# SOLITONS AND OTHER SOLUTIONS TO NONLINEAR PDEs USING $(\frac{G'}{G})$-EXPANSION METHOD

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**Abstract.** In this article, we apply $(\frac{G'}{G})$-expansion method to construct exact traveling wave solutions with parameters of four nonlinear PDEs having non-integer balance numbers, namely, the convection-diffusion-reaction equation with power-law nonlinearity with density-independent (or density-dependent) diffusion, and the generalized KdV-mKdV equation with any order (or with higher-order) nonlinear terms. When the parameters take up special values, the solitary wave solutions as well as the trigonometric and rational function solutions are derived from the exact traveling wave solutions.The used method in this article presents a wider applicability for handling nonlinear wave equations.

**Keywords:** homogeneous balance principle; $(\frac{G'}{G})$-expansion method; nonlinear PDEs; exact traveling wave solutions; solitary wave solutions; non-integer balance numbers; trigonometric function solutions; rational function solutions.

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## 1. Introduction

It is significant to looking for the exact traveling wave solutions for nonlinear partial differential equations (PDEs) in nonlinear sciences. These equations are widely used to describe many important phenomena and dynamic processes in physics, chemistry, biology, fluid mechanics plasma, optical fibers and other areas of engineering. As mathematical models of the phenomena, the investigation of exact traveling wave solutions of these equations will help us to understand these phenomena better. In recent decades, various effective approaches have been developed to construct the exact traveling wave solutions of these nonlinear equations, such as the exp-function methods [1]-[7], the sine-cosine method [8], [9], the homogeneous balance method [10], [11], the tanh-sech method [12], [13], the extended tanh-coth method [14], [15], the $(\frac{G'}{G})$-expansion method [16]-[18], the modified simple equation method [19], [20], the multiple exp-function method [21], [22], the modified Kudryashov method [23], the method of soliton ansatz [24]-[46], and so on.

The objective of this article is to use $(\frac{G'}{G})$-expansion method [47]-[49] to find the exact traveling wave solutions and the solitary wave solutions of nonlinear PDEs having non-integer balance numbers, such as the convection-diffusion-reaction equation [50] with power-law nonlinearity and with density-independent (or density-dependent) diffusion, and the generalized KdV-mKdV equation [51]-[53] with any order (or with higher-order) nonlinear terms.

This article is organized as follows: In Section 2, the description of $(\frac{G'}{G})$-expansion method is given. In Section 3, we apply this method to solve the four nonlinear PDEs having non-integer balance numbers indicated above. In Section 4, physical explanations of some results are presented. In Section 5, conclusions are obtained.

## 2. Description of $(\frac{G'}{G})$-expansion method

Suppose that a nonlinear PDE has the form

(2.1) $F(u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0,$

where $u = u(x,t)$ is an unknown function, $F$ is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved.

The main steps of the used method are described as follows [47]-[49]:

**Step 1.** We use the wave transformation

(2.2) $u(x,t) = U(\xi), \;\; \xi = kx + wt,$

where $k$ and $w$ are nonzero constants, to reduce Eq. (2.1) into the following nonlinear ordinary differential equation (ODE):

(2.3) $P(U, U', U'', ...) = 0,$

where $P$ is a polynomial in $U(\xi)$ and its total derivatives $U', U'', ...$, such that $U' = \frac{dU}{d\xi}$, $U'' = \frac{d^2U}{d\xi^2}$ and so on.

**Step 2.** We suppose that Eq. (2.3) has the formal solution:

(2.4) $U(\xi) = A\left[\frac{G'(\xi)}{G(\xi)}\right]^m,$

and $G(\xi)$ satisfies the linear ODE:

(2.5) $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0,$

where $A$, $m$ ,$\lambda$ and $\mu$ are constants to be determined later.

**Step 3.** Balancing the highest order derivatives with the highest nonlinear terms in Eq. (2.3), we can find the balance number $m$ in Eq. (2.4).

**Step 4.** Substituting Eq. (2.4) along with Eq. (2.5) into Eq. (2.3) and setting all the coefficients of powers of $(\frac{G'}{G})$ to zero, we have a set of algebraic equations which can be solved to find the values of $A$, $\lambda$, $\mu$, $k$ and $w$.

**Step 5.** We solve the ODE (2.5) to find the values of the ratio $(\frac{G'}{G})$. Consequently, we can find the exact traveling wave solutions and the solitary wave solutions of Eq. (2.1).

## 3. Applications

In this section, we will apply the method described in Sec.2, to find exact traveling wave solutions and the solitary wave and singular solitary wave solutions of the following nonlinear PDEs:

**Example 3.1.** The convection-diffusion-reaction equation with power-law nonlinearity and with density-independent diffusion.

This equation is well-known [54] and has the form:

$$(3.1)\quad u_t = [u(b_0 + b_1 u^p)]_x + u_{xx} + u(1-u^p)(c_0 + c_1 u^p)\ ,\ p > 0,$$

where $p, b_0, b_1, c_0, c_1$ are real numbers. Eq. (3.1) is a general equation which can include the Fisher equation and a number of other well-known convection-diffusion-reaction equations, such as the Newell-whitehead equation, the Zeldovich equation and the Nagumo equation. Eq. (3.1) has been discussed in [54] using a simplest equation method based on the Ricatti and Bernoulli equations, where exact traveling wave solutions have been found. Eq. (3.1) has not been discussed elsewhere using the method of Sec.2. Let us now, solve Eq. (3.1) by using the method of Section 2. To this aim, we use the wave transformation

$$(3.2)\quad u(x,t) = U(\xi), \qquad \xi = x - ct,$$

where $c$ is a nonzero constant to be determined later referred as the velocity of the traveling wave propagation, to reduce Eq. (3.1) into the following nonlinear ODE:

$$(3.3)\quad U'' + b_1(p+1)U'U^p + (c+b_0)U' + c_0 U + (c_1 - c_0)U^{p+1} - c_1 U^{2p+1} = 0.$$

Balancing $U''$ with $U^{2p+1}$ yields the balance number m as follows:

$$(3.4)\quad m = \frac{1}{p},\ \ p > 0,$$

which is non-integer. From (2.4) and (3.4), we assume that the solution of Eq. (3.3) has the form

$$(3.5)\quad U(\xi) = A\left[\frac{G'(\xi)}{G(\xi)}\right]^{\frac{1}{p}},\ \ p > 0,$$

where $G(\xi)$ satisfies Eq. (2.5). Substituting (3.5) along with Eq. (2.5) into Eq. (3.3) and according to the homogeneous balance principle [55], collecting all the coefficients of powers of $\left[\frac{G'(\xi)}{G(\xi)}\right]$ and setting them to zero, we have the following algebraic equations:

