WEAKLY EQUIVALENCE PRESERVING FUNCTORS

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Abstract. Let R be a commutative ring and T be an additive covariant functor from the category of R-modules to itself. We recall that an R-module M is said to be T-acyclic (resp. T-coacyclic) if the n-th right (resp. left) derived functor of T on M is zero for any positive integer n. Assume that T is a left (resp. right) exact functor and X, Y are two bounded to the left (resp. right) R-complexes of T-acyclic (resp. T-coacyclic) R-modules such that $X \simeq Y$. We will show that $T(X) \simeq T(Y)$. As an application, we extend the main results of Sazeedeh, Divaani-Aazar and Mohammadi that provide some new ways for computing local (co)homology modules.

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1. Introduction

Throughout this paper R is a commutative ring. We denote the category of R-complexes (resp. R-modules) by $\mathcal{C}(R)$ (resp. $\mathcal{C}_0(R)$) and we use the symbol $\simeq$ for denoting quasi-isomorphisms in the category $\mathcal{C}(R)$. Let T be an additive covariant functor from the category of R-modules to itself and M be an R-module. We recall that M is said to be T-*acyclic* (resp. T-*coacyclic*) if the n-th right (resp. left) derived functor of T on M is zero, i.e. $R^nT(M) = 0$ (resp. $L_nT(M) = 0$) for any positive integer n. Assume that T is a left (resp. right) exact functor and N is an R-module. We know that for computing the n-th right (resp. left) derived functor of T on N, we can use right (resp. left) resolutions of N by T-acyclic (resp. T-coacyclic) R-modules. More precisely, let I and P be an injective resolution and projective resolution of N, respectively. Assume that E (resp. Q) is a right (resp. left) resolution of N by T-acyclic (resp. T-coacyclic) R-modules. Then we have $I \simeq E$ (resp. $P \simeq Q$) and then $T(I) \simeq T(E)$ (resp. $T(P) \simeq T(Q)$). So, for computing $R^nT(N)$ (resp. $L_nT(N)$), we can use right (resp. left) resolutions of N by T-acyclic (resp. T-coacyclic) R-modules. Now, we want to replace R-modules N by R-complexes and we will show that for an R-complex X,

i) If T is a left exact functor and $H_i(X) = 0$ for all $i \gg 0$, then for any bounded to the left R-complex E of T-acyclic R-modules such that $X \simeq E$, we have $T(E) \simeq T(I)$, where I is any injective resolution of X.

ii) If T is a right exact functor and $H_i(X) = 0$ for all $i \ll 0$, then for any bounded to the right R-complex Q of T-coacyclic R-modules such that $X \simeq Q$, we have $T(Q) \simeq T(P)$, where P is any projective resolution of X.

We say that T is a weakly equivalence preserving functor over $\mathcal{C}(R)$ if for any two R-complexes X and Y such that $X \simeq Y$, then $T(X) \simeq T(Y)$. So part i) (resp. part ii)) induces that the left (resp. right) exact functor T is a weakly equivalence preserving functor over the full subcategory of bounded to the left (resp. right) R-complexes of T-acyclic (resp. T-coacyclic) R-modules. These results are very important for computing derived functors and specially, when we apply them for famous functors such as section functor and completion functor and we get some ways for computing local (co)homology modules. Our methods extend [7, Remark, Page 2890], [6, Theorem 2.5] and [4, Corollary 4.7].

2. Prerequisites

Let R be a commutative ring. An *R-complex* X is a sequence of R-modules X_n and R-homomorphisms ∂_n^X

$$X = \cdots \longrightarrow X_{n+1} \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \longrightarrow \cdots ,$$

such that $\partial_n^X \partial_{n+1}^X = 0$ for all $n \in \mathbb{Z}$. The R-complex X is said to be *bounded to the left* (resp. *right*), if there is an integer u such that $X_n = 0$ for all $n > u$ (resp. $n < u$). An R-module M is thought of as the R-complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots ,$$

with M in degree zero. More information about complexes can be found in [2] and [3].

