, k_m^3)$
$\vdots$
where $k_j^{l-1} \in X$ or $k_j^{l-1} = S^m(t_1, k_1^l, \dots, k_m^l)$
where $k_j^l \in X$ for some $l \in \mathbb{N}$.

For the last case, we obtain the following.

Proposition 3.23. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) > 1$, $t_1 = f(s_1, \dots, s_m)$ and $t_2 = g(s'_1, \dots, s'_n)$ where $s_1, \dots, s_m, s'_1, \dots, s'_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:*

(1) *σ_{t_1,t_2} is idempotent.*

(2) *the following statements hold:*

(2.1) *if $x_j \in var(t_1)$ where $1 \leq j \leq m$, then $s_j = x_j$;*

(2.2) *if* $x_k \in var(t_2)$ *where* $1 \leq k \leq n$*, then* $s'_k = x_k$*.*

Proof. (1) $\Rightarrow$ (2) : Assume that $x_j \in \text{var}(t_1)$ where $1 \leq j \leq m$. Suppose that $s_j \neq x_j$. If $s_j \in W_{(m,n)}(X) \setminus X$, then

$$\text{op}(t_1) < \text{op}(S^m(\sigma_{t_1,t_2}(f), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m])).$$

Otherwise, we consider,

$$t_1 = S^m(t_1\hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]).$$

Since $x_j \in \text{var}(t_1)$, we replace x_j in the term t_1 by $\hat{\sigma}_{t_1,t_2}[s_j] \neq x_j$. This implies

$$S^m(t_1\hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]) \neq t_1,$$

this is a contradiction. Hence, $s_j = x_j$. Similarly, we can prove that if $x_k \in \text{var}(t_2)$ where $1 \leq k \leq n$, then $s'_k = x_k$.

(2) $\Rightarrow$ (1) : Consider

$$\hat{\sigma}_{t_1,t_2}[t_1] = S^m(t_1, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]).$$

Then we replace x_j in the term t_1 by $\hat{\sigma}_{t_1,t_2}[s_j]$. Since $s_j = x_j$, $\hat{\sigma}_{t_1,t_2}[s_j] = x_j$. Thus, $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. Similarly, we can show that $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. ∎

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