(3.6) $$\left(\frac{G'}{G}\right)^{\frac{1}{p}} : (\lambda^2+2\mu)-\lambda P(c+b_0)+c_0p^2-\mu b_1p(p+1)A^p=0,$$

(3.7) $$\left(\frac{G'}{G}\right)^{\frac{1}{p}-1} : [\lambda(2-p)-p(c+b_0)]\mu=0,$$

(3.8) $$\left(\frac{G'}{G}\right)^{\frac{1}{p}+1} : \lambda(2+p)-p(c+b_0)+[p^2(c_1-c_0)-p\lambda b_1(p+1)]A^p=0,$$

(3.9) $$\left(\frac{G'}{G}\right)^{\frac{1}{p}-2} : \mu^2(1-p)=0,$$

(3.10) $$\left(\frac{G'}{G}\right)^{\frac{1}{p}+2} : (1+p)-b_1p(p+1)A^p-c_1p^2A^{2p}=0$$

From (3.7) or (3.9) we deduce that

(3.11) $$\mu=0\ .$$

From (3.8) and (3.10) we have the equation

(3.12) $$\lambda(2+p)-\lambda[(p+1)-c_1p^2A^{2p}]-p(c+b_0)+p^2(c_1-c_0)A^p=0,$$

which can be simplified to become

(3.13) $$\lambda\, c_1p^2A^{2p}+p^2(c_1-c_0)A^p-p(c+b_0)+\lambda=0.$$

Substituting (3.11) into (3.6) we have

(3.14) $$\lambda^2-\lambda p(c+b_0)+p^2c_0=0.$$

Multiply Eq.(3.13) by $\lambda$ and using Eq.(3.14) we have

(3.15) $$c_1p^2\lambda^2A^{2p}+\lambda p^2A^p(c_1-c_0)-c_0p^2=0.$$

From (3.10) and (3.15) we have the condition

(3.16) $$\frac{c_1p^2\lambda^2}{c_1p^2}=\frac{\lambda p^2(c_1-c_0)}{b_1p(p+1)}=\frac{c_0p^2}{p+1},$$

which gives

(3.17) $$\lambda^2=\frac{\lambda p(c_1-c_0)}{b_1(p+1)}=\frac{c_0p^2}{p+1},$$

where $c_1 \neq c_0$. By solving Eq. (3.15) and using relations (3.17) we deduce that

$$(3.18)\quad A^P = \frac{c_0}{\lambda c_1} \quad or \quad A^P = \frac{-1}{\lambda},$$

where

$$(3.19)\quad \lambda = \frac{p(c_1 - c_0)}{b_1(p+1)} \quad and \quad \lambda^2 = \frac{c_0 p^2}{p+1}.$$

Now, from (3.14) and (3.19) we deduce that the velocity $c$ is given by

$$(3.20)\quad c = \frac{b_1 c_0 (p+2)}{(c_1 - c_0)} - b_0,$$

and consequently $\xi$ is given by

$$(3.21)\quad \xi = x - ct = x - \left[\frac{c_0 b_1 (p+2)}{(c_1 - c_0)} - b_0\right] t.$$

The exact traveling wave solution of Eq. (3.1) is given by

$$(3.22)\quad U(\xi) = \left[-\frac{c_0}{c_1}\left(\frac{A_1\, e^{-\lambda\xi}}{A_0 + A_1\, e^{-\lambda\xi}}\right)\right]^{\frac{1}{p}},$$

or

$$(3.23)\quad U(\xi) = \left[\frac{A_1\, e^{-\lambda\xi}}{A_0 + A_1\, e^{-\lambda\xi}}\right]^{\frac{1}{p}},$$

where $A_0$ and $A_1$ are constants of integration. Setting $\frac{A_0}{A_1} = \pm e^{-\lambda\xi_0}$, where $\xi_0$ is a constant, then we have the solitary wave solutions of Eq. (3.1) in the forms

$$(3.24)\quad U(\xi) = \left\{\frac{-c_0}{2c_1}\left[1 - \tanh\frac{\lambda(\xi - \xi_0)}{2}\right]\right\}^{\frac{1}{p}},$$

or

$$(3.25)\quad U(\xi) = \left\{\frac{1}{2}\left[1 - \tanh\frac{\lambda(\xi - \xi_0)}{2}\right]\right\}^{\frac{1}{p}},$$

while, the singular solitary wave solutions of Eq. (3.1) have the forms

$$(3.26)\quad U(\xi) = \left\{\frac{-c_0}{2c_1}\left[1 - \coth\frac{\lambda(\xi - \xi_0)}{2}\right]\right\}^{\frac{1}{p}},$$

or

$$(3.27)\quad U(\xi) = \left\{\frac{1}{2}\left[1 - \coth\frac{\lambda(\xi - \xi_0)}{2}\right]\right\}^{\frac{1}{p}},$$

where $c_0/c_1 < 0$.

**Example 3.2.** The convection-diffusion-reaction equation with power-law nonlinearity and with density-dependent diffusion

This equation is well-known [54] and has the form:

$$u_t = [u(b_0 + b_1 u^p)]_x + (u^{p+1})_{xx} + u^{1-p}(1-u^p)(c_0 + c_1 u^p), \; p > 0 \;, \tag{3.28}$$

where $p, b_0, b_1, c_0, c_1$ are real numbers. Eq. (3.28) has been discussed in [54] using a simplest equation method based on the Riccati and Bernoulli equations. Eq. (3.28) has not been investigated elsewhere using the method of Section 2. Let us now solve Eq. (3.28) using the method of Section 2. To this aim, we use the wave transformation (3.2) to reduce Eq. (3.28) into the following ODE:

$$\begin{aligned}(p+1)U^pU'' + b_1(p+1)U'U^p + (c+b_0)U' + p(p+1)U^{p-1}U'^2 \\ + c_0U^{1-p} + (c_1 - c_0)U - c_1U^{p+1} = 0.\end{aligned} \tag{3.29}$$

