ON p-NILPOTENCY OF FINITE GROUPS

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Abstract. A subgroup H of a finite group G is said to be weakly s-supplementently embedded subgroup in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{se}$, where H_{se} is an s-quasinormally embedded subgroup of G contained in H. In this paper we investigate the structure of G under the assumption that some subgroups of P are weakly s-supplementently embedded in G, and some new criteria are obtained.

Keywords: p-nilpotent group, Sylow subgroup, weakly s-supplementently embedded subgroup.

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1. Introduction

It is well known that the normalizer of Sylow subgroups of a group play an important role in the structure of groups. Let P be a Sylow subgroup of a group G. An interesting question is what one can say about G if some properties of the normalizer $N_G(P)$ of P are known. For example, the well known Burnside's Theorem asserts that if $N_G(P) = C_G(P)$, then G is p-nilpotent. Hall in [1] got the generalization of Burnside's theorem: if p'-elements of $N_G(P)$ are commute

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to the elements of P and the class size of P is less than p, then G is p-nilpotent. In [2], Wielandt showed that a group G is p-nilpotent if it has a regular Sylow p-subgroup whose G-normalizer is p-nilpotent. On the other hand, local analytic theory of groups is somehow substantial in studying the structure of finite groups and normalizers play a very important role in the local analysis theory. Therefore, it is of interest to study the structure of finite groups from properties of the normalizer of a Sylow subgroup.

Let G be a group and H a subgroup of G. H is said to be s-permutable (or s-quasinormal, π-quasinormal) in G if H permutes with every Sylow subgroup of G; H is called c-normal in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G; H is called weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H; H is said to be s-permutably embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G. More recently, authors in [3] introduced the following concept, which covers both weakly s-permutably embedded property and weakly s-permutability.

Definition 1.1. Let H be a subgroup of G. H is called a weakly s-supplemently embedded subgroup of G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{se}$, where H_{se} is an s-permutably embedded subgroup H_{se} of G contained in H.

Let G be a group, p a prime and P a Sylow p-subgroup of G, we introduce the two families of subgroups:

$\mathcal{H}(P) = \{H \leq P \mid P' \leq H \leq \Phi(P)\}$

$\mathcal{K}(P) = \{K \leq G \mid K$ is p-closed and the Sylow p-subgroup of K is contained in $\mathcal{H}(P)\}$.

It is obvious that $\mathcal{H}(P) \subseteq \mathcal{K}(P)$ and each element in $\mathcal{H}(P)$ is normal in P. Now, we consider the structure of G under the assumption that some subgroups of P are weakly s-supplemently embedded in G and some new criteria are obtained. For example, we prove the following results:

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$. Then G is p-nilpotent if and only if there exists $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in G and $N_G(P)$ is p-nilpotent.*

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups $\mathcal{U}$, and assume that G is a group with a normal subgroup E satisfies $G/E \in \mathcal{F}$. Suppose that for any prime p dividing $|E|$ and $P \in Syl_p(E)$, there exists $K \in \mathcal{K}(P)$ such that K is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$, then $G \in \mathcal{F}$.*

Theorem 3.6. *Let G be a group and P a Sylow p-subgroup of G, where p is a prime of $|G|$. Then G is p-nilpotent if and only if there exists a subgroup $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in G and $N_G(H)$ is p-nilpotent.*

All groups in this paper are finite. Let H be a subgroup of a group G, H_{se} denote an s-permutably embedded subgroup of G contained in H. The other notations and terminology are standard, as in [2].

2. Preliminaries

Lemma 2.1. ([4])

(a) *An s-permutable subgroup of G is subnormal in G.*

(b) *If $H \leq K \leq G$ and H is s-permutable in G, then H is s-permutable in K.*

(c) *Let $K \trianglelefteq G$. If H is s-permutable in G, then HK/K is s-permutable in G/K.*

(d) *If P is an s-permutable p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.2. ([5, Lemma 2.1]) *Suppose that U is s-permutably embedded in a group G, and that $H \leq G$ and $N \trianglelefteq G$.*

(a) *If $U \leq H$, then U is s-permutably embedded in H.*

(b) *UN is s-permutably embedded in G and UN/N is s-permutably embedded in G/N.*

Lemma 2.3. ([6, Lemma 2.5]) *Suppose that H is s-permutable in a group G, P a Sylow p-subgroup of H, where p is a prime. If either $H_G = 1$ or $P \leq O_p(G)$, then P is s-permutable in G.*

