A NEW CORRECTOR-PREDICTOR ALGORITHM FOR CONVEX QUADRATIC SEMIDEFINITE OPTIMIZATION

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Abstract. In this paper, we propose a new corrector-predictor algorithm for convex quadratic semidefinite optimization problem based on a new proximity measure. The search direction is obtained by an equivalent algebraic transformation of the centering equation. At each iteration, the algorithm is composed of a corrector step and a predictor step. The predictor step uses line search schemes requiring the reduction of the duality gap, while the corrector step is used to restore the iterates to the neighborhood of the central path. Finally, the algorithm has the currently best-known iteration complexity.

Keywords: convex quadratic semidefinite optimization; corrector-predictor algorithm; iteration complexity.

1. Introduction

In this paper, we consider the standard form of convex quadratic semidefinite optimization (CQSDO) problem

$$
\begin{aligned}
(P) \qquad & \min\, C \bullet X + \frac{1}{2} X \bullet \Omega(X) \\
& A_i \bullet X = b_i,\ i = 1, 2, \ldots, m, \\
& X \succeq 0,
\end{aligned}
$$

and its dual problem

$$
\begin{aligned}
(D) \qquad & \max\, b^T y - \frac{1}{2} X \bullet \Omega(X) \\
& \sum_{i=1}^{m} y_i A_i - \Omega(X) + S = C, \\
& X \succeq 0,\ S \succeq 0,
\end{aligned}
$$

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$A_i \in \mathbf{S}^n$, $i = 1, 2, \ldots, m$, $b \in \mathbb{R}^m$, $C \in \mathbf{S}^n$ and $\Omega(X) : \mathbf{S}^n \longrightarrow \mathbf{S}^n$ is a self-adjoint semidefinite linear operator, i.e., for any $A, B \in \mathbf{S}^n$, we have $\Omega(A) \bullet B = A \bullet \Omega(B)$ and $\Omega(A) \bullet A \geq 0$. Here, $\mathbf{S}^n$ denotes the vector space of $n \times n$ real symmetric matrices endowed with the inner product $A \bullet B := \mathrm{Tr}(AB)$. The sign $X \succeq 0$ ($X \succ 0$) means that matrix X is positive semidefinite (positive definite). To simplify analysis, as in [23] and [25], we will restrict ourselves to the following special case

$$\Omega(X) = \sum_{i=1}^{l} H_i^T X H_i.$$

Here, $H_i \in \mathbb{R}^{n \times n}$ is matrix and l is an integer not greater than $l \leq n^2$. It is easy to verity that $\Omega(X)$ satisfies the following conditions

$$\Omega(X) = \Omega(X)^T, \quad \Omega(X) \bullet X \geq 0, \quad X \in \mathbb{R}^{n \times n}.$$

Remark 1. If $\Omega(X) = 0$, then CQSDO problem reduces to semidefinite optimization (SDO) problem.

CQSDO problem is an extension of SDO problem and convex quadratic optimization (CQO) problem, CQSDO problem was first proposed in the researches of Kojima et al. [10]. And they also proved that the CQSDO problem can be transformation as semidefinite linear complementarity problem (SDLCP) in [11]. Additionally, the CQSDO problem has some important applications, such as the nearest Euclidean distance matrix problem [1], the nearest correlation matrix problem [20]. During the past years, there are many efficient interior-point algorithms have been proposed to solving CQSDO problem. Nie et al. [16] proposed a potential reduction algorithm for solving CQSDO problem and obtained the iteration bound as $O(\sqrt{n} \log \frac{n}{\varepsilon})$. In other article [17], they developed a predictor-corrector algorithm for CQSDO problem by using Dikin-type and Newton centering steps. For their computation they cited the conjugate gradient method. Toh [21] proposed an inexact primal-dual path-following algorithm for CQSDO problem. By numerical experiment, he also shown that the proposed algorithm is efficient and robust. Wang et al. [23] presented a new primal-dual interior point algorithm for CQSDO problem based on a parametric kernel function for large-and small-update methods. Based on a different technique for finding search direction, a new small-update interior point algorithm is proposed by Bai et al. in [4]. Later on, Lin [12] proposed an inexact spectral bundle method for CQSDO problem, where at each iteration of the corresponding algorithm, an eigenvalue problem is inexactly. Zhang et al. [24] presented a full-step interior point algorithm for CQSDO problem based on a simple univariate kernel function. At almost the same time, a large-update interior point algorithm was proposed for CQSDO problem based on a new nonself-regular function by Zhang in [25]. Kheirfam et al. [9] suggested a large-update feasible interior-point algorithm for CQSDO problem based on a parametric kernel function, and obtained the best-known iteration complexity. Finally, Mohamed et al. [13] proposed a full-NT step feasible primal-dual path-following interior point algorithm for CQSDO problem.