Balancing $U^pU''$ with $U^{1-p}$ gives the balance number $m$ as follows:

$$m = \frac{-1}{p}, \;\; p > 0, \tag{3.30}$$

which is non-integer. From (2.4) and (3.30), we assume that the solution of Eq. (3.29) has the form

$$U(\xi) = A\left[\frac{G'(\xi)}{G(\xi)}\right]^{\frac{-1}{p}}, \;\; p > 0, \tag{3.31}$$

where $G(\xi)$ satisfies Eq. (2.5). Substituting (3.31) along with Eq. (2.5) into Eq. (3.29) and according to the homogeneous balance principle [55], collecting all the coefficients of powers of $[\frac{G'(\xi)}{G(\xi)}]$ and setting them to zero, we have the following algebraic equations:

$$\left(\frac{G'}{G}\right)^{\frac{-1}{p}} : A^p\left[b_1p(p+1) + \lambda(p+1)(p+2)\right] + \lambda P(c+b_0) - (c_0 - c_1)p^2 = 0, \tag{3.32}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^{\frac{-1}{p}-1} &: \mu\, p(c+b_0) - A^p[c_1\, p^2 - \lambda\, b_1 p(p+1) \\ &- (p+1)(\lambda^2 + 2\mu) - \lambda^2\, p(p+1) - 2\mu p(p+1)] = 0,\end{aligned} \tag{3.33}$$

$$\left(\frac{G'}{G}\right)^{\frac{-1}{p}+1} : p(c+b_0) + c_0\, p^2 A^{-p} + (p+1)A^p = 0, \tag{3.34}$$

$$\left(\frac{G'}{G}\right)^{\frac{-1}{p}-2} : A^p\mu\left[b_1p(p+1) + \lambda(p+1)(p+2) + 2\lambda p(p+1)\right] = 0, \tag{3.35}$$

$$\left(\frac{G'}{G}\right)^{\frac{-1}{p}-3} : \mu^2(p+1)(2p+1)A^{p+1} = 0. \tag{3.36}$$

From (3.35) or (3.36) we deduce that

(3.37) $\mu = 0.$

Now, Eq. (3.33) reduces to

(3.38) $\lambda^2(p+1)^2 + \lambda b_1 p(p+1) - c_1 p^2 = 0.$

Multiply Eq. (3.34) by $A^p$, we get the equation

(3.39) $(p+1)A^{2p} + p(c+b_0)A^p + c_0\, p^2 = 0.$

Multiply Eq. (3.39) by λ and use Eq. (3.32) we have the equation

(3.40) $\lambda(p+1)A^{2p} + [(c_0 - c_1)\, p^2 - \{b_1 p(p+1) + \lambda(p+1)(p+2)\}A^p]A^p + \lambda c_0 p^2 = 0,$

which can be rewritten in the form

(3.41) $(p+1)[\lambda(p+1) + b_1 p]A^{2p} - (c_0 - c_1)\, p^2 A^p - \lambda c_0 p^2 = 0.$

With the aid of (3.38), we can solve Eq. (3.41) to get

(3.42) $A^p = \dfrac{\lambda c_0}{c_1} \quad or \quad A^p = -\lambda,$

where λ can be determined by solving Eq. (3.38) to get the values

(3.43) $\lambda = \dfrac{p}{2(p+1)}\left[-b_1 \pm \sqrt{b_1^2 + 4c_1}\right] \quad and \quad c_1 \neq 0.$

In the case of $A^p = \frac{\lambda c_0}{c_1}$, we deduce from (3.39) that the velocity $c$ is given by

(3.44) $c = -\left[\dfrac{\lambda^2 c_0(p+1) + c_1^2 p^2}{\lambda c_1 p} + b_0\right],$

while in the case of $A^p = -\lambda$ we deduce from(3.39) that the velocity $c$ is given by

(3.45) $c = \left[\dfrac{\lambda^2(p+1) + c_0 p^2}{\lambda p} - b_0\right].$

Note that the values (3.44) and (3.45) can be found also if we use the Eqs. (3.32) and (3.38). Now, the exact traveling wave solutions of Eq. (3.28) are

(3.46) $U(\xi) = \left[-\dfrac{c_1}{c_0}\left(\dfrac{A_1\, e^{-\lambda\xi_1}}{A_0 + A_1\, e^{-\lambda\xi_1}}\right)\right]^{\frac{-1}{p}},$

where $\xi_1$ is given by

(3.47) $\xi_1 = x - ct = x + \left[\dfrac{\lambda^2 c_0(p+1) + c_1^2 p^2}{\lambda c_1 p} + b_0\right] t,$

or

$$U(\xi)=\left[\frac{A_1\ e^{-\lambda\xi_2}}{A_0+A_1\ e^{-\lambda\xi_2}}\right]^{\frac{-1}{p}},$$

where $\xi_2$ is given by

$$\xi_2=x-ct=x-\left[\frac{\lambda^2(p+1)+c_0p^2}{\lambda p}-b_0\right]t, \tag{3.48}$$

and $\lambda$ is given by (3.43). Setting $\frac{A_0}{A_1}=\pm e^{-\lambda\xi_0}$, where $\xi_0$ is a constant, then we have the solitary wave solutions of Eq. (3.28) as follows:

$$U(\xi)=\left\{\frac{-c_1}{2c_0}\left[1-\tanh\frac{\lambda(\xi_1-\xi_0)}{2}\right]\right\}^{\frac{-1}{p}}, \tag{3.49}$$

or

$$U(\xi)=\left\{\frac{1}{2}\left[1-\tanh\frac{\lambda(\xi_2-\xi_0)}{2}\right]\right\}^{\frac{-1}{p}}, \tag{3.50}$$

while the singular solitary wave solutions of Eq. (3.28)are given by:

$$U(\xi)=\left\{\frac{-c_1}{2c_0}\left[1-\coth\frac{\lambda(\xi_1-\xi_0)}{2}\right]\right\}^{\frac{-1}{p}}, \tag{3.51}$$

or

$$U(\xi)=\left\{\frac{1}{2}\left[1-\coth\frac{\lambda(\xi_2-\xi_0)}{2}\right]\right\}^{\frac{-1}{p}}, \tag{3.52}$$

where $c_1/c_0<0$.