We denote the category of R-complexes (resp. R-modules) by $\mathcal{C}(R)$ (resp. $\mathcal{C}_0(R)$). The full subcategory of bounded to the right (resp. left) R-complexes of projective (resp. injective) R-modules is denoted by $\mathcal{C}_{\sqsupset}^{P}(R)$ (resp. $\mathcal{C}_{\sqsubset}^{I}(R)$). Also $\mathcal{C}_{\sqsupset}^{F}(R)$ denotes the full subcategory of bounded to the right R-complexes of flat R-modules. Let $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ be an additive functor. We denote the full subcategory of bounded to the right (resp. left) R-complexes of T-coacyclic (resp. T-acyclic) R-modules by $\mathcal{C}_{\sqsupset}^{T-coacy.}(R)$ (resp. $\mathcal{C}_{\sqsubset}^{T-acy.}(R)$).

Let $\alpha : X \to Y$ be a morphism of R-complexes. It is called *quasi-isomorphism* if the induced morphism $H(\alpha) : H(X) \to H(Y)$ is an isomorphism, and we denote it by $\alpha : X \xrightarrow{\simeq} Y$. The *mapping cone* complex of α, $\mathrm{Cone}\,(\alpha)$, is defined by the formulas $(\mathrm{Cone}(\alpha))_l = Y_l \oplus X_{l-1}$ and $\partial_l^{\mathrm{Cone}\,(\alpha)}(y_l, x_{l-1}) = (\partial_l^Y(y_l) + \alpha_{l-1}(x_{l-1}), -\partial_{l-1}^X(x_{l-1}))$, for all integers l. Also, we say that two R-complexes X

and Y are *equivalent* and we denote it by $X \simeq Y$ if and only if there exist an R-complex Z and two quasi-isomorphisms $X \xrightarrow{\simeq} Z \xleftarrow{\simeq} Y$. One can see that the relation $\simeq$ is an equivalence relation in the category $\mathcal{C}(R)$.

Assume that X is an R-complex such that $H_i(X) = 0$ for all $i \ll 0$. A *projective resolution* of X is a quasi-isomorphism $\alpha : P \xrightarrow{\simeq} X$ where $P \in \mathcal{C}^P_\sqsupset(R)$. By [1, Remarks 1.7 1)], the projective resolution of X exists. Also, assume that Y is an R-complex such that $H_i(Y) = 0$ for all $i \gg 0$. An *injective resolution* of Y is a quasi-isomorphism $\alpha : Y \xrightarrow{\simeq} I$ where $I \in \mathcal{C}^I_\sqsubset(R)$. By [1, Remarks 1.7 2)], the injective resolution of Y exists.

3. The results

We start this section by recalling the following definition of Foxby [3, 6.37].

Definition 3.1. Let ξ and ξ' be two subcategories of $\mathcal{C}(R)$. A functor $T : \xi \longrightarrow \xi'$ is said to be *equivalence preserving* if

$$X \simeq Y \Longrightarrow T(X) \simeq T(Y)$$

for all $X, Y \in \xi$.

In the following examples, we provide some equivalence preserving functors.

Example 3.2. As for any two R-modules M and N, we have

$$M \simeq N \Longleftrightarrow M \cong N,$$

then any functor $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ is an equivalence preserving functor.

Example 3.3. The following functors are equivalence preserving:

i) $\mathrm{Hom}_R(P, -) : \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$ for any R-complex $P \in \mathcal{C}^P_\sqsupset(R)$,

ii) $\mathrm{Hom}_R(-, I) : \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$ for any R-complex $I \in \mathcal{C}^I_\sqsubset(R)$,

iii) $F \otimes_R - : \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$ for any R-complex $F \in \mathcal{C}^F_\sqsupset(R)$,

iv) $\mathrm{Hom}_R(-, W) : \mathcal{C}^P_\sqsupset(R) \longrightarrow \mathcal{C}(R)$ for any R-complex W,

v) $\mathrm{Hom}_R(U, -) : \mathcal{C}^I_\sqsubset(R) \longrightarrow \mathcal{C}(R)$ for any R-complex U,

vi) $V \otimes_R - : \mathcal{C}^F_\sqsupset(R) \longrightarrow \mathcal{C}(R)$ for any R-complex V.

Proof. See [3, 6.13, 6.15, 7.10, 6.17, 6.19, 7.14]. □

Now, we introduce weakly equivalence preserving functors.