Recently, Potra [19] proposed two corrector-predictor interior point algorithms for solving monotone linear complementarity problem (MLCP) based on the negative infinity norm neighborhood, and shown that algorithms have $O(\sqrt{n}L)$ iteration complexity. Gurtuna et al. [8] proposed a corrector-predictor method for solving sufficient LCP that does not depend on the handicap of the matrix so that it can be applied for any sufficient LCP. Very recently, Kheirfam [6] proposed a corrector-predictor path-following algorithm for SDO problem, and presented the iteration bound as $O(\sqrt{n}\log\frac{n}{\varepsilon})$ with small-update methods. Subsequently, he generated this algorithm to convex quadratic symmetric cone optimization (CQSCO) problem [5].

The aim of this paper is to propose a corrector-predictor algorithm for CQSDO problem based on a new proximity measure. The algorithm uses corrector and predictor steps. In the corrector step, we use full-step which has the advantage that no line searches are needed, the purpose of corrector step is to restore to the neighborhood of the central path. While in the predictor step, the algorithm operates one damped NT-step, which is used to reduce the duality gap. Finally, by using new tools, the favorable iteration complexity is reported.

The remainder paper is organized as follows: In Section 2, we recall some basic concepts regards the central path and introduce the new direction. In Section 3, we present a corrector-predictor algorithm for CQSDO problem. Analysis of the algorithm is presented in Section 4. In Section 5, we derive the iteration complexity. Finally, some concluding remarks follow in Section 6.

2. Preliminaries

2.1. The central path

It is common in interior point algorithms theory to assume that (P) and (D) satisfy the interior-point condition (IPC), i.e., there exists (X^0, y^0, S^0) such that

$$A_i \bullet X^0 = b_i, \quad X^0 \succ 0, \quad i = 1, 2, ..., m, \quad \sum_{i=1}^{m} y_i^0 A_i - \Omega(X^0) + S^0 = C, \quad S^0 \succ 0,$$

and the matrices A_i, $i = 1, 2, ..., m$ are linearly independent, the same assumptions can be found in [9], [12], [13], [23], [24], [25]. Under the assumption of IPC, finding an optimal solution of (P) and (D) is equivalent to solving the following system

$$\begin{aligned} & A_i \bullet X = b_i, \quad i = 1, 2, ..., m, \quad X \succeq 0, \\ & \sum_{i=1}^{m} y_i A_i - \Omega(X) + S = C, \quad S \succeq 0, \\ & XS = 0. \end{aligned} \tag{1}$$

The basic idea of primal-dual interior point algorithms is to replace the third equation in the system (1), the so-called complementarity condition for (P) and (D), by the parameterized equation $XS = \mu E$, E denotes the $n \times n$ identity matrix.

Then, we consider the following system

$$(2)\qquad \begin{aligned} &A_i \bullet X = b_i, \quad i = 1, 2, ..., m, \quad X \succ 0, \\ &\sum_{i=1}^{m} y_i A_i - \Omega(X) + S = C, \quad S \succ 0, \\ &XS = \mu E, \end{aligned}$$

where $\mu > 0$. Since the matrices A_i are linearly independent and the IPC holds, the parameterized system (2) has a unique solution, denoted as $(X(\mu), y(\mu), S(\mu))$ for each $\mu > 0$. We call $X(\mu)$ the μ-center of (P) and $(y(\mu), S(\mu))$ the μ-center of (D). The set of μ-centers (with μ running through all the positive real numbers) gives a homotopy path, which is called the central path of (P) and (D). If μ goes to zero, then the limit of the central path exists and since the limit points satisfy the complementarity condition, the limit yields an optimal solution for (P) and (D) (see [23]).

2.2. The new search direction

Here, we introduce the new search direction for CQSDO problem. In [22], Wang et al. presented a new technique for finding a class of search directions for SDO problem. He replaces the standard centering equation $XS = \mu E$ with

$$\psi(\frac{XS}{\mu}) = \psi(E),$$

where

$$\psi \in C^1, \quad \psi : \mathbb{R}^+ \to \mathbb{R}^+,$$

and require that the inverse function ψ^{-1} exists. Similar to the SDO problem case, if we replace the standard centering equation $XS = \mu E$ with $\psi(\frac{XS}{\mu}) = \psi(E)$, then system (2) can be written the following equivalent form

$$(3)\qquad \begin{aligned} &A_i \bullet X = b_i, \quad i = 1, 2, ..., m, \quad X \succ 0, \\ &\sum_{i=1}^{m} y_i A_i - \Omega(X) + S = C, \quad S \succ 0, \\ &\psi\Big(\frac{XS}{\mu}\Big) = \psi(E). \end{aligned}$$

Applying Newton's method to system (3) and neglecting the term $\Delta X \Delta S$, we have

$$(4)\qquad \begin{aligned} &A_i \bullet \Delta X = 0, \quad i = 1, 2, ..., m, \\ &\sum_{i=1}^{m} \Delta y_i A_i - \Omega(\Delta X) + \Delta S = 0, \\ &\Delta X + X \Delta S S^{-1} = \mu \Big(\psi'\Big(\frac{XS}{\mu}\Big)\Big)^{-1} \Big(\psi(E) - \psi\Big(\frac{XS}{\mu}\Big)\Big) S^{-1}. \end{aligned}$$

