A NUMERICAL SOLUTION OF THE ARBITRARY ORDER WEAKLY SINGULAR INTEGRAL USING BLOCK-PULSE FUNCTIONS AND APPLICATIONS

Hao Song[1]
Jun Huang
Mingxu Yi
Yalin Pan

School of Aeronautic Science and Technology
Beihang University
Beijing 100191
P.R.China

Abstract. The purpose of this paper is to obtain the approximation of the arbitrary order weakly singular integral using Block-Pulse functions. The obtained results can be used to solve the numerical solution of higher order linear and nonlinear weakly singular Volterra integral equation of the second kind. Furthermore, the initial equations are transformed into a system of algebraic equations. Finally, some examples are given to demonstrate the validity and applicability of this approach, results of these examples show that this new method is an efficient algorithm.

Keywords: weakly singular integral; weakly singular integral equation; Block Pulse functions; operational matrix; numerical solution.

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1. Introduction

Many engineering and scientific problems can be reduced into integral equations. These integral equations are always singular, some are even supersingular. Especially, all of the natural boundary integral equations can result to singular integral equations [10], [19], [18]. Due to the singularity of the kernel, many quadrature rules for the singular integrals are less accurate than their counterparts for Riemann integrals. The weakly singular integral in many equations can be observed [8], [16], [4], [9], [1], [11], [12]. The weakly singular Volterra integral equations are also found in a lot of physical, chemical, and biological problems. For example, reaction-diffusion problems, crystal growth etc [5], [7], [13]. Therefore, it is necessary to study the weakly singular integral.

[1]Corresponding author. E-mail: buaasonghao@163.com

As the Block-Pulse functions have many good properties [2], [14], they are used to get the quadrature formula of the arbitrary order weakly singular integral in this paper. The numerical examples show that the precision is very high. Therefore, this method and operational matrix of the Block-Pulse functions can be used to solve a class of higher order linear and nonlinear weakly singular Volterra integral equation. In this work, the operational matrix of the Block-Pulse functions are obtained using the operational matrix of Legendre wavelet. Finally, the equation is translated into a linear or nonlinear system of algebraic equations which are easier to get the solutions.

In this study, considering the following linear weakly singular Volterra integral equation of the second kind:

$$\sum_{i=0}^{n} a_i(t)y^{(i)}(t) + \lambda \int_0^t (t-s)^{-\alpha} y(s)ds = f(t). \tag{1.1}$$

where $a_i(t)$, $f(t)$ are continuous functions on $[0,1]$ and $y^{(i)}(t)$ stands for the i th-order derivative of $y(t)$. λ is a real constants.

The numerical solutions of the nonlinear weakly singular Volterra integral equation of the second kind can also got using this method:

$$\sum_{i=0}^{n} a_i(t)y^{(i)}(t) + \lambda \int_0^t (t-s)^{-\alpha} [y(s)]^p ds = f(t). \tag{1.2}$$

2. The quadrature formula of the arbitrary order weakly singular integral

The Block-Pulse functions and some properties are introduced in this part. The set of these functions, over interval $[0,T)$, is defined as [14]:

$$b_i(x) = \begin{cases} 1, & \frac{iT}{m} \le x < \frac{(i+1)T}{m}; \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

where $i = 0, 1, 2, \cdots m-1$, with a positive integer value for m. In this paper, it is assumed that $T = 1$. The useful properties of the Block-Pulse functions:

1. *Disjointness:*

$$b_i(x)b_j(x) = \begin{cases} b_i(x), & i = j; \\ 0, & i \ne j. \end{cases} \tag{2.2}$$

2. *Orthogonality:*

$$\int_0^1 b_i(x)b_j(x)dx = \begin{cases} 1/m, & i = j; \\ 0, & i \ne j. \end{cases} \tag{2.3}$$

3. *Completeness:* For any $f \in L^2([0,1])$, the sequence $\{b_i\}$ is complete if $\int b_i f = 0$ results in $f = 0$ almost everywhere. Because of completeness of $\{b_i(x)\}$, Parsevals identity holds, i.e. we have $\int_0^1 f^2(x)dx = \sum_{i=0}^{\infty} f_i^2 \|b_i(x)\|^2$, for every real bounded function $f(x) \in L^2([0,1])$ and

$$f_i = m \int_0^1 b_i(x) f(x) dx. \tag{2.4}$$

Arbitrary order weakly singular integral is shows as following formula:

$$I(t) = \int_0^t \frac{g(s)}{(t-s)^\alpha} ds, 0 \leq t \leq 1, \ 0 < \alpha < 1. \tag{2.5}$$

where $g(s) \in L^2([0,1])$.