**Example 3.3.** The generalized KdV-mKdV equation with any order nonlinear terms

This equation is well-known [51]-[53] and has the form:

$$u_t+(\alpha+\beta u^p+\gamma u^{2p})_{u_x}+u_{xxx}=0,\ p>0\ , \tag{3.53}$$

where $\alpha,\beta,\gamma$ and $p$ are constants. Eq. (3.53) has been investigated in [51] using the homogeneous balance principle combined with a sub-ODE method and in [53] using the $(\frac{G'}{G})$-expansion method which is absolutely different from $(\frac{G'}{G})$-expansion method described in Section 2. Eq. (3.53) has not been discussed elsewhere using the method in Section 2. Let us now solve it using the method of Section 2. To this aim, we use the wave transformation (3.2) to reduce Eq. (3.53) into the following ODE:

$$(\alpha-c)U'+\beta U'U^p+\gamma U'U^{2p}+U'''=0. \tag{3.54}$$

Integrating (3.54) once with respect to $\xi$ with vanishing the constant of integration, we get

$$(3.55)\ \ (\alpha-c)U+\left(\frac{\beta}{p+1}\right)U^{p+1}+\left(\frac{\gamma}{2p+1}\right)U^{2p+1}+U''=0.$$

Balancing $U''$ with $U^{2p+1}$ yields the balance number $m$ as follows:

$$(3.56)\ \ m=\frac{1}{p},\ \ p>0,$$

which is non-integer. From (2.4) and (3.56), we assume that the solution of Eq. (3.55) has the form

$$(3.57)\ \ U(\xi)=A\left[\frac{G'(\xi)}{G(\xi)}\right]^{\frac{1}{p}},\ \ p>0,$$

where $G(\xi)$ satisfies Eq. (2.5). Substituting (3.57) along with Eq. (2.5) into Eq. (3.55) and according to the homogeneous balance principle [55], collecting all the coefficients of powers of $\left[\frac{G'(\xi)}{G(\xi)}\right]$ and setting them to zero, we have the following algebraic equations:

$$(3.58)\ \ \left(\frac{G'}{G}\right)^{\frac{1}{p}-2}: A\mu^2(1-p^2)(1+2p)=0,$$

$$(3.59)\ \ \left(\frac{G'}{G}\right)^{\frac{1}{p}+2}: \gamma p^2A^{2p}(1+p)+(1+2p)(p+1)^2=0,$$

$$(3.60)\ \ \left(\frac{G'}{G}\right)^{\frac{1}{p}+1}: \beta p^2A^{p}(1+2p)+\lambda(1+2p)(p+1)(p+2)=0,$$

$$(3.61)\ \ \left(\frac{G'}{G}\right)^{\frac{1}{p}}: (\alpha-c)p^2(p+1)(1+2p)+(\lambda^2+2\mu)(1+2p)(p+1)=0\ ,$$

$$(3.62)\ \ \left(\frac{G'}{G}\right)^{\frac{1}{p}-1}: \lambda\mu(2-p)(p+1)(1+2p)=0.$$

From (3.58) or (3.62) we deduce that

$$(3.63)\ \ \mu=0.$$

Eq. (3.59) gives

$$(3.64)\ \ A^{2p}=-\frac{(p+1)(1+2p)}{\gamma p^2}\ ,$$

while (3.60) gives

$$(3.65)\ \ A^{p}=-\frac{\lambda(p+1)(2+p)}{\beta p^2}\ ,$$

From (3.64) and (3.65) we conclude that

$$(3.66)\quad \lambda^2 = -\frac{\beta^2 p^2(2p+1)}{\gamma(p+1)(p+2)^2}.$$

Substituting (3.63) and (3.66) into Eq. (3.61) we have the velocity $c$ as follows:

$$(3.67)\quad c = \alpha - \frac{\beta^2(2p+1)}{\gamma(p+1)(p+2)^2} \;,$$

and consequently $\xi$ is given by

$$(3.68)\quad \xi = x - ct = x - \left[\alpha - \frac{\beta^2(2p+1)}{\gamma(p+1)(p+2)^2}\right] t.$$

From (3.65) and (3.66) we deduce that

$$(3.69)\quad A^p = \frac{\beta(2p+1)}{\lambda\gamma(p+2)}.$$

Now, the exact traveling wave solution of Eq. (3.53) is given by

$$(3.70)\quad U(\xi) = \left[-\frac{\beta(2p+1)}{\gamma(p+2)}\left(\frac{A_1\, e^{-\lambda\xi}}{A_0 + A_1\, e^{-\lambda\xi}}\right)\right]^{\frac{1}{p}},$$

where $A_0$ and $A_1$ are arbitrary constants of integration.

Choose $\frac{A_0}{A_1} = \pm e^{-\lambda\xi_0}$, where $\xi_0$ is a constant, we have the solitary wave solution of Eq. (3.53) in the form

$$(3.71)\quad U(\xi) = \left\{-\frac{\beta(2p+1)}{2\gamma(p+2)}\left[1 - \tanh\frac{\lambda(\xi-\xi_0)}{2}\right]\right\}^{\frac{1}{p}},$$

and the singular solitary wave solution of Eq. (3.53) in the form

$$(3.72)\quad U(\xi) = \left\{-\frac{\beta(2p+1)}{2\gamma(p+2)}\left[1 - \coth\frac{\lambda(\xi-\xi_0)}{2}\right]\right\}^{\frac{1}{p}},$$

provided $\beta/\gamma < 0$, and $\lambda$, $\xi$ are given by (3.66) and (3.68) respectively.

**Example 3.4.** The generalized KdV-mKdV equation with higher-order nonlinear terms

This equation is well-known [52] and has the form:

$$(3.73)\quad u_t + (\varepsilon u^{-2p} + \delta u^{-p} + \alpha + \beta u^p + \gamma u^{2p})_{u_x} + u_{xxx} = 0, \quad p > 1\;,$$

where $\varepsilon, \delta, \alpha, \beta, \gamma$ and $p$ are real numbers. If $\varepsilon = \delta = 0$, then Eq. (3.73) reduces to the generalized KdV-mKdV equation (3.53) with any order nonlinear terms. Eq. (3.73) has been investigated in [52] using the homogeneous balance principle combined with a subsidiary ODE method, and its solutions have been found. Eq. (3.73) has not been solved elsewhere using the method of Section 2. Let us now solve it using the method of Section 2. To this aim, we use the wave transformation (3.2) to reduce Eq. (3.73) into the following nonlinear ODE:

$$(3.74)\quad (\alpha - c)U' + \varepsilon U'U^{-2p} + \delta U'U^{-p} + \beta U'U^{p} + \gamma U'U^{2p} + U''' = 0.$$

Integrating (3.74) with respect to $\xi$ with vanishing the constant of integration, we get

$$(3.75)\quad \begin{aligned} (\alpha - c)U \; &+ \left(\frac{\varepsilon}{1-2p}\right)U^{1-2p} + \left(\frac{\delta}{1-p}\right)U^{1-p} + \left(\frac{\beta}{1+p}\right)U^{1+p} \\ &+ \left(\frac{\gamma}{1+2p}\right)U^{1+2p} + U'' = 0. \end{aligned}$$