Lemma 2.4. ([3, Lemma 2.3]) *Let U be a weakly s-supplemently embedded subgroup of a group G and N a normal subgroup of G. Then:*

(a) *If $U \leq H \leq G$, then U is weakly s-supplemently embedded in H.*

(b) *If $N \leq U$, then U/N is weakly s-supplemently embedded in G/N.*

(c) *Let π be a set of primes, U a π-subgroup and N a π' -subgroup. Then UN/N is weakly s-supplemently embedded in G/N.*

Lemma 2.5. *Let P be a Sylow p-subgroup of a group G, H the normal Sylow p-subgroup of a subgroup K of G and $H \leq \Phi(P)$. If K is weakly s-supplemently embedded in G, then H is s-permutably embedded in G.*

Proof. By hypothesis, there is a subgroup A of G and an s-permutably embedded subgroup K_{se} of G contained in K such that $G = KA$ and $K \cap A \leq K_{se}$. Since $H \trianglelefteq K$, for P, there exists a Sylow p-subgroup P_1 of A such that $P = HP_1 \leq \Phi(P)P_1 \leq P_1$, so $H \leq P \leq A$ and then H is a Sylow p-subgroup of K_{se}. It follows from the definition of the s-permutably embedded subgroup that H is an s-permutably embedded subgroup of G. ∎

Lemma 2.6. ([7]) *If P is a Sylow p-subgroup of G and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.7. *Let $P/\Phi(P)$ be a minimal normal subgroup of a group $G/\Phi(P)$, where p is a prime divisor of $|G|$ and P a Sylow p-subgroup of G. If every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in G, then P is cyclic.*

Proof. Let P_1 be a proper subgroup of P. If P_1 is weakly s-supplemently embedded in G, we claim that $P_1 \leq \Phi(P)$. Let T be a supplement of P_1 in G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{se}$. Then $G = P_1T$ and $P = P \cap G = P \cap P_1T = P_1(P \cap T)$. Since $P/\Phi(P)$ is abelian, $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$, and hence $(P \cap T)\Phi(P) \trianglelefteq G$. Since $P/\Phi(P)$ be a minimal normal Sylow p-subgroup of $G/\Phi(P)$, $P \cap T \leq \Phi(P)$ or $P \cap T = P$. If $P \cap T \leq \Phi(P)$, then $P = P_1(P \cap T) = P_1$, a contradiction. Now assume that $P \cap T = P$. Then $P_1 \leq P_1 \cap T \leq (P_1)_{se} \leq O_p(G) = P$. Hence P_1 is s-permutable in G by Lemma 2.3. So $P_1\Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$ follows from Lemma 2.1(c) and so $N_{G/\Phi(P)}(P_1\Phi(P)/\Phi(P)) \geq O^p(G/\Phi(P))$. Note that $P/\Phi(P)$ is abelian, $P_1\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. By the minimal normality of $P/\Phi(P)$ in $G/\Phi(P)$ again, we have $P_1 \leq \Phi(P)$.

If every maximal subgroup of P is weakly s-supplemently embedded in G, then by the above argument P has a unique maximal subgroup, which implies that P is cyclic.

If every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in G, then we also have $|P/\Phi(P)| = p$ and then P is cyclic. Otherwise, let $K/\Phi(P)$ be any non-trivial cyclic subgroup of $P/\Phi(P)$. Let $x \in K \setminus \Phi(P)$ such that $T = \langle x\rangle\Phi(P)$. Then by the above argument, $\langle x\rangle \leq \Phi(P)$ and so $T = \Phi(P)$, a contradiction. This contradiction completes the proof the lemma. ∎

Lemma 2.8. *Let G be a group and P a normal Sylow p-subgroup of G, where $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in G.*

Proof. If G is p-nilpotent, then G has a normal p-complement T. Let P_1 be a subgroup of P. If P_1 is maximal in P, then P_1T is normal in G for $|G : P_1T| = p$. Hence P_1 is weakly s-supplemently embedded in G. Now assume that P_1 is a cyclic subgroup of P with prime order and order 4 . It follows that P_1 is a Sylow p-subgroup of P_1T. Let Q be a Sylow q-subgroup of G, where $q \neq p$ is a prime divisor of $|G|$. Then $Q \leq T$ and so $P_1TQ = QP_1T = P_1T$. By hypothesis, P is normal in G and so $P_1TP = PT$ is a subgroup of G, which implies that P_1T is an s-permutable subgroup of G and P_1 is s-permutably embedded in G. Hence P_1 is weakly s-supplemently embedded in G.