From the orthogonality property of the Block-Pulse functions, it is possible to expand functions into their Block-Pulse series, so it can be written as:

$$g(s) \cong \sum_{i=0}^{m-1} c_i b_i(s) = c^T B_m(s). \tag{2.6}$$

where $c = (c_0, c_1, \cdots, c_{m-1})^T$, $B_m(s) = (b_0(s), b_1(s), \cdots, b_{m-1}(s))^T$.

Since equation (2.6) is substituted into equation (2.5), then, we have:

$$I(t) = \int_0^t \frac{g(s)}{(t-s)^\alpha} ds = c^T \int_0^t \frac{B_m(s)}{(t-s)^\alpha} ds = c^T D(t) \tag{2.7}$$

where

$$D(t) = \int_0^t \frac{B_m(s)}{(t-s)^\alpha} ds. \tag{2.8}$$

Combining equation (2.1) and equation (2.8), we can obtain

$$\begin{aligned} (2.9)\quad D(t) &= \int_0^t \frac{B_m(s)}{(t-s)^\alpha} ds \\ &= \int_0^{1/m} \frac{B_m(s)}{(t-s)^\alpha} ds + \int_{1/m}^{2/m} \frac{B_m(s)}{(t-s)^\alpha} ds + \cdots + \int_{i/m}^{t} \frac{B_m(s)}{(t-s)^\alpha} ds \\ &= \begin{pmatrix} -\frac{(t-1/m)^{1-\alpha} - t^{1-\alpha}}{1-\alpha} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{(t-2/m)^{1-\alpha} - (t-1/m)^{1-\alpha}}{1-\alpha} \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{(t-i/m)^{1-\alpha}}{1-\alpha} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= \left(-\frac{(t-\frac{1}{m})^{1-\alpha} - t^{1-\alpha}}{1-\alpha} - \frac{(t-\frac{2}{m})^{1-\alpha} - (t-\frac{1}{m})^{1-\alpha} 1-\alpha}{\cdots} \frac{(t-\frac{i}{m})^{1-\alpha} 1-\alpha}{0} \cdots 0 \right)^T \end{aligned}$$

where $D(0) = (0, 0, ..., 0)^T$.

Let $t = k/m,\ k \in 1, 2, ..., m-1$. The following results can be obtained by using equation (2.10)

$$D\left(\frac{k}{m}\right) = \left(-\frac{(\frac{k}{m}-\frac{1}{m})^{1-\alpha}-(\frac{k}{m})^{1-\alpha}}{1-\alpha}, -\frac{(\frac{k}{m}-\frac{2}{m})^{1-\alpha}-(\frac{k}{m}-\frac{1}{m})^{1-\alpha}}{1-\alpha}, ..., \frac{(\frac{k}{m}-\frac{i}{m})^{1-\alpha}}{1-\alpha}, 0, ..., 0\right)^T.$$

At this time, $i = k-1$.

Then, combining equation (2.7) and equation (2.10), the numerical solution of equation (2.5) can be obtained.

3. Applied method

Consider the following linear weakly singular Volterra integral equation:

$$\sum_{i=0}^{n} a_i(t)y^{(i)}(t) + \lambda \int_0^t (t-s)^{-\alpha} y(s)ds = f(t), \tag{3.1}$$

under the initial conditions

$$y^{(n-1)}(0) = y_{n-1}, ..., y(0) = y_0,\ 0 \le t \le 1, 0 < \alpha < 1. \tag{3.2}$$

where $a_i(t)$, $f(t)$ are continuous functions on $[0, 1]$ and $y^{(i)}(t) \in L^2([0, 1])$ stands for the i th-order derivative of $y(t)$. λ and $y_k (k = 0, 1, 2, ..., n-1)$ are real constants.