Balancing $U''$ with $U^{2p+1}$ yields the balance number m as follows:

$$(3.76)\quad m = \frac{1}{p}, \;\; p > 1 \,,$$

which is non-integer. From (2.4) and (3.76), we assume that the solution of Eq. (3.75) is given by

$$(3.77)\quad U(\xi) = A\left[\frac{G'(\xi)}{G(\xi)}\right]^{\frac{1}{p}},$$

where $G(\xi)$ satisfies Eq. (2.5). Substituting(3.77) along with Eq. (2.5) into Eq. (3.75) and according to the homogeneous balance principle [55], collecting all the coefficients of powers of $\left[\frac{G'(\xi)}{G(\xi)}\right]$ and setting them to zero, we have the following algebraic equations:

$$(3.78)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}} : (\alpha - c)p^2 + (\lambda^2 + 2\mu p) = 0,$$

$$(3.79)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}+1} : \beta p^2 A^p + \lambda(1+p)(2+p) = 0,$$

$$(3.80)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}+2} : \gamma p^2 A^{2p} + (1+p)(1+2p) = 0,$$

$$(3.81)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}-1} : \delta p^2 A^{-p} + \lambda\mu(1-p)(2-p) = 0,$$

$$(3.82)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}-2} : \varepsilon p^2 A^{-2p} + \mu^2(1-p)(1-2p) = 0.$$

Eqs. (3.79) and (3.80) give

$$(3.83)\quad A^p = -\frac{\lambda(1+p)(2+p)}{\beta p^2} \quad \textit{and} \quad A^{2p} = -\frac{(1+p)(1+2p)}{\gamma p^2},$$

while from (3.81) and (3.82), we have

$$(3.84)\quad A^{-p} = -\frac{\lambda\mu(1-p)(2-p)}{\delta p^2} \quad \textit{and} \quad A^{-2p} = -\frac{\mu^2(1-p)(1-2p)}{\varepsilon p^2}.$$

From (3.83) and (3.84), we note that the interchanges $p \leftrightarrow -p$ yields the relations

$$(3.85)\quad \delta = \mu\ \beta \quad \textit{and} \quad \varepsilon = \mu^2\ \gamma\,.$$

From (3.83) we have

$$(3.86)\quad \lambda^2 = -\frac{\beta^2\ p^2\ (1+2p)}{\gamma\ (1+p)(2+p)^2}\,,$$

while from (3.84) we have

$$(3.87)\quad \lambda^2 = -\frac{\delta^2\ p^2\ (1-2p)}{\varepsilon\ (1-p)(2-p)^2}\,.$$

Under conditions (3.85) we see the right-hand-side of (3.86) is the same as the right-hand-side of (3.87). Now, the exact traveling wave solution of Eq. (3.73) is given by

$$(3.88)\quad U(\xi) = \left[\frac{\beta(2p+1)}{\lambda\,\gamma(p+2)}\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^{\frac{1}{p}}\,,$$

where, with the aid of (3.78) and (3.86) we can determine the velocity $c$, and consequently we have

$$(3.89)\quad \xi = x - c\,t = x - \left[\alpha - \frac{\beta^2(1+2p)}{\gamma(1+p)(2+p)^2} + \frac{2\mu}{p}\right]t\,.$$

It is well-known that, the ratio $\left(\frac{G'}{G}\right)$ can be determined with the help of the solutions of Eq. (2.5) to get

**(i) If** $\lambda^2 - 4\mu > 0$, then

$$(3.90)\quad \left(\frac{G'}{G}\right) = -\frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{C_1\exp\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) - C_2\exp\left(\frac{-\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{C_1\exp\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + C_2\exp\left(\frac{-\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right),$$

where $C_1$ and $C_2$ are arbitrary constants, while $\xi$ is given by (3.89).
From (3.88) and (3.90), the exact traveling wave solution of Eq. (3.73) is given by

(3.91)

$$U(\xi) = \left\{-\frac{\beta(2p+1)}{2\,\gamma(p+2)}\left[1 - \frac{\sqrt{\lambda^2-4\mu}}{\lambda}\left(\frac{C_1\exp\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) - C_2\exp\left(\frac{-\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{C_1\exp\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + C_2\exp\left(\frac{-\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right)\right]\right\}^{\frac{1}{p}},$$

where $\beta/\gamma < 0$. Choosing $\frac{C_2}{C_1} = \pm\, e^{-\sqrt{\lambda^2-4\mu}\,\xi_0}$, where $\xi_0$ is a constant, then the solitary wave solution of Eq. (3.73) follows from (3.91) and has the form

$$(3.92)\quad U(\xi) = \left\{ -\frac{\beta(2p+1)}{2\,\gamma(p+2)} \left[ 1 - \frac{\sqrt{\lambda^2-4\mu}}{\lambda} \tanh\left( \frac{1}{2}\sqrt{\lambda^2-4\mu}\,(\xi-\xi_0) \right) \right] \right\}^{\frac{1}{p}},$$

which is agreement with (3.71) if $\mu = 0$ (i.e., if $\varepsilon = \delta = 0$), while the singular solitary wave solution of Eq. (3.73) follows also from Eq. (3.91) and has the form

$$(3.93)\quad U(\xi) = \left\{ -\frac{\beta(2p+1)}{2\,\gamma(p+2)} \left[ 1 - \frac{\sqrt{\lambda^2-4\mu}}{\lambda} \coth\left( \frac{1}{2}\sqrt{\lambda^2-4\mu}\,(\xi-\xi_0) \right) \right] \right\}^{\frac{1}{p}},$$

which is agreement with (3.72) if $\mu = 0$ $(i.e.\ if\ \ \varepsilon = \delta = 0)$ . Note that

$$(3.94)\quad \lambda^2 - 4\mu = -\left[ \frac{\beta^2 p^2 (2p+1)}{\gamma(1+p)(p+2)^2} + 4\mu \right],$$

which is positive if $\gamma < 0$ and $\mu < 0$ and negative if $\gamma > 0$ and $\mu > 0$. Consequently, we have the following results:

**(ii) If** $\lambda^2 - 4\mu < 0$, then

$$(3.95)\quad \left(\frac{G'}{G}\right) = -\frac{\lambda}{2} + \frac{1}{2}\sqrt{4\mu-\lambda^2} \left( \frac{-C_1\, sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + C_2\, cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{C_1\, cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + C_2\, sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)} \right),$$

where $C_1$ and $C_2$ are arbitrary constants, while $\xi$ is given by (3.89). From (3.88) and (3.95), we have the exact traveling wave solution of Eq. (3.73) in the form:

(3.96)

$$U(\xi) = \left\{ -\frac{\beta(2p+1)}{2\,\gamma(p+2)} \left[ 1 - \frac{\sqrt{4\mu-\lambda^2}}{\lambda} \left( \frac{-C_1\, sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + C_2\, cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{C_1\, cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + C_2\, sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)} \right) \right] \right\}^{\frac{1}{p}},$$

where $\beta/\gamma < 0$ .