A well-known problem with the above Newton system is that ΔX is not necessarily symmetric. There are many ways are suggested for symmetrizing the third equation in system (4) such that the new system has an unique symmetric solution. In this paper, we use the Nesterov Todd (NT) symmetrization scheme in [14], [15]. Let us define

$$P := X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}} = S^{-\frac{1}{2}}(S^{\frac{1}{2}}XS^{\frac{1}{2}})^{\frac{1}{2}}S^{-\frac{1}{2}}. \tag{5}$$

Then, system (4) is replaced by the system

$$\begin{aligned} &A_i \bullet \triangle X = 0, \quad i = 1, 2, ..., m, \\ &\sum_{i=1}^{m} \triangle y_i A_i - \Omega(\Delta X) + \triangle S = 0, \\ &\triangle X + P\triangle SP^T = \mu S^{-1} - X. \end{aligned} \tag{6}$$

Now, we can conclude that system (6) has a unique symmetric solution. And also define $D := P^{\frac{1}{2}}$, where for any symmetric position matrix G, the exponent $G^{\frac{1}{2}}$ denotes its symmetric square root, the role of matrix D is to scale both matrices X and S to the same matrix V by

$$V := \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} DSD. \tag{7}$$

Note that the matrices D and V are symmetric and positive definite. Moreover, we have

$$V^2 := \frac{1}{\mu} D^{-1} XSD. \tag{8}$$

Let us further define

$$\begin{aligned} \bar{A}_i &:= \frac{1}{\sqrt{\mu}} DA_iD, \quad i = 1, 2, ..., m, \\ D_X &:= \frac{1}{\sqrt{\mu}} D^{-1}\triangle X D^{-1}, \\ D_S &:= \frac{1}{\sqrt{\mu}} D\triangle SD. \end{aligned} \tag{9}$$

Consequently, system (6) reduces to

$$\begin{aligned} &\bar{A}_i \bullet D_X = 0, \quad i = 1, 2, ..., m, \\ &\sum_{i=1}^{m} \triangle y_i \bar{A}_i - \bar{\Omega}(D_X) + D_S = 0, \\ &D_X + D_S := P_V, \end{aligned} \tag{10}$$

where

$$\begin{aligned} &\bar{\Omega}(D_X) := \sum_{i=1}^{l} DH_i^T DD_X DD_i D, \\ &P_V = \sqrt{\mu} D^{-1}(D\psi'(V^2)D^{-1})^{-1}(\psi(E) - D\psi(V^2)D^{-1})S^{-1}D^{-1}. \end{aligned}$$

Note that, if we choose a different function $\psi(t)$, then it results a different P_V, such as

- $\psi(t) = t$ yields $P_V = V^{-1} - V$, which gives the classical search direction (see [2], [3], [7], [18]);
- $\psi(t) = t^2$ yields $P_V = \frac{1}{2}(V^{-3} - V)$ (see [2], [3], [18]);
- $\psi(t) = t^{\frac{q+1}{2}}$, $q \geq 1$ yields $P_V = \frac{2}{q+1}V^{-q} - V$ (see [2], [3], [18]).

In this paper, we restrict the analysis to the case where $\psi(t) = \sqrt{t}$. This yields

$$P_V := 2(E - V).$$

For the analysis of the algorithm, we also define a new norm-based proximity measure $\delta(X, S; \mu)$ as follows

$$\delta(X, S; \mu) := \delta(V) := \frac{1}{2}\|P_V\|_F = \|E - V\|_F.$$

By using the first two equations of the system (10) and $\Omega(X)$ is a self-adjoint positive semidefinite linear operation, we have

$$D_X \bullet D_S = D_S \bullet D_X = \frac{1}{\mu}\Delta X \Omega(\Delta X) \geq 0. \tag{11}$$

Remark 2. Note that, for the SDO problem case [6], [22], we have $D_X \bullet D_S = 0$. It is the main difference between the CQSDO problem and the SDO problem in the analysis. This makes the analysis different.