Before solving equation (3.1), the operational matrix of Block-Pulse functions can be given using the operational matrix of Legendre wavelet.

It is generally known that the Legendre wavelet in the interval $[0, 1)$ can be defined as [15]:

$$\psi_{nm}^{(k)}(x) = \begin{cases} \sqrt{2m+1}2^{\frac{k}{2}} P_m(2^{k+1}x - 2n + 1), & x \in [\frac{n-1}{2^k}, \frac{n}{2^k}); \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

P_m is said Legendre polynomial.

Set P is the Legendre wavelet operational matrix of integration, where

$$P = \frac{1}{2^k}\begin{bmatrix} L & F & F & \cdots & F \\ O & L & F & \cdots & F \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & O & O & L & F \\ O & O & O & O & L \end{bmatrix}_{m\times m},$$

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{M\times M}$$

and

$$L=\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{3\sqrt{5}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{7}}{5\sqrt{5}} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-5}} & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{\sqrt{2M-1}}{(2M-3)\sqrt{2M-5}} & 0 \end{bmatrix}_{M\times M}$$

where $m = 2^{k-1}M$.

Let $t_i = \frac{i-\frac{1}{2}}{2^{k-1}M}$, $i = 1, 2, ..., 2^{k-1}M$, the Legendre wavelet matrix [14] can be obtained:

$$\Phi_{m\times m} = [\Psi(t_1), \Psi(t_2), ..., \Psi(t_{2^{k-1}M})] \tag{3.4}$$

where

$$\Psi(t) = [\psi^{(k)}_{1,0}(t), ..., \psi^{(k)}_{1,M-1}(t), ..., \psi^{(k)}_{2^{k-1},0}(t), ..., \psi^{(k)}_{2^{k-1},M-1}(t)]^T.$$

There is a relation between the Block-Pulse functions and Legendre wavelet, we have found:

$$\Psi(t) = \Phi_{m\times m} B_m(t). \tag{3.5}$$

From Equation (3.5), we have

$$\begin{aligned}\int_0^t B_m(s)ds &= \int_0^t \Phi^{-1}_{m\times m}\Psi(s)ds = \Phi^{-1}_{m\times m}\int_0^t \Psi(s)ds \\ &= \Phi^{-1}_{m\times m}P\Psi(t) = \Phi^{-1}_{m\times m}P\Phi_{m\times m}B_m(t).\end{aligned} \tag{3.6}$$

Let $Q = \Phi^{-1}_{m\times m}P\Phi_{m\times m}$, Q is called the Block-Pulse operational matrix of integration, namely

$$\int_0^t B_m(s)ds = QB_m(t). \tag{3.7}$$

Since $y^{(n)}(t) \in L^2([0, 1])$ it is supposed that

$$y^{(n)}(t) \cong \sum_{i=0}^{m-1} d_i b_i(t) = d^T B_m(t). \tag{3.8}$$

Then

$$\begin{aligned}y^{(n-1)}(t) &= \int_0^t y^{(n)}(s)ds + y^{(n-1)}(0) = d^T QB_m(t) + y^{(n-1)}(0) \\ &= [d^T Q + A^T y^{(n-1)}(0)]B_m(t)\end{aligned} \tag{3.9}$$

$$y^{(n-2)}(t) = \int_0^t y^{(n-1)}(s)ds + y^{(n-2)}(0) \tag{3.10}$$
$$= [d^TQ^2 + A^Ty^{(n-1)}(0)Q + A^Ty^{(n-1)}(0)]B_m(t)$$
$$\vdots$$
$$y(t) = \int_0^t y'(s)ds + y(0) = [d^TQ^n + A^Ty^{(n-1)}(0)Q^{n-1} \tag{3.11}$$
$$+ \cdots + A^Ty'(0)Q + A^Ty(0)]B_m(t),$$

and, $\forall i \in \{0, 1, 2, ..., n\}$,

$$y^{(i)}(t) = [d^TQ^{n-i} + A^Ty^{(n-1)}(0)Q^{n-i-1} + \cdots + A^Ty^{(i)}(0)]B_m(t) \tag{3.12}$$

where $A = m\int_0^1 B_m(t)dt$.