Choosing $\frac{C_2}{C_1} = \tan\,(\frac{\xi_0}{2}\sqrt{4\mu-\lambda^2})$, where $\xi_0$ is a constant, then we have the trigonometric function solution

$$(3.97)\quad U(\xi) = \left\{ -\frac{\beta(2p+1)}{2\,\gamma(p+2)} \left[ 1 + \frac{\sqrt{4\mu-\lambda^2}}{\lambda} \tan\left( \frac{1}{2}\sqrt{4\mu-\lambda^2}\,(\xi-\xi_0) \right) \right] \right\}^{\frac{1}{p}},$$

while if $\frac{C_1}{C_2} = \tan\ (\frac{\xi_0}{2}\sqrt{4\mu-\lambda^2})$, where $\xi_0$ is a constant, then we have the trigonometric function solution

$$(3.98)\quad U(\xi) = \left\{ -\frac{\beta(2p+1)}{2\,\gamma(p+2)}\left[1-\frac{\sqrt{4\mu-\lambda^2}}{\lambda}\cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,(\xi+\xi_0)\right)\right]\right\}^{\frac{1}{p}},$$

where $\xi$ is given by (3.89).

**(iii) If** $\lambda^2 - 4\mu = 0$, then we get

$$(3.99)\quad \left(\frac{G'}{G}\right) = -\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi},$$

where $C_1$ and $C_2$ are arbitrary constants. With the aid of (3.89) we deduce that

$$(3.100)\quad \xi = x - \left[\alpha - \frac{\beta^2(1+2p)}{2\gamma(1+p)(2+p)}\right] t\,.$$

Now, we have the rational function solution of Eq. (3.73) as follows:

$$(3.101)\quad U(\xi) = \left\{ -\frac{\beta(2p+1)}{2\,\gamma(p+2)}\left[1-\frac{C_2}{\lambda(C_1+C_2\xi)}\right]\right\}^{\frac{1}{p}},$$

where $\xi$ is given by (3.100), and $\beta/\gamma < 0$.

## 4. Physical explanations for some results

In this section, we present some graphs of the solitary wave solutions by taking suitable values of the parameters to visualize the mechanism of the original equations (3.1), (3.28), (3.53) and (3.73). Their solutions (3.24), (3.49), (3.71) and (3.92) are solitary wave solutions, while their solutions (3.26), (3.51), (3.72) and (3.93) are singular solitary wave solutions respectively. Eq. (3.73) has also the trigonometric solution (3.97) and (3.98) as well as the rational function solution (3.101). For more convenience, the graphical representations of some of these solutions are shown in Figs. 1-3 as follows:

## 5. Conclusions

We have derived the exact traveling wave solutions including the solitary wave and singular solitary wave solutions for the nonlinear PDEs (3.1), (3.28), (3.53) and (3.73) having non-integer balance numbers by using $(\frac{G'}{G})$-expansion method with the aid of the homogeneous balance principle described in Section 2 of this paper. In view of mathematical analysis, we see that the used method is an efficient method of integrability for constructing these solutions. On comparing the used method described in Section 2 of this paper with the other methods used in [27]-[46], [51]-[54] we deduce that our method is different and to our knowledge has been used in the first time to solve the equations (3.1), (3.28), (3.53) and

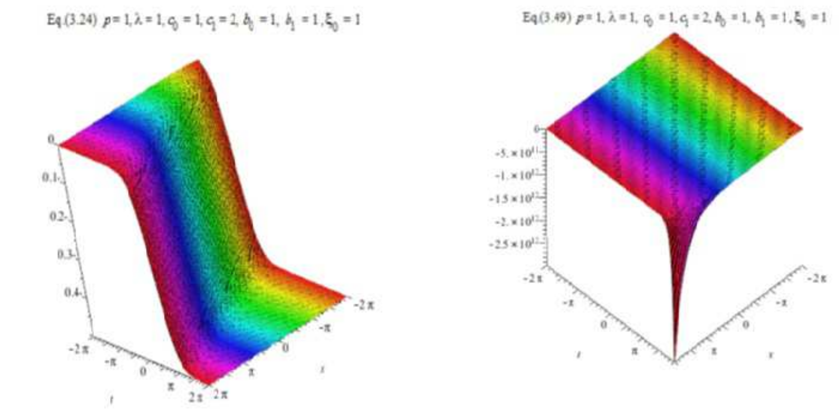


Figure 1: The solitary wave solutions (3.24) and (3.49)

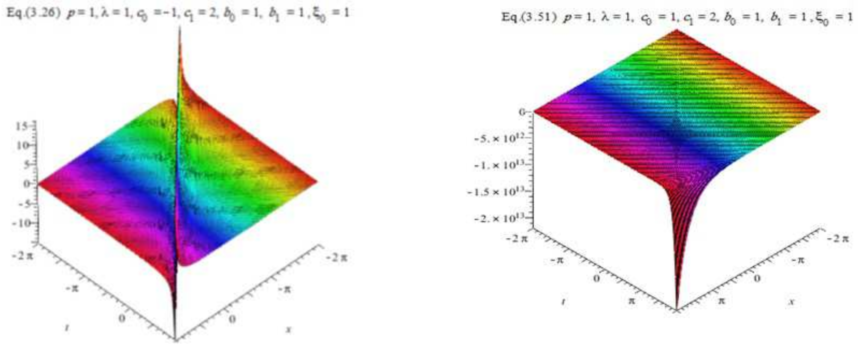


Figure 2: The singular solitary wave solutions (3.26) and (3.51)