Definition 3.4. Let ξ be a subcategory of $\mathcal{C}(R)$. A functor $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ is said to be *weakly equivalence preserving* on ξ if its induced functor $T : \xi \longrightarrow \mathcal{C}(R)$ is an equivalence preserving functor.

Example 3.3 gives us some weakly equivalence preserving functors, as follows.

Example 3.5.

i) The functor $\operatorname{Hom}_R(-, W)$ is a weakly equivalence preserving on $\mathcal{C}_{\sqsupset}^{P}(R)$ for any R-module W.

ii) The functor $\operatorname{Hom}_R(U, -)$ is a weakly equivalence preserving on $\mathcal{C}_{\sqsubset}^{I}(R)$ for any R-module U.

iii) The functor $V \otimes_R -$ is a weakly equivalence preserving on $\mathcal{C}_{\sqsupset}^{F}(R)$ for any R-module V.

We need the following lemma for proving our main result.

Lemma 3.6. *Let $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ be an additive functor. Any morphism $\alpha : X \to Y$ of R-complexes yields an isomorphism $\phi : \operatorname{Cone}(T(\alpha)) \to T(\operatorname{Cone}(\alpha))$ of R-complexes.*

Proof. We assume that T is covariant. The proof for the other case is similar. Assume that $\lambda_i^1 : Y_i \to \operatorname{Cone}(\alpha)_i$ and $\lambda_i^2 : X_i \to \operatorname{Cone}(\alpha)_{i+1}$ are the natural monomorphisms and $\pi_i^1 : \operatorname{Cone}(\alpha)_i \to Y_i$ and $\pi_i^2 : \operatorname{Cone}(\alpha)_{i+1} \to X_i$ are the natural epimorphisms for any integer $i \in \mathbb{Z}$. So, one has $\pi_i^1\lambda_i^1 = 1_{Y_i}$, $\pi_i^2\lambda_i^2 = 1_{X_i}$ and $\lambda_i^1\pi_i^1 + \lambda_{i-1}^2\pi_{i-1}^2 = 1_{\operatorname{Cone}(\alpha)_i}$. Define

$$\phi_i : T(Y_i) \oplus T(X_{i-1}) \to T(Y_i \oplus X_{i-1})$$

by

$$\phi_i(v, w) := T(\lambda_i^1)(v) + T(\lambda_{i-1}^2)(w)$$

for all $(v, w) \in T(Y_i) \oplus T(X_{i-1})$. We claim that

$$\phi := (\phi_i)_i : \operatorname{Cone}(T(\alpha)) \to T(\operatorname{Cone}(\alpha))$$

is an isomorphism of R-complexes. For each integer i, one can easily examine that

$$\lambda_{i-1}^1\alpha_{i-1} - \lambda_{i-2}^2\partial_{i-1}^X = \partial_i^{\operatorname{Cone}(\alpha)}\lambda_{i-1}^2$$

and $\lambda_{i-1}^1\partial_i^Y = \partial_i^{\operatorname{Cone}(\alpha)}\lambda_i^1$. Now, it is straightforward to check that the diagram

$$\begin{array}{ccc}
\operatorname{Cone}(T(\alpha))_i & \xrightarrow{\partial_i^{\operatorname{Cone}(T(\alpha))}} & \operatorname{Cone}(T(\alpha))_{i-1} \\
\downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i-1}} \\
T(\operatorname{Cone}(\alpha)_i) & \xrightarrow{T(\partial_i^{\operatorname{Cone}(\alpha)})} & T(\operatorname{Cone}(\alpha)_{i-1})
\end{array}$$

is commutative for all integers i, and so ϕ is a morphism of R-complexes. It remains to show that each ϕ_i is an isomorphism. To this end, we define

$$\psi_i : T(Y_i \oplus X_{i-1}) \to T(Y_i) \oplus T(X_{i-1})$$

by

$$\psi_i(u) := (T(\pi_i^1)(u), T(\pi_{i-1}^2)(u))$$

for any $u \in T(Y_i \oplus X_{i-1})$. One can see that $\psi_i\phi_i = 1_{\operatorname{Cone}(T(\alpha))_i}$ and $\phi_i\psi_i = 1_{T(\operatorname{Cone}(\alpha)_i)}$ for all integers i. ∎