3. The corrector-predictor algorithm

In this section, we propose a corrector-predictor algorithm for CQSDO problem. Firstly, we define a τ-neighborhood of the central path as follows

$$\mathcal{N}(\tau) = \Big\{(X, S) : A_i \bullet X = b_i,\ i = 1, 2, ..., m,\ X \succeq 0; \sum_{i=1}^{m} y_i A_i - \Omega(X) + S = C, \\ S \succeq 0;\ \delta(X, S; \mu) \leq \tau\Big\}.$$

The algorithm starts with (X, y, S) in the τ-neighborhood $\mathcal{N}(\tau)$, which this certainly holds at (X^0, y^0, S^0) since $\delta(X^0, S^0; \mu^0) = 0$. If, for the current iterate

(X, y, S) in the τ-neighborhood, $X \bullet S > \varepsilon$, then the algorithm performs corrector step and predictor step. In the corrector step, we define

$$\begin{aligned} D &:= \left[X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}}\right]^{\frac{1}{2}} = \left[S^{-\frac{1}{2}}(S^{\frac{1}{2}}XS^{\frac{1}{2}})^{\frac{1}{2}}S^{-\frac{1}{2}}\right]^{\frac{1}{2}}, \\ V &:= \frac{1}{\sqrt{\mu}}D^{-1}XD^{-1} = \frac{1}{\sqrt{\mu}}DSD, \\ \bar{A}_i &:= \frac{1}{\sqrt{\mu}}DA_iD, \ i = 1, 2, ..., m, \end{aligned}$$

and obtain D_X, Δy and D_S by solving the following system

$$\begin{aligned} &\bar{A}_i \bullet D_X = 0, \quad i = 1, 2, ..., m, \\ &\sum_{i=1}^{m} \triangle y_i \bar{A}_i - \bar{\Omega}(D_X) + D_S = 0, \\ &D_X + D_S = 2(E - V). \end{aligned} \tag{12}$$

Then, we obtain

$$\Delta X = \sqrt{\mu}DD_XD, \quad \Delta S = \sqrt{\mu}D^{-1}D_SD^{-1},$$

and let

$$X_+ := X + \Delta X, \quad y_+ := y + \Delta y, \quad S_+ := S + \Delta S.$$

In the predictor step, we define

$$\begin{aligned} V_+ &:= \frac{1}{\sqrt{\mu}}D_+^{-1}X_+D_+^{-1} = \frac{1}{\sqrt{\mu}}D_+S_+D_+, \\ \bar{A}_i^+ &= \frac{1}{\sqrt{\mu}}D_+A_iD_+, \end{aligned}$$

and obtain D_X^p and D_S^p by solving the following system

$$\begin{aligned} &\bar{A}_i^+ \bullet D_X^p = 0, \quad i = 1, 2, ..., m, \\ &\sum_{i=1}^{m} \triangle^p y_i^+ \bar{A}_i^+ - \bar{\Omega}(D_X^p) + D_S^p = 0, \\ &D_X^p + D_S^p = -2V_+. \end{aligned} \tag{13}$$

Then, we get

$$\Delta^p X = \sqrt{\mu}D_+D_X^pD_+, \quad \Delta^p S = \sqrt{\mu}D_+^{-1}D_S^pD_+^{-1},$$

and let

$$X^p := X_+ + \theta\Delta^p X, \quad y^p := y_+ + \theta\Delta^p y, \quad S^p := S_+ + \theta\Delta^p S, \quad \mu^p := (1 - 2\theta)\mu,$$

for each $\theta \in (\theta, \frac{1}{2})$. The iterate (X^p, y^p, S^p) will be in the τ-neighborhood $\mathcal{N}(\tau)$ again. This procedure is repeated until μ is small enough, say until $n\mu \leq \varepsilon$.

The framework of corrector-predictor algorithm for CQSDO problem is shown as follows:

Algorithm: Corrector-predictor algorithm for CQSDO problem

Input:
A threshold parameter $\tau \geq 1$;
an accuracy parameter $\varepsilon > 0$;
a barrier update parameter θ, $0 < \theta < 1/2$;
a strictly feasible point (X^0, S^0) and μ^0 such that $\delta(X^0, S^0; \mu^0) \leq \tau$;
begin
$X := X^0$; $S := S^0$; $\mu := \mu^0$;
while $n\mu > \varepsilon$ do
begin
corrector step
solve system (12) and let
$(X_+, y_+, S_+) = (X, y, S) + (\Delta X, \Delta y, \Delta S)$;
predictor step
solve system (13) and let
$(X_+^p, y_+^p, S_+^p) = (X_+, y_+, S_+) + \theta(\Delta^p X, \Delta^p y, \Delta^p S)$;
μ-update
$\mu^p := (1 - 2\theta)\mu$;
$(X, y, S) := (X_+^p, y_+^p, S_+^p)$, $\mu := \mu^p$;
end
end

4. Analysis of the algorithm

In this section, we focus on the analysis of the above algorithm. Let

$$Q_V := D_X - D_S,$$

it is easy to obtain

$$D_X = \frac{P_V + Q_V}{2}, \quad D_S = \frac{P_V - Q_V}{2}, \quad D_X D_S + D_S D_X = \frac{P_V^2 - Q_V^2}{2}.$$

Note that D_X and D_S are nonorthogonal by (11), therefore

$$\begin{aligned} \|Q_V\| &= \sqrt{\mathrm{Tr}^2(D_X - D_S)} \\ &= \sqrt{\|P_V\|_F^2 - 4 D_X \bullet D_S} \\ &\leq 2\delta(V). \end{aligned}$$

In the following, we recall some results without proof.