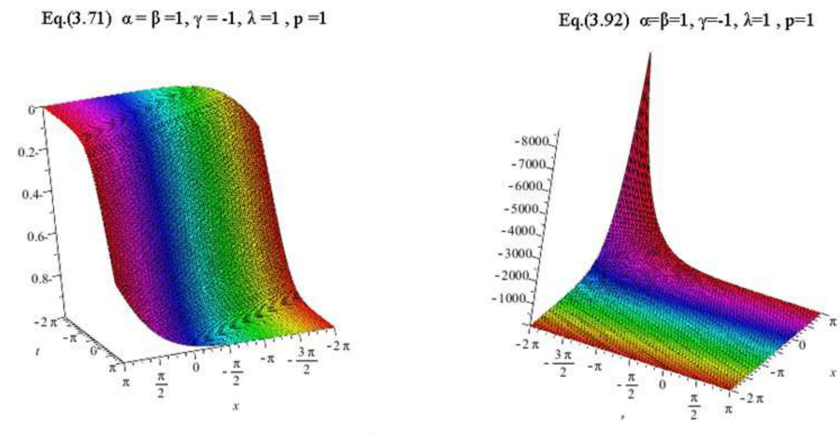


Figure 3: The solitary wave solutions (3.71) and (3.92)

(3.73). Consequently, our results in this paper are new and not reported elsewhere. The obtained solutions may be significant important for the explanations of some special physical phenomena. Finally, our results in this article have been checked by using the Maple by putting them back into the original equations (3.1), (3.28), (3.53) and (3.73).

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## References

[1] WAZWAZ, A.M., *Solitary wave solutions of the generalized shallow water wave (GSWW) equation by Hirota's method, tanh-coth method and Exp-function method*, App. Math. Comput., 202 (2008), 275-286.

[2] HE, J.H., WU, X.H., *Exp-function method for nonlinear wave equations*, Chaos, Solitons & Fractals. 30 (2006), 700-708.

[3] WU, X.H., HE, J.H., *Solitary solutions periodic solutions and compacton-like solutions using Exp-function method*, Comput. Math. Appl., 54 (2007), 966-986.

[4] HE, J.H., ZHANG, L.N., *Generalized solitary solution and compacton-like solution of the Jaulent-Miodek equations using the Exp-function method*, Phys. Lett. A, 372 (2008), 1044-1047.

[5] ZHU, S.D., *Exp-function method for the discrete mKdV lattice*, Int. J. Nonlinear Sci. Numer. Solut., 8 (2007), 465-468.

[6] ZHANG, S., *Application of Exp-function method to high-dimensional nonlinear evolution equation*, Chaos Solitons Fract., 38 (2008), 270-276.

[7] GANJI, D.D., ASGARI, A., GANJI, Z.Z., *Exp-function based solution of nonlinear Radhakrishnan, Kundu and Laskshmanan (RKL) equation*, Acta Appl. Math., 104 (2008), 201-209.

[8] WAZWAZ, A.M., *The tanh and the sine-cosine methods for a reliable treatment of the modified equal width equation and its variants*, Commun. Nonlinear Sci. Numer. Simul., 11 (2006), 148-160.

[9] WAZWAZ, A.M., *The tanh method and the sine-cosine method for solving the KP-MEW equation*, Int. J. Comput. Math., 82 ( 2005), 235-246.

[10] FAN, E., ZHANG, H., *A note on the homogeneous balance method*, Phys. Lett. A, 246 (1998) 403-406.

[11] ZAYED, E.M.E., ALURRFI, K.A.E., *The homogeneous balance method and its applications for finding the exact solutions for nonlinear evolution equations*, Italian J. Pure Appl. Math., 33 (2014), 307-318.

[12] MALFIET, W., HEREMAN, W., *The tanh method: I. Exact solutions of nonlinear evolution and wave equations*, Phys., Scr., 54 (1996), 563-568.

[13] MALFIET, W., HEREMAN, W., *The tanh method: II. Perturbation technique for conservative systems*, Phys. Scr., 54 (1996), 569-575.

[14] FAN, E., *Extended tanh-function method and its applications to nonlinear equations*, Phys. Lett. A, 277 (2000), 212.

[15] ABDOU, M.A. , *The extended tanh method and its applications for solving nonlinear physical models*, Appl. Math. Comput. 190 (2007), 988-996.

[16] WANG, M.L., LI, X., ZHANG, J., *The* $(\frac{G'}{G})$*-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics*, Phys. Lett., A 372 (2008), 417-423.

[17] ZAYED, E.M.E., GEPREEL, K.A., *The* $(\frac{G'}{G})$*-expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics*, J. Math. Phys. 50 (2009), 013502-013513.

[18] ZAYED, E.M.E., *New traveling wave equations for higher dimensional nonlinear evolution equations using a generalized* $(\frac{G'}{G})$*-expansion method*, J. Phys. A: Math. Theor., 42 (2009), 195202-195214.

[19] JAWAD, A.J.M., PETKOVIC, M.D., BISWAS, A., *Modified simple equation method for nonlinear evolution equations*, Appl. Math. Comput., 217 (2010), 869-877.

[20] ZAYED, E.M.E., HODA IBRAHIM, S.A. , *Exact solutions of nonlinear evolution equations in mathematical physics using the modified simple equation method*, Chinese Phys, Lett. 29 (2012) 060201-060204.

[21] MA, W.X., HUANG, T., ZHANG, Y., *A multiple exp-function for nonlinear differential equations and its application*, Phys. Script., 82 (2010), 065003.

[22] ZAYED, E.M.E., ABDUL-GHANI, AL-N., *The multiple exp-function method and the linear superposition principle for solving the (2+1)-dimensional Calogero-Bogoyavleskii-Schiff equation*, Z. Naturforsh. 70a (2015), 775-779.

[23] ZAYED, E.M.E., ALURRFI, K.A.E., *The modified Kudryashov method for solving some seventh order nonlinear PDEs in mathematical physics*, World J. Model. Simula. 11 (2015), 308- 319.

[24] BISWAS, A., MILOVIC, M., EDWARDS, M., *Mathematical Theory of Dispersion-Managed Optical Solitons*, Springer-Verlag, New York, 2010.

[25] SARMA, A.K., SAHA, M., BISWAS, A., *Optical solitons with law nonlinearity and Hamiltonian perturbations: An exact solution*, J. Infrared Mili Terahz waves, 31 (2010), 1048-1056.

[26] BISWAS, A., *1-soliton solution of Benjamin-Bona-Mahoney equation with dual-power law non-linearity*, Comm. Nonlin. Sci. Numer. Simul., 15 (2010), 2744-2746.