Proposition 3.7. *Any covariant additive exact functor $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ is weakly equivalence preserving on $\mathcal{C}(R)$.*

Proof. Assume that $X \simeq Y$ for two R-complexes $X, Y \in \mathcal{C}(R)$. Then there exist a third R-complex Z and two quasi-isomorphisms, $X \overset{\alpha}{\longrightarrow} Z \overset{\beta}{\longleftarrow} Y$. By [2, Lemma A.1.19], Cone(α) and Cone(β) are exact R-complexes and so $T(\text{Cone}(\alpha))$ and $T(\text{Cone}(\beta))$ are exact R-complexes, too. Lemma 3.6 yields that $\text{Cone}(T(\alpha))$ and $\text{Cone}(T(\beta))$ are exact as well. Applying again [2, Lemma A.1.19] yields that $T(\alpha)$ and $T(\beta)$ are quasi-isomorphisms and so $T(X) \simeq T(Y)$. ∎

Next, we present the main result of this paper.

Theorem 3.8. *Let $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ be an additive covariant functor.*

i) *Assume that T is a right exact functor. Let Q be a bounded to the right R-complex of T-coacyclic R-modules and P be a projective resolution of Q. Then $T(Q) \simeq T(P)$.*

ii) *Assume that T is a left exact functor. Let E be a bounded to the left R-complex of T-acyclic R-modules and I be an injective resolution of E. Then $T(E) \simeq T(I)$.*

Proof. The proofs of i) and ii) are entirely similar, so we prove only part i). Set $s := \inf\{i \in \mathbb{Z} | Q_i \neq 0\}$ and $t := \inf\{i \in \mathbb{Z} | P_i \neq 0\}$. Set $u := \min\{t, s\}$. Without loss of generality, we may assume that $u = s$. As P is a projective resolution of Q, there exists a quasi-isomorphism $\alpha : P \longrightarrow Q$. Therefore, the following

$$\begin{aligned}\text{Cone}(\alpha) = \cdots \longrightarrow Q_i \oplus P_{i-1} &\overset{\partial_i^{\text{Cone}(\alpha)}}{\longrightarrow} Q_{i-1} \oplus P_{i-2} \overset{\partial_{i-1}^{\text{Cone}(\alpha)}}{\longrightarrow} \cdots \\ &\overset{\partial_{s+2}^{\text{Cone}(\alpha)}}{\longrightarrow} Q_{s+1} \oplus P_s \overset{\partial_{s+1}^{\text{Cone}(\alpha)}}{\longrightarrow} Q_s \longrightarrow 0,\end{aligned}$$

is an exact R-complex by [2, Lemma A.1.19]. Clearly, any projective R-module is T-coacyclic and the class of T-coacyclic R-modules is closed under finite direct sums. Because of this, the exact R-complex Cone(α) is a R-complex of T-coacyclic R-modules. Now, by splitting Cone(α) into short exact sequences, we have the following short exact sequences of T-coacyclic R-modules

$$0 \longrightarrow K_{s+1} \longrightarrow Q_{s+1} \oplus P_s \overset{\partial_{s+1}^{\text{Cone}(\alpha)}}{\longrightarrow} Q_s \longrightarrow 0,$$

and

$$0 \longrightarrow K_{i+1} \longrightarrow Q_{i+1} \oplus P_i \overset{\partial_{i+1}^{\text{Cone}(\alpha)}}{\longrightarrow} K_i \longrightarrow 0,$$

where $K_i = \ker(\partial_i^{\text{Cone}(\alpha)})$ for any $i > s$. By applying T over the above short exact sequences and using the assumption, we obtain the following exact sequences

$$0 \longrightarrow T(K_{s+1}) \longrightarrow T(Q_{s+1} \oplus P_s) \overset{T(\partial_{s+1}^{\text{Cone}(\alpha)})}{\longrightarrow} T(Q_s) \longrightarrow 0,$$

and

$$0 \longrightarrow T(K_{i+1}) \longrightarrow T(Q_{i+1} \oplus P_i) \overset{T(\partial_{i+1}^{\mathrm{Cone}(\alpha)})}{\longrightarrow} T(K_i) \longrightarrow 0.$$

By adjoining the above short exact sequences to each other, one can see that $T(\mathrm{Cone}(\alpha))$ is an exact R-complex. Hence, Lemma 3.6 yields that the R-complex $\mathrm{Cone}(T(\alpha))$ is also exact and so $T(\alpha) : T(P) \longrightarrow T(Q)$ is a quasi-isomorphism by [2, Lemma A.1.19] and so $T(P) \simeq T(Q)$. ∎

In the following corollary, we extend parts ii) and iii) of Example 3.5. Also, [6, Theorem 2.5] and [4, Corollary 4.7] are special cases of the following corollary that we mention them in Corollary 3.10.