Lemma 4.1. (Lemma 6.2 and 7.3 in [7]) *Let $D_X, D_S \in \mathbf{S}^n$ be two vectors such that $Tr(D_X D_S) \geq 0$. Then we have*

$$\|D_{XS}\|_\infty \leq \frac{1}{4}\|D_X + D_S\|_F^2, \quad \|D_{XS}\|_F \leq \frac{1}{2\sqrt{2}}\|D_X + D_S\|_F^2,$$

where $D_{XS} := \frac{1}{2}(D_X D_S + D_S D_X)$.

Lemma 4.2. (Modification of Lemma 6.1 in [7]) *Let $X(\alpha) = X + \alpha\Delta X$, $S(\alpha) = S + \alpha\Delta S$ for $0 \leq \alpha \leq 1$, which $X \succ 0$ and $S \succ 0$. If one has*

$$det(X(\alpha)S(\alpha)) > 0, \ \forall \alpha \in [0, \bar{\alpha}],$$

then $X(\bar{\alpha}) \succ 0$ and $S(\bar{\alpha}) \succ 0$.

Lemma 4.3. (Lemma 4.4 in [6]) *Let $X, S \succ 0$ and $\mu > 0$. Assume that $\delta := \delta(V)$, one has*

$$\sqrt{1-\delta} \leq \lambda_i(V) \leq \sqrt{1+\delta}, \ i = 1, 2, \ldots, n \quad and \quad \|V\|_F^2 \leq n(1+\delta).$$

Lemma 4.4. (Lemma A.1 in [18]) *Let $Q \in \mathbf{S}_{++}^n$ and $M \in \mathbb{R}^{n\times n}$ be skew-symmetric (i.e., $M = -M^T$). Then* $\det(Q + M) > 0$. *Moreover, if the eigenvalues of $Q + M$ are real, then*

$$0 < \lambda_{min}(Q) \leq \lambda_{min}(Q + M) \leq \lambda_{max}(Q + M) \leq \lambda_{max}(Q),$$

which implies $Q + M \succ 0$.

4.1. Corrector step

The following lemma shows that the strict feasibility of iterates after a full NT-step.

Lemma 4.5. (Modification of Lemma 6.3 in [22]) *If $\delta := \delta(X, S; \mu) < 1$, then (X_+, y_+, S_+) are strictly feasible.*

Lemma 4.6. (Modification of Lemma 6.4 in [23]) *If $\delta := \delta(X, S; \mu) < 1$, one has*

$$\delta_+ := \delta(X_+, S_+; \mu) \leq \frac{\delta^2}{1 + \sqrt{1-\delta^2}}.$$

Thus $\delta(X_+, S_+, \mu) \leq \delta^2$, which shows the quadratical convergence of the algorithm.

The following lemma gives an upper bound of the duality gap after a full NT-step.

Lemma 4.7. (Modification of Lemma 3.8 in [19]) *After a full NT-step, we have*

$$X_+ \bullet S_+ \leq n\mu.$$

4.2. Predictor step

Here, we present a sufficient condition for strict feasibility after a predictor step.

Lemma 4.8. *Let $X_+ \succ 0$, $S_+ \succ 0$ and $\mu > 0$ such that $\delta_+ := \delta(X_+, S_+; \mu) < 1$. Moveover, let $X^p := X_+ + \theta\Delta^p X$ and $S^p := S_+ + \theta\Delta^p S$, $\theta \in (0, \frac{1}{2})$, denote the iterates after a predictor step. Then $X^p \succ 0$ and $S^p \succ 0$ if $\mathcal{K}(\delta_+, \theta, n) > 0$, where*

$$\mathcal{K}(\delta_+, \theta, n) := (1 - \delta_+) - \frac{n\theta^2(1 + \delta_+)}{1 - 2\theta}.$$

Proof. Let us introduce

$$X^p(\alpha) := X_+ + \alpha\theta\Delta^p X, \quad S^p(\alpha) := S_+ + \alpha\theta\Delta^p S,$$

for $0 \leq \alpha \leq 1$. Moreover, we have

$$X^p(\alpha) = \sqrt{\mu} D_+(V_+ + \alpha\theta D_X^p)D_+, \quad S^p(\alpha) = \sqrt{\mu} D_+^{-1}(V_+ + \alpha\theta D_S^p)D_+^{-1}.$$

Therefore, we have

$$\begin{aligned} X^p(\alpha)S^p(\alpha) &= \mu D_+(V_+ + \alpha\theta D_X^p)(V_+ + \alpha\theta D_S^p)D_+^{-1} \\ &\sim \mu(V_+ + \alpha\theta D_X^p)(V_+ + \alpha\theta D_S^p), \end{aligned}$$