Conversely, assume that every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in G and G is non-p-nilpotent. Let G be a counterexample with minimal order. Let M be a proper subgroup of G.

Then $P \cap M$ is a normal Sylow p-subgroup of M. It follows from Lemma 2.4 that every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in M and so M is p-nilpotent. By [8, VI, Theorem 24.2], $P/\Phi(P)$ is a G-chief factor of P. Now by Lemma 2.7, P is cyclic and G is p-nilpotent follows from Burside's Theorem, a contradiction. Now assume that every maximal subgroup P_1 of P is weakly s-supplemently embedded in G and G is non-p-nilpotent. Let G be a counterexample with minimal order. Let N be a minimal normal subgroup of G contained in P. It is easy to see that G/N is p-nilpotent. By a routine argument, we have that $N = P$. Now by Lemma 2.7 again, we have that G is p-nilpotent, a contradiction. The lemma is proved. ∎

Lemma 2.9. *Let Q be a normal Sylow q-subgroup of a group G such that G/Q is supersolvable, where q is a prime divisor of $|G|$. If every maximal subgroup of Q or every minimal subgroup of Q with prime order and order 4 is weakly s-supplemently embedded in G, then G is supersolvable.*

Proof. Assume the result is false and let G be a counterexample with minimal order. By Lemma 2.4, it is easy to see that if every minimal subgroup of Q with prime order and order 4 is weakly s-supplemently embedded in G and G is non-supersolvable, then G is a minimal non-supersolvable, i.e., each proper subgroup of G is supersolvable and G is non-supersolvable. Now by [10], G has a normal Sylow p-subgroup P such that $G = PM$, where M is a supersolvable maximal subgroup of G and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. If $P \neq Q$, then $G \lesssim G/P \times G/Q$ is supersolvable, a contradiction. Hence $P = Q$. Now Q is cyclic by Lemma 2.7 and then G is supersolvable, a contradiction.

So we may assume that every maximal subgroup of Q is weakly s-supplemently embedded in G. Let N be a minimal normal subgroup of G contained in Q. Let Q_1/N be a maximal subgroup of Q/N, then Q_1 is maximal in Q and by hypothesis and Lemma 2.4, Q_1/N is weakly s-supplemently embedded in G/N. So G/N satisfies the hypothesis and by induction G/N is supersolvable. It follows that N is the unique minimal normal subgroup of G contained in Q and $N \nleq \Phi(G)$. Hence $N = Q$. Now by Lemma 2.7 again, we have that Q is cyclic and so G is supersolvable, a contradiction. This contradiction completes the proof. ∎

Lemma 2.10. *Let G be a group and $P \in Syl_p(G)$ where $p \in \pi(G)$. If P is abelian and $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

Proof. Since $N_G(P)$ is p-nilpotent, $N_G(P) = P \times H$, where H is a normal p-complement of P in $N_G(P)$, so $H \leq C_G(P)$. On the other hand, by the hypothesis, P is abelian, hence $P \leq C_G(P)$. Thus we have $N_G(P) = C_G(P)$. By the Burnside theorem, G is p-nilpotent. ∎

3. Main results

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$. Then G is p-nilpotent if and only if there exists $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in G and $N_G(P)$ is p-nilpotent.*

Proof. If G is p-nilpotent, then $N_G(P)$ is p-nilpotent and G has a normal p-complement T such that $G = PT$. It follows that $P'T$ is normal in G and P' is a Sylow p-subgroup of $P'T$, which implies that P' is a weakly s-supplemently embedded subgroup of G. Obviously $P' \in \mathcal{H}(P)$. Now we prove that if there exists $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in G and $N_G(P)$ is p-nilpotent, then G is p-nilpotent. Assume this is false and let G be a counterexample with minimal order. We derive a contradiction in several steps.

Step 1. H is a non-identity s-permutably embedded subgroup of G and G is not a non-abelian simple group.

By Lemma 2.5, H is an s-permutably embedded subgroup of G. If $H = 1$, then $P' \leq H = 1$ implies that P is abelian. Now by Lemma 2.10, G is p-nilpotent, a contradiction. Let A be an s-permutable subgroup of G such that H is a Sylow p-subgroup of A. So $A \neq 1$. Since $H < P$, $A < G$. Therefore A is a non-trivial abnormal subgroup of G, which implies that G is not a non-abelian simple group.