Corollary 3.9. *Let $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ be an additive covariant functor.*

i) *Assume that T is a right exact functor. Then it is weakly equivalence preserving on $\mathcal{C}_{\sqsupset}^{T-coacy.}(R)$.*

ii) *Assume that T is a left exact functor. Then it is weakly equivalence preserving on $\mathcal{C}_{\sqsubset}^{T-acy.}(R)$.*

Proof. We only prove part i), because the proof of part ii) is similar. Assume that $Q \simeq Q'$ for any two R-complexes $Q, Q' \in \mathcal{C}_{\sqsupset}^{T-coacy.}(R)$. Let P be a projective resolution of Q. Then $P \simeq Q \simeq Q'$, and so P is a projective resolution of Q' by [1, 1.1.P and 1.4.P]. Hence, $T(Q') \simeq T(P) \simeq T(Q)$ by Theorem 3.8 i). ∎

As an application of our main theorem, we prove the completion (resp. section) functor is weakly equivalence preserving on the full subcategory of bounded to the right (resp. left) R-complexes of Gorenstein flat (resp. injective) R-modules. By using these, we give some methods for computing local (co)homology R-modules which are the main theorems of [6] and [4].

Before doing this, we recall that an R-module M is said to be *Gorenstein flat* if there exists an exact R-complex F of flat R-modules such that $M \cong \mathrm{im}(F_0 \longrightarrow F_{-1})$ and $F \otimes_R I$ is exact for all injective R-modules I. Any flat R-module is Gorenstein flat. Also, an R-module N is said to be *Gorenstein injective* if there exists an exact R-complex I of injective R-modules such that $N \cong \mathrm{im}(I_1 \longrightarrow I_0)$ and $\mathrm{Hom}_R(E, I)$ is exact for all injective R-modules E. Any injective R-module is Gorenstein injective.

Let $\mathcal{D}(R)$ denote the derived category of R-complexes. Assume that $\mathfrak{a}$ is an ideal of R. The left derived functor of $\mathfrak{a}$-adic completion functor, $\Lambda^{\mathfrak{a}}(-) = \varprojlim_n (R/\mathfrak{a}^n \otimes_R -)$, exists in $\mathcal{D}(R)$ and is denoted by $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$. For any R-complex X with $H_i(X) = 0$ for all $i \ll 0$, we have $\mathbf{L}\Lambda^{\mathfrak{a}}(X) := \Lambda^{\mathfrak{a}}(P)$, where P is a (every) projective resolution of X, see e.g. [5]. For any integer i, the i-th local homology module of X with respect to $\mathfrak{a}$ is defined by $H_i^{\mathfrak{a}}(X) := H_i(\mathbf{L}\Lambda^{\mathfrak{a}}(X))$. We recall that the $\mathfrak{a}$-section functor, $\Gamma_{\mathfrak{a}}(-) = \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{a}^n, -)$ is a left exact functor and its right derived functor exists in $\mathcal{D}(R)$ which is denoted by $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$. For any R-complex X with $H_i(X) = 0$ for all $i \gg 0$, we have $\mathbf{R}\Gamma_{\mathfrak{a}}(X) := \Gamma_{\mathfrak{a}}(I)$, where I is an (every) injective resolution of X, see e.g. [5]. For any integer i, the i-th local cohomology module of X with respect to $\mathfrak{a}$ is defined by $H_{\mathfrak{a}}^i(X) := H_{-i}(\mathbf{R}\Gamma_{\mathfrak{a}}(X))$.