Substituting equation (3.12), equation (3.11) and equation (2.7) into equation (3.1), we have

$$\begin{aligned} &d^T\left(\sum_{i=0}^{n} a_i(t)Q^{n-i}B_m(t) + \lambda Q^nD(t)\right) \\ &= f(t) - \sum_{i=0}^{n} a_i(t)(A^Ty^{(n-1)}(0)Q^{n-i-1} + \cdots + A^Ty^{(i)}(0))B_m(t) \\ &- \lambda(A^Ty^{(n-1)}(0)Q^{n-1} + \cdots + A^Ty(0))D(t) \end{aligned} \tag{3.13}$$

Discreting equation (3.13) by taking step $\Delta = \frac{1}{m}$ of t, a linear system of algebraic equations can be easily obtained. Then d^T can be obtained. $y(t)$ can be obtained by using equation (3.11).

Consider the following nonlinear weakly singular Volterra integral equation:

$$\sum_{i=0}^{n} a_i(t)y^{(i)}(t) + \lambda\int_0^t (t-s)^{-\alpha}[y(s)]^p ds = f(t) \tag{3.14}$$

under the initial conditions equation (3.2).

Let $\beta^T = d^TQ^n + A^Ty^{(n-1)}(0)Q^{n-1} + \cdots + A^Ty'(0)Q + A^Ty(0)$, namely

$$\beta = (\beta_0, \beta_1, \cdots, \beta_{m-1})^T.$$

Equation (3.11) can be translated into:

$$f(t) = \beta^TB_m(t). \tag{3.15}$$

According to the properties of the Block-Pulse functions, we have

$$[f(t)]^p = [\beta^p]^TB_m(t) \tag{3.16}$$

where $\beta^p = (\beta_0^p, \beta_1^p, \cdots, \beta_{m-1}^p)^T$.

Substituting equation (3.12), equation (3.16) and equation (2.7) into equation (3.14), we have:

$$\beta^T \sum_{i=0}^{n} a_i(t) Q^{-i} B_m(t) + \lambda[\beta^p]^T D(t) = f(t) + \sum_{i=0}^{n} a_i(t)(A^T y^{(i-1)} Q^{-1} + \cdots + A^T y(0) Q^{-i}) B_m(t) \tag{3.17}$$

when $p = 1$, equation (3.17) is equation (3.13).

Discreting equation (3.17) by taking step $\Delta = \frac{1}{m}$ of t, a linear system of algebraic equations can be easily obtained. Then β can be obtained. $y(t)$ can be obtained by using equation (3.15).

4. Numerical examples

Example 1. Consider the weakly singular integral [17]:

$$I_1(t) = \int_0^t \frac{s^n}{\sqrt{t-s}} ds \tag{3.18}$$

The exact solution is

$$\frac{\sqrt{\pi} t^{(\frac{1}{2}+n)} \Gamma(n+1)}{\Gamma(n+\frac{3}{2})}.$$

Taking $m = 16$, $m = 32$, and making use of MATLAB2011a, Fig.1 and Fig.2 are comparison of the approximations with the exact.

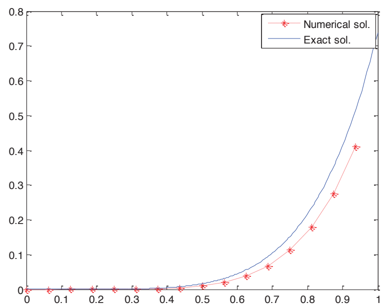

Figure 1: $m = 16$, $n = 5$

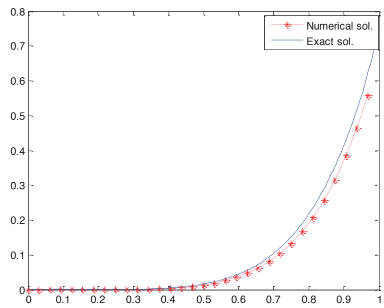

Figure 2: $m = 16$, $n = 5$

Example 2. Consider the weakly singular integral [6]:

$$I_2(t) = \int_0^t \frac{y(s)}{\sqrt{t-s}} ds, \ 0 \le t < 1, \tag{3.19}$$

where

$$y(s) = \frac{2^{2r-1}}{\pi} r \frac{(\Gamma(r))^2}{\Gamma(2r)} s^{r-1/2}.$$

The exact solution is x^r. Making use of MATLAB2011a, Fig.3 and Fig.4 are comparison of the approximations solution with the exact in this paper.