Using the third equation of system (13), we get

$$\begin{aligned} \frac{X^p(\alpha)S^p(\alpha)}{\mu} &\sim (V_+ + \alpha\theta D_X^p)(V_+ + \alpha\theta D_S^p) \\ &= V_+^2 + \alpha\theta(V_+ D_S^p + D_X^p V_+) + \alpha^2\theta^2 D_X^p D_S^p \\ &= V_+^2 + \alpha\theta(V_+ D_S^p + V_+ D_X^p) + \alpha\theta(D_X^p V_+ - V_+ D_X^p) + \alpha^2\theta^2 D_X^p D_S^p \\ &= V_+^2 + \alpha\theta(-2V_+^2) + \alpha\theta(D_X^p V_+ - V_+ D_X^p) + \alpha^2\theta^2 D_X^p D_S^p \\ &= (1 - 2\alpha\theta)V_+^2 + \alpha\theta(D_X^p V_+ - V_+ D_X^p) + \alpha^2\theta^2 D_X^p D_S^p. \end{aligned}$$

Furthermore, we have

$$\frac{X^p(\alpha)S^p(\alpha)}{\mu(1-2\alpha\theta)} \sim V_+^2 + \frac{\alpha^2\theta^2}{1-2\alpha\theta} D_{XS}^p + \frac{\alpha\theta}{1-2\alpha\theta}(\alpha\theta M + (1-\alpha\theta)(D_X^p V_+ - V_+ D_X^p)), \tag{14}$$

where

$$M := (D_X^p V_+ - V_+ D_X^p) + \frac{1}{2}(D_X^p D_S^p - D_S^p D_X^p).$$

One can easily verify that the matrix $\alpha\theta M + (1 - \alpha\theta)(D_X^p V_+ - V_+ D_X^p)$ is skew-symmetric. And it follows from Lemma 4.4 that

$$\begin{aligned} \lambda_{\min}\left(\frac{X^p(\alpha)S^p(\alpha)}{\mu(1 - 2\alpha\theta)}\right) &\geq \lambda_{\min}\left(V_+^2 + \frac{\alpha^2\theta^2}{1 - 2\alpha\theta} D_{XS}^p\right) \\ &\geq \lambda_{\min}(V_+^2) - \frac{\alpha^2\theta^2}{1 - 2\alpha\theta}\|D_{XS}^p\|_\infty \\ &\geq \lambda_{\min}(V_+^2) - \frac{\theta^2}{1 - 2\theta}\|D_{XS}^p\|_\infty, \end{aligned}$$

the last inequality follows by $f(\alpha) = \frac{\alpha^2\theta^2}{1-2\alpha\theta}$ is strictly increasing for $0 \le \alpha \le 1$. By Lemmas 4.1 and 4.3, we have

$$\|D^p_{XS}\|_\infty \le \frac{1}{4}\|D^p_X + D^p_S\|_F^2 = \frac{1}{4}\| - 2V_+\|_F^2 \le n(1+\delta_+).$$

Therefore,

$$\lambda_{\min}\left(\frac{X^p(\alpha)S^p(\alpha)}{\mu(1-2\alpha\theta)}\right) \ge (1-\delta_+) - \frac{n\theta^2(1+\delta_+)}{1-2\theta} := \mathcal{K}(\delta_+,\theta,n) > 0. \tag{15}$$

This implies that $\det(X^p(\alpha)S^p(\alpha)) > 0$ for $0 \le \alpha \le 1$.

In addition, since $X^p(0) = X_+ \succ 0$ and $S^p(0) = S_+ \succ 0$, Lemma 4.2 implies that $X^p(1) = X^p \succ 0$ and $S^p(1) = S^p \succ 0$. This proves the lemma.

In the next lemma, we investigate the effect on the proximity measure of a predictor step by an update of the parameter μ.

Lemma 4.9. *Let $\delta_+ := \delta(X_+, S_+; \mu) < 1$, $\mu^p := (1-2\theta)\mu$ for each $0 < \theta < \frac{1}{2}$, $\mathcal{K}(\delta_+,\theta,n) > 0$. Moreover, let X^p, S^p denote the iterates after a predictor step, i.e., $X^p = X_+ + \theta\Delta^p X$ and $S^p = S_+ + \theta\Delta^p S$, $\theta \in (0, \frac{1}{2})$. Then*

$$\delta^p := \delta(X^p, S^p; \mu^p) \le \delta_+ + \frac{\sqrt{2}n\theta^2(1+\delta_+)}{1-2\theta}.$$