Corollary 3.10. *Let $\mathfrak{a}$ be an ideal of the Noetherian ring R.*

i) $\Lambda^{\mathfrak{a}}(-)$ *is weakly equivalence preserving on the full subcategory of bounded to the right R-complexes of Gorenstein flat R-modules. Hence, for any R-complex X with $H_i(X) = 0$ for all $i \ll 0$ and any bounded to the right R-complex Q of Gorenstein flat R-modules such that $X \simeq Q$, we have $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \Lambda^{\mathfrak{a}}(Q)$, and so $H_i^{\mathfrak{a}}(X) \cong H_i(\Lambda^{\mathfrak{a}}(Q))$ for all $i \in \mathbb{Z}$.*

ii) $\Gamma_{\mathfrak{a}}(-)$ *is weakly equivalence preserving on the full subcategory of bounded to the left R-complexes of Gorenstein injective R-modules. Hence, for any R-complex X with $H_i(X) = 0$ for all $i \gg 0$ and any bounded to the left R-complex E of Gorenstein injective R-modules such that $X \simeq E$, we have $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \Gamma_{\mathfrak{a}}(E)$, and so $H^i_{\mathfrak{a}}(X) \cong H_{-i}(\Gamma_{\mathfrak{a}}(E))$ for all $i \in \mathbb{Z}$.*

Proof. i) Since for any integer i, $H_i(\mathbf{L}H_0^{\mathfrak{a}}(-)) \cong H_i(\mathbf{L}\Lambda^{\mathfrak{a}}(-))$ and any Gorenstein flat R-module is $\Lambda^{\mathfrak{a}}$-coacyclic by [6, Lemma 2.2 i)], then every Gorenstein flat R-module is $H_0^{\mathfrak{a}}$-coacyclic. Assume that X and Y are two bounded to the right R-complexes of Gorenstein flat R-modules such that $X \simeq Y$. Corollary 3.9 i) and [6, Lemma 2.2 ii)] imply that

$$\Lambda^{\mathfrak{a}}(X) \simeq H_0^{\mathfrak{a}}(X) \simeq H_0^{\mathfrak{a}}(Y) \simeq \Lambda^{\mathfrak{a}}(Y),$$

and so $\Lambda^{\mathfrak{a}}(-)$ is weakly equivalence preserving on the full subcategory of bounded to the right R-complexes of Gorenstein flat R-modules. Now, assume that X is an R-complex such that $H_i(X) = 0$ for all $i \ll 0$ and P is a projective resolution of X. Then $P \simeq Q$ and so the first part of our proof yields that $\Lambda^{\mathfrak{a}}(P) \simeq \Lambda^{\mathfrak{a}}(Q)$, as desired.

ii) By [7, Theorem 3.1], any Gorenstein injective R-module is $\Gamma_{\mathfrak{a}}$-acyclic. Note that in the statement of that result, the ring is assumed to be Gorenstein. But, the author has not used this assumption. So, the first part of our assertion follows from Corollary 3.9 ii). Now, assume that X is an R-complex such that $H_i(X) = 0$ for all $i \gg 0$ and I is an injective resolution of X. Then $I \simeq E$ and so $\Gamma_{\mathfrak{a}}(I) \simeq \Gamma_{\mathfrak{a}}(E)$ by Corollary 3.9 ii). ∎

In the next results, we present the analogues of Theorem 3.8 and Corollary 3.9 for additive contravariant functors. We leave their proofs to the reader.

Theorem 3.11. *Let $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ be an additive contravariant functor.*

i) *Assume that T is a right exact functor. Let E be a bounded to the left R-complex of T-acyclic R-modules and I be an injective resolution of E. Then $T(E) \simeq T(I)$.*

ii) *Assume that T is a left exact functor. Let Q be a bounded to the right R-complex of T-coacyclic R-modules and P be a projective resolution of Q. Then $T(Q) \simeq T(P)$.*

Part ii) of the following corollary extends Example 3.5 i).

Corollary 3.12. *Let $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ be an additive contravariant functor.*

i) *Assume that T is a right exact functor. Then it is weakly equivalence preserving on $\mathcal{C}_{\sqsubset}^{T-acy.}(R)$.*

ii) *Assume that T is a left exact functor. Then it is weakly equivalence preserving on $\mathcal{C}_{\sqsupset}^{T-coacy.}(R)$.*

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