Step 2. G has a unique minimal normal subgroup N, G/N is p-nilpotent. Furthermore, $O_{p'}(G) = 1$ and $N \nleq \Phi(G)$.

Let N be a minimal normal subgroup of G, consider the quotient group $\overline{G} = G/N$. Then $\overline{P} = PN/N$ is a Sylow p-subgroup of $\overline{G}$. Certainly, $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ and $(\overline{P})' \leq \overline{P'} \leq \overline{H} \leq \overline{\Phi(P)} \leq \Phi(\overline{P})$. It follows that $(\overline{P})' \leq \overline{H} \leq \Phi(\overline{P})$. Hence $\overline{H} \in \mathcal{H}(\overline{P})$. By Step 1 and Lemma 2.2, it is easy to see that G/N satisfies the hypotheses, so by induction G/N is p-nilpotent. Obviously N is the unique minimal normal subgroup of N. Furthermore, $O_{p'}(G) = 1$ and $N \nleq \Phi(G)$.

Step 3. $O_p(G) = 1$.

Assume that $O_p(G) \neq 1$. Then $N \leq O_p(G)$ and by Step 2 $N \cap \Phi(G) = 1$, it follows that $O_p(G) \cap \Phi(G) = 1$. Now, by [9, Lemma 2.6] we have $N = O_p(G)$.

Now, we claim that $N \leq \Phi(P)$. Let A be an s-permutable subgroup of G such that H is a Sylow p-subgroup of A. If $A_G \neq 1$, then $O_p(G) = N \leq H \leq \Phi(P)$. If $A_G = 1$, then by Lemma 2.3 H is an s-permutable subgroup of G. It follows from Lemma 2.1(e) that $O^p(G) \leq N_G(H)$ and so $G = PO^p(G) \leq N_G(H)$, which implies that $H \trianglelefteq G$. Hence either $H = 1$ or $N \leq H$. If $H = 1$, then $P' = 1$ and so by Lemma 2.10 G is p-nilpotent, a contradiction. Hence $N \leq H$ and it follows that $N \leq \Phi(P)$. Assume that $G = \langle M, N \rangle$ for $M \subseteq G$. Then $G = \langle M \rangle N$. Since $N \leq P$, $P = P \cap N\langle M \rangle = N(P \cap \langle M \rangle) = P \cap \langle M \rangle$, so $P = P \cap \langle M \rangle$ and $G = \langle M \rangle$. Hence $N \leq \Phi(G)$, which contradicts Step 2. So $O_p(G) = 1$.

Step 4. N is p-nilpotent, the finial contradiction.

If $NP < G$, then NP satisfies the hypotheses and by induction NP is p-nilpotent. Therefore N is p-nilpotent, which contradicts Step 2 and Step 3. Hence $G = NP$.

By Step 1, H is a Sylow p-subgroup of an s-permutably embedded subgroup A of G. If $A_G = 1$, then by Lemma 2.3 H is s-permutable in G and so $H \leq O_p(G)$, which contradicts Step 3. So $A_G \neq 1$. It follows from the uniqueness of N that

$N \leq A_G \leq A$ and so $H \cap N$ is a Sylow p-subgroup of N. Noting that $P \cap N$ is also a Sylow p-subgroup of N and $H \cap N \leq P \cap N$, $P \cap N = H \cap N \leq H \leq \Phi(P)$. By Lemma 2.6, N is p-nilpotent, which contradicts Step 2 and Step 3. The finial contradiction completes the proof. ∎

Theorem 3.2. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if there exists $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$.*

Proof. In view of Lemma 2.8 and Theorem 3.1, the result is obvious. ∎

Theorem 3.3. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if there exists $K \in \mathcal{K}(P)$ such that K is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$.*

Proof. This theorem follows from Lemma 2.5 and Theorem 3.2. ∎

Theorem 3.4. *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. Suppose that E is a normal subgroup of G such that G/E is p-nilpotent. Let P be a Sylow p-subgroup of E. If there exists $K \in \mathcal{K}(P)$ such that K is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$, then G is p-nilpotent.*