Proof. From Lemma 4.8, we know that the predictor step is strictly feasible. Letting $\alpha = 1$, it follows from (14) that

$$(V^p)^2 \sim \frac{X^pS^p}{\mu^p} \sim V_+^2 + \frac{\theta^2}{1-2\theta}D^p_{XS} + \frac{\theta}{1-2\theta}\tilde{M}, \tag{16}$$

where $\tilde{M} := \theta M + (1-\theta)(D^p_X V_+ - V_+ D^p_X)$ is skew-symmetric. From (15), we know

$$\lambda_{min}((V^p)^2) \ge \mathcal{K}(\delta_+,\theta,n) > 0.$$

Using (16) and properties of the Frobenius norm, we get

$$\begin{aligned}
(\delta^p)^2 = \|E - (V^p)^2\|_F^2 &= \sum_{i=1}^n \left(\lambda_i\left(V_+^2 + \frac{\theta^2}{1-2\theta}D^p_{XS} + \frac{\theta}{1-2\theta}\tilde{M}\right) - 1\right)^2 \\
&= \frac{1}{(1-2\theta)^2}\sum_{i=1}^n(\lambda_i((1-2\theta)V_+^2 + \theta^2 D^p_{XS} + \theta\tilde{M})^2 \\
&+ (1-2\theta)^2 - 2(1-2\theta)\lambda_i((1-2\theta)V_+^2 + \theta^2 D^p_{XS} + \theta\tilde{M})) \\
&= \frac{1}{(1-2\theta)^2}[\mathrm{Tr}(((1-2\theta)V_+^2 + \theta^2 D^p_{XS} + \theta\tilde{M})^2) \\
&+ n(1-2\theta)^2 - 2(1-2\theta)\mathrm{Tr}((1-2\theta)V_+^2 + \theta^2 D^p_{XS})] \\
&\le \frac{1}{(1-2\theta)^2}[\mathrm{Tr}(((1-2\theta)V_+^2 + \theta^2 D^p_{XS})^2) + n(1-2\theta)^2 \\
&- 2(1-2\theta)\mathrm{Tr}((1-2\theta)V_+^2 + \theta^2 D^p_{XS})]
\end{aligned}$$

$$= \frac{1}{(1-2\theta)^2}\mathrm{Tr}(((1-2\theta)(E-V_+^2)-\theta^2 D_{XS}^p)^2)$$
$$= \frac{1}{(1-2\theta)^2}\|((1-2\theta)(E-V_+^2)-\theta^2 D_{XS}^p)\|_F^2.$$

Taking the second root and using the triangle inequality, we obtain

$$\delta^p \leq \|E-V_+^2\|_F + \frac{\theta^2}{1-2\theta}\|D_{XS}^p\|_F. \tag{17}$$

Furthermore, by using Lemma 4.1 and the third equation in (13) we have

$$\begin{aligned} \|D_{XS}^p\|_F &\leq \frac{1}{2\sqrt{2}}\|D_X^p + D_S^p\|_F^2 \leq \frac{1}{2\sqrt{2}}\|-2V_+\|_F^2 \\ &= \frac{1}{2\sqrt{2}}\sum_{i=1}^n \lambda_i(2V_+)^2 \leq \sqrt{2}n(1+\delta_+). \end{aligned} \tag{18}$$

Using (17) and (18), we get

$$\delta^p \leq \delta_+ + \frac{\sqrt{2}n\theta^2(1+\delta_+)}{1-2\theta}.$$

This completes the proof of lemma.

4.3. The effect on duality gap after an iteration

The following lemma gives the effect on duality gap after a main iteration.

Lemma 4.10. *After an iteration, we have*

$$X^p \bullet S^p \leq \frac{n\mu^p}{1-2\theta}$$

Proof. Letting $\alpha = 1$ in (14), we have

$$\frac{X^p S^p}{\mu(1-2\theta)} \sim V_+^2 + \frac{\theta^2}{1-2\theta}D_{XS}^p + \frac{\theta}{1-2\theta}(\theta M_+ + (1-\theta)(D_X^p V_+ - V_+ D_X^p)). \tag{19}$$

Note that $\theta M_+ + (1-\theta)(D_X^p V_+ - V_+ D_X^p)$ is skew-symmetry, we obtain

$$\frac{1}{\mu(1-2\theta)}\mathrm{Tr}(X^p S^p) = \mathrm{Tr}(V_+^2) + \frac{\theta^2}{1-2\theta}\mathrm{Tr}(D_{XS}^p). \tag{20}$$

It follows from the third equation of (13) that

$$D_{XS}^p = 2V_+^2 - \frac{(D_X^p)^2 + (D_S^p)^2}{2}.$$

Taking trace, we get

$$\mathrm{Tr}(D_{XS}^p) = \frac{1}{2}\mathrm{Tr}\left(\frac{X_+ S_+}{\mu}\right) - \frac{\|D_X^p\|_F^2 + \|D_S^p\|_F^2}{2} \leq \frac{1}{2}\mathrm{Tr}\left(\frac{X_+ S_+}{\mu}\right). \tag{21}$$

Using (20), (21), Lemma 4.7 and $\mu^p = (1-2\theta)\mu$, one has

$$\mathrm{Tr}(X^pS^p) \leq \left(1 - 2\theta + \frac{\theta^2}{2}\right)\mathrm{Tr}(X_+S_+) \leq \mathrm{Tr}(X_+S_+) \leq n\mu = \frac{n\mu^p}{1-2\theta},$$

which proves the desired result.