From the above results, the approximations are in good agreement with exact solution, and the value of is bigger, the precision is higher.

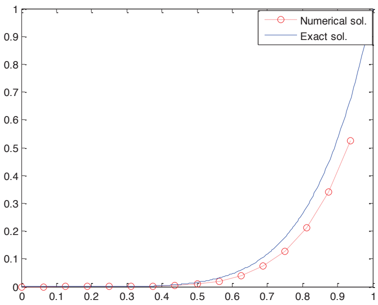

Figure 3: $m = 16$, $r = 6$

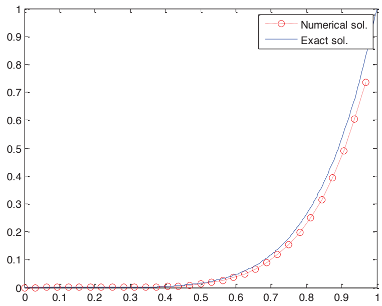

Figure 4: $m = 32$, $r = 6$

Example 3. Consider the weakly singular Volterra integral equation [3]:

$$y(t) + \int_0^t y(s)(t-s)^{-1/2} ds = \frac{1}{2}\pi t + \sqrt{t}, \ 0 \leq t < 1. \tag{3.20}$$

The exact solution is $\sqrt{t}$. Table 1 can be obtained through taking $m = 16, 32, 64$, and applying the above method in solving numerical solution of this problem.

Table 1: The exact solution and absolute error for different of m.

t	$m = 16$	$m = 32$	$m = 64$	$m = 128$	Exact solution
0	0.0000	0.0000	0.0000	0.0000	0.0000
1/8	2.23e-002	1.66e-002	7.54e-003	4.24e-004	0.3536
2/8	2.58e-002	1.28e-002	6.89e-003	4.78e-004	0.5000
3/8	2.25e-002	1.09e-002	6.03e-003	6.34e-004	0.6124
4/8	1.99e-002	9.71e-003	9.25e-004	8.69e-005	0.7071
5/8	1.81e-002	8.82e-003	8.37e-004	7.36e-005	0.7906
6/8	1.67e-002	8.22e-003	8.04e-004	6.85e-005	0.8660
7/8	1.56e-002	7.62e-003	7.57e-004	5.33e-005	0.9354

Example 4. Consider the weakly singular Volterra integral equation [3]:

$$y(t) + \int_0^t y(s)(t-s)^{-1/2} ds = 1, \ 0 \leq t < 1. \tag{3.21}$$

The exact solution is $y(t) = \exp(\pi t) erfc(\sqrt{\pi t})$. The numerical solution and exact solution, shown in Table 2, can be obtained through taking $m = 16, 32, 64$, and applying the above method in solving numerical solution of this problem by MATLAB2011a.

As can be seen in Table 1 and Table 2, a good approximation with the exact solution using above method is achieved. Besides, with the increase of m, the errors become smaller and smaller.

Table 2: The numerical solution and exact solution for different of m.

t	$m = 16$	$m = 32$	$m = 64$	$m = 128$	Exact solution
0	1.0000	1.0000	1.0000	1.0000	1.0000
1/8	0.5429	0.5389	0.5558	0.5560	0.5561
2/8	0.4411	0.4504	0.4605	0.4607	0.4608
3/8	0.3903	0.3980	0.4052	0.4052	0.4053
4/8	0.3556	0.3615	0.3670	0.3671	0.3671
5/8	0.3292	0.3340	0.3385	0.3385	0.3385
6/8	0.3083	0.3122	0.3160	0.3130	0.3160
7/8	0.2910	0.2943	0.2975	0.2975	0.2975

5. Conclusion

This paper uses the Block-Pulse functions and their good properties to solve the arbitrary order weakly singular integral. The numerical solution of higher order linear and nonlinear weakly singular Volterra integral equation of the second kind can be obtained using this method and the Block-Pulse functions operational matrix of integration. Furthermore, this equation is transformed into a system of algebraic equations which is easily to be solved. Numerical examples show that this new method is an efficient algorithm.

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