Proof. Assume that the result is false and let G with subgroup E be a minimal counterexample to the theorem in respect to $|G| + |E|$. By Lemma 2.4 and Theorem 3.3, E is p-nilpotent. Let T be the normal p-complement of E, then $T \trianglelefteq G$. If $T \neq 1$, we consider G/T with subgroup E/T. It is easy to see that $E = PT$ and $(|P|, |T|) = 1$. With a similar argument as in Step 2 of Theorem 3.1, we know that the hypothesis is still true for G/T with subgroup E/T, hence the minimal choice of G implies that G/M is p-nilpotent. Thus G is p-nilpotent, a contradiction. So we may assume that $T = 1$, i.e., $E = P$ is a p-group. Let K/P be the normal p-complement of G/P, this makes sense as $G/P = G/E$ is p-nilpotent. It is clear that there exists $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in K and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_K(P)$, whence K is p-nilpotent by Theorem 3.3, so that $K_{p'} char T \trianglelefteq G$ yielding that $K_{p'}$ is also a normal Hall p'-subgroup of G, i.e., G is p-nilpotent, a contradiction. This contradiction completes the proof. ∎

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups $\mathcal{U}$, and assume that G is a group with a normal subgroup E satisfies $G/E \in \mathcal{F}$. Suppose that for any prime p dividing $|E|$ and $P \in Syl_p(E)$, there exists $K \in \mathcal{K}(P)$ such that K is weakly s-supplemently embedded in G and* *every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$, then $G \in \mathcal{F}$.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. By Theorem 3.3, G is a Sylow-tower group. Let $q = max\pi(G)$ and $Q \in Syl_q(G)$. Then $Q \trianglelefteq G$. By Lemmas 2.5 and 2.4, it is easy to see that G/Q satisfies the hypothesis and by induction G/Q is supersolvable. Now by Lemma 2.9, G is supersolvable. ∎

Theorem 3.6. *Let G be a group and P a Sylow p-subgroup of G, where p is a prime of $|G|$. Then G is p-nilpotent if and only if there exists a subgroup $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in G and $N_G(H)$ is p-nilpotent.*

Proof. We only need to prove the sufficient part. Assume it is false and let G be a counterexample with minimal order. Then

(1) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Consider $G/O_{p'}(G)$. Then by Lemma 2.4 and [11, Lemma 3.6.10], we have $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. The choice of G yields that $G/O_{p'}(G)$ is p-nilpotent, which implies that G is p-nilpotent, a contradiction.

(2) H is an s-permutably embedded subgroup of G. Let A be an s-permutable subgroup of G such that H is a Sylow p-subgroup of A. Then $A_G \neq 1$.

By Lemma 2.5, H is an s-permutably embedded subgroup of G. If $A_G = 1$, then H is s-permutable in G by Lemma 2.3 and so $O^p(G) \leq N_G(H)$. Notice that H is normal in P, $G = PO^p(G) \leq N_G(H)$ is p-nilpotent, a contradiction. Hence $A_G \neq 1$.

(3) G is p-nilpotent, the final contradiction.

Since H is a Sylow p-subgroup of HA_G, $N_{G/A_G}(HA_G/A_G) = N_G(H)A_G/A_G$ follows from [11, Lemma 3.6.10] and $HA_G/A_G \in \mathcal{H}(PA_G/A_G)$. It is easy to see that G/A_G satisfies the hypothesis of the theorem and by the minimal choice of G, we have that G/A_G is p-nilpotent.

Since $P \cap A_G = H \cap A_G \leq \Phi(P)$, A_G is p-nilpotent by Lemma 2.6. By (1), $A_G \leq H \leq \Phi(P)$ and so $A_G \leq \Phi(G)$, which implies that G is p-nilpotent, the final contradiction. ∎

From Theorem 3.6, we can obtain

Theorem 3.7. *Let G be a group. Then G is nilpotent if and only if there exists a subgroup $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in G and $N_G(H)$ is p-nilpotent for any $P \in Syl_p(G)$.*

4. Some applications

Let G be a group and P a Sylow p-subgroup of G, where $p \in \pi(G)$. If $N_G(P)$ is p-nilpotent, then it is easy to see that $P' \in Syl_p((N_G(P))')$ and $\Phi(P) \in Syl_p(\Phi(N_G(P)))$. So Theorem A has the following corollaries:

Corollary 4.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if P' is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$.*

Corollary 4.2. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if $\Phi(P)$ is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$.*

Corollary 4.3. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if $(N_G(P))'$ is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$.*

Corollary 4.4. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if $\Phi(N_G(P))$ is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$.*

Corollary 4.5. ([3, Theorem 3.1]) *Let G be a group and assume p is a prime dividing the order of G with $(|G|, p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly s-supplemently embedded in $N_G(P)$ and if P' is s-quasinormal in G, then G is p-nilpotent.*

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