[27] ANTONOVA, M., BISWAS, A., *Adiabatic parameter dynamics of perturbed solitary waves*, Comm. Nonlin. Sci. Numer. Simul., 14 (2009), 734-748.

[28] GIRGIS, L., BISWAS, A., *Soliton perturbation theory for nonlinear wave equations*, Appl. Math. Comput., 216 (2010), 2226-2231.

[29] GIRGIS, L., BISWAS, A., *A study of solitary waves by He's semi-inverse variational principle*, Waves in Random and Complex Media, 21 (2011), 96-104.

[30] BISWAS, A., *Solitary wave solution for KdV equation with power law nonlinearty and time-dependent coefficients*, Nonli. Dyna., 58 (2009), 345-348.

[31] BISWAS, A., Solitary waves for power-law regularized long-wave equation and $R(m;n)$ equation, Nonli. Dyna., 59 (2010), 423-426.

[32] KRISHNAN, E.V., TRIKI, H., LABIDI, M., BISWAS, A., *A study of shallow water waves with gardner's equation*, Nonli. Dyna., 66 (2011), 497-507.

[33] TRIKI, H., KARA, A.H., BHRAWY, A., BISWAS, A., *Soliton solution and conservation law of Gear-Grimshaw model for shallow water waves*, Acta Physi. Polon., A. 125 (2014), 1099-1106.

[34] ISMAIL, M.S., BISWAS, A., *1-soliton solution of the generalized KdV equation with generalized evolution*, Appl. Math. Compu., 216 (2010), 1673-1679.

[35] BISWAS, A., ISMAIL, M.S., *1-soliton solution of the coupled KdV equation and Gear-Grimshaw model*, Appl. Math. Compu., 216 (2010), 3662-3670.

[36] BHRAWY, A.H., ABDELKAWY, M.A., BISWAS, A., *Cnoidal and snoidal wave solutions to coupled nonlinear wave equations by the extended Jacobi's elliptic function method*, Commun. in Nonli. Sci. and Numer. Simul., 18 (2013), 915-925.

[37] BHRAWY, A.H., ABDELKAWY, M.A., BISWAS, A., *Topological solutions and cnoidal waves to a few nonlinear wave equations in theoretical physics*, Indian J. of Phys., 87 (2013), 1125-1131.

[38] EBADI, G., FARD, N.Y., BHRAWY, A.H., KUMAR, S., TRIKI, S.H., YILDIRIM, A., BISWAS, A., *Solitons and other solutions to the (3+1)-dimensional extended Kadomtsev-Petviashvili equation with power law nonlinearity*, Roman. Rep. in Phys., 65 (2013), 27-62.

[39] BHRAWY, A.H., ABDELKAWY, M.A., KUMAR, S., BISWAS, A., *Solitons and other solutions to Kadomtsev-Petviashvili equation of B-type*, Roman. J. of Phys., 58 (2013), 729-748.

[40] BISWAS, A., BHRAWY, A.H., ABDELKAWY, M.A., ALSHAERY, A.A., HILAL, E.M., *Symbolic computation of some nonlinear fractional differential equations*, Roman. J. of Phys., 59 (2014), 433-442.

[41] TRIKI, H., MIRZAZADEH, M., BHRAWY, A.H., RAZBOROVA, P., BISWAS, A., *Soliton and other solutions to long-wave short-wave interaction equation*, Roman. J.of Phys., 60 (2015), 72-86.

[42] ABDELKAWY, M.A., BHRAWY, A.H., ZERRAD, E., BISWAS, A., *Application of tanh method to complex coupled nonlinear evolution equations*, Acta Phys. Polon., A. 129 (2016), 278-283.

[43] BHRAWY, A.H., ALZAIDY, J.F., ABDELKAWY, M.A., BISWAS, A., *Jacobi spectral collocation approximation for multi-dimensional time fractional Schrödinger's equation*, To appear in Nonli. Dyna., DOI 10.1007/s11071-015-2588-x, 6 January (2016), 1-15.

[44] BISWAS, A., KRISHNAN, E.V., SUAREZ, P., KARA, A.H., KUMAR, S., *Solitary waves and conservation law of Bona-Chen equation*, Indi. J. of Phys., 87 (2013), 169-175.

[45] INC, M., ULUTAS, E., BISWAS, A., Singular solitons and other solutions to a couple of nonlinear wave equations, Chin. Phys., B. 22 (2013), 060204.

[46] INC, M., ULUTAS, E., CAVLAK, E., BISWAS, A., *Singular 1-soliton solution of the K(m; n) equation with generalized evolution and its subsidaries*, Acta Phys. Polon., B. 44 (2013), 1825-1836.

[47] ZHANG, H., *New application of the* $(\frac{G'}{G})$*-expansion method*, Commu. Nonl. Sci. Numer. Simul., 14 (2009), 3220-3225.

[48] ZAYED, E.M.E., EL-MALKY, M.A.S., *The* $(\frac{G'}{G})$*-expansion method for solving nonlinear Klein-Gordon equations*, AIP Conf. Proc., 1389 (2011), 2020-2024.

[49] ZAYED, E.M.E., *Equivalence of the* $(\frac{G'}{G})$*-expansion method and the tanh-coth function method*, AIP Conf. Proc., 1281 (2010), 2225-2228.

[50] ZHANG, J.L., WANG, M.L., *Exact solutions to a class of nonlinear Schrödinger-type equations*, Pramane J. Phys., 67 (2006) 1011-1022.

[51] ZHANG, S., WANG, W., TONG, J.L., *The improved Sub-ODE method for a generalized KdV-mKdV equation with nonlinear terms of any order*, Phys. Lett., A 372 (2008), 3808-3813.

[52] LI, Z.L., *Periodic wave solutions of a generalized KdV-mKdV equation with higher-order nonlinear terms*, Z. Naturfersch, 65a (2010), 649-657.

[53] LI, Z.L., *Constructing of new exact solutions to the GKdV-mKdV equation with any order nonlinear terms by* $(\frac{G'}{G})$*-expansion method*, Appl. Math. Compu., 217 (2010), 1398-1403.

[54] HAYEK, M., *Exact and traveling wave solutions for convection-diffusion-reaction equation with power-law nonlinearity*, Appl. Math. Compu., 218 (2011), 2407-2420.

[55] ZHANG, J.L. , WANG, M.L., LI, X.Z., *The subsidiary ordinary differential equations and the exact solutions of the higher order dispersive nonlinear Schrödinger equation*, Phys. Lett., A 357 (2006), 188-195.

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