4.4. Fixing the parameters

Here, we choose the parameters τ and θ, which guarantee that after a main iteration, the proximity measure will not exceed the proximity parameter τ. Let (X, y, S) be the iterate at the start of a main iteration and $\delta := \delta(X, S; \mu) \leq \tau$. After a correct step, by Lemma 4.6, we have

$$\delta_+ = \delta(X_+, S_+; \mu) \leq \frac{\delta^2}{1 + \sqrt{1-\delta^2}}.$$

Since the right-hand side of the above inequality is monotonically increasing with respect to δ, thus

$$\delta_+ \leq \frac{\tau^2}{1 + \sqrt{1-\tau^2}}.$$

Following a predictor step and an μ-update, by Lemma 4.8, one has

$$\delta^p \leq \delta_+ + \frac{\sqrt{2}n\theta^2(1+\delta_+)}{1-2\theta}. \tag{22}$$

Note that the right-hand side of (22) is monotonically increasing with respect to δ_+, we obtain

$$\delta^p \leq \delta_+ + \frac{\sqrt{2}n\theta^2(1+\delta_+)}{1-2\theta} \leq \frac{\tau^2}{1+\sqrt{1-\tau^2}} + \frac{\sqrt{2}n\theta^2(2-\sqrt{1-\tau^2})}{1-2\theta}.$$

To keep $\delta^p \leq \tau$, if suffices to have

$$\frac{\tau^2}{1+\sqrt{1-\tau^2}} + \frac{\sqrt{2}n\theta^2(2-\sqrt{1-\tau^2})}{1-2\theta} \leq \tau.$$

At this state, we let $\tau = \frac{1}{8}$ and $\theta = \frac{1}{8\sqrt{n}}$, after some simple calculation, we have $\delta^p \leq \frac{1}{8}$. Thus, the algorithm is well defined.

5. Iteration complexity

In the present section, we derive worst-case iteration complexity of **Algorithm**.

Lemma 5.1. *Let X^0 and S^0 be strictly feasible, $n\mu^0 = X^0 \bullet S^0$ and $\delta(X^0, S^0; \mu^0) \leq \tau$. Moveover, let X^k and S^k be the iterates obtained after k iterations. Then, $X^k \bullet S^k \leq \varepsilon$ for*

$$k \geq 1 + \left\lceil \frac{1}{2\theta} \log \frac{X^0 \bullet S^0}{\varepsilon} \right\rceil.$$

Proof. It follows from Lemma 4.10 that

$$X^k \bullet S^k \leq \frac{n\mu^k}{1-2\theta} = n(1-2\theta)^{k-1}\mu^0 = (1-2\theta)^{k-1}(X^0 \bullet S^0).$$

In order to prove $X^k \bullet S^k \leq \varepsilon$, it is sufficient to prove

$$(1-2\theta)^{k-1}(X^0 \bullet S^0) \leq \varepsilon.$$

Taking natural logarithms, we get

$$(k-1)\log(1-2\theta) + \log(X^0 \bullet S^0) \leq \log \varepsilon.$$

Using $\log(1-2\theta) \leq -2\theta$ for $0 < \theta < \frac{1}{2}$, we observe that the above inequality holds if

$$-2\theta(k-1) + \log(X^0 \bullet S^0) \leq \log \varepsilon.$$

This implies the desired result.

Lemma 5.2. *Let $\tau = \frac{1}{8}$ and $\theta = \frac{1}{8\sqrt{n}}$. Then the algorithm is well-defined and the algorithm requires at most*

$$O\left(\sqrt{n}\log\frac{X^0 \bullet S^0}{\varepsilon}\right)$$

iterations. The output is a primal-dual pair (X, y, S) satisfying $X \bullet S \leq \varepsilon$.

Proof. Let $\tau = \frac{1}{8}$ and $\theta = \frac{1}{8\sqrt{n}}$, the desired result follows immediately from Lemma 5.1.

Corollary 5.3. *If one takes $X^0 = S^0 = E$, the iteration bound becomes*

$$O\left(\sqrt{n}\log\frac{n}{\varepsilon}\right),$$

which is the currently best-known iteration bound for the algorithm with small-update methods.

6. Concluding remarks

In this paper, we proposed a new corrector-predictor algorithm for CQSDO problem. Each iteration is composed of a corrector step and a predictor step. Some good properties of our algorithm are: (1) The search direction is obtained by an equivalent algebraic reformulation of the centering equation. (2) In the corrector-step, the step length need not be calculated because we have full-step. (3) The iteration complexity obtained for proposed algorithm coincides with the currently best-known complexity bound with small-update interior point algorithms. For further research, this algorithm may be possible extended to the Cartesian $P_*(\kappa)$-LCPs over symmetric cones.

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