DIFFERENTIAL SANDWICH THEOREMS FOR p-VALENT FUNCTIONS ASSOCIATED WITH A CERTAIN GENERALIZED DIFFERENTIAL OPERATOR AND INTEGRAL OPERATOR

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Abstract. The purpose of this paper is to derive some subordination and superordination results for functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ which are $p-$valent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ by using certain differential operator $A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)$ and integral operator $F^m_p(\rho, \vartheta)f(z)$. Some special cases are also considered.

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1. Introduction

Let $\mathcal{H}$ denote the class of functions which are *analytic* in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\},$$

and let $\mathcal{H}[a, p]$ denote the subclass of the functions $f \in \mathcal{H}$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots \quad (a \in \mathbb{C};\ p \in \mathbb{N} := \{1, 2, ...\}). \tag{1.1}$$

For simplicity, $\mathcal{H}[a]=\mathcal{H}[a,1]$. Also, let $\mathcal{A}(p)$ denote the subclass of $\mathcal{H}$ consisting of functions of the form:

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k, \qquad (a_k\geq 0; p\in\mathbb{N}:=\{1,2,...\}), \tag{1.2}$$

which are $p-$valent in $\mathbb{U}$.

Given two functions f, $g\in\mathcal{H}$. The function $f(z)$ is said to be *subordination* to $g(z)$ in $\mathbb{U}$, written $f(z)\prec g(z)$, if there exists a function $h(z)$, analytic in $\mathbb{U}$, with $h(0)=0$ and $|h(z)|<1$ for all $z\in\mathbb{U}$, such that $f(z)=g(h(z))$ for all $z\in\mathbb{U}$. Furthermore, if the function g is *univalent* in $\mathbb{U}$, then we have the following equivalence (see [7] and [9]):

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0) \text{ and } f(\mathbb{U})\subset g(\mathbb{U}).$$

Let $\phi:\mathbb{C}^3\times\mathbb{U}\rightarrow\mathbb{C}$ and $\psi(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the second-order differential subordination:

$$\phi\left(p(z),zp'(z),z^2p''(z);z\right)\prec\psi(z), \tag{1.3}$$

then $p(z)$ is a solution of the differential subordination (1.3). The univalent function $q(z)$ is called a *dominant* of the solutions of the differential subordination (1.3) if $p(z)\prec q(z)$ for all $p(z)$ satisfying (1.3). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q}\prec q$ for all dominants of (1.3) is called the *best dominant.* If $p(z)$ is univalent in $\mathbb{U}$ and satisfies the second-order differential superordination:

$$\psi(z)\prec\phi\left(p(z),zp'(z),z^2p''(z);z\right), \tag{1.4}$$

then $p(z)$ is a solution of the differential supordination (1.4). A univalent function $q(z)$ is called a *subordinant* of the solutions of the differential superordination (1.4) if $q(z)\prec p(z)$ for all $p(z)$ satisfying (1.4). A univalent subordinant $\tilde{q}$ that satisfies $q\prec\tilde{q}$ for all subordinants of (1.4) is called the *best subordinant.* Using the results of Miller and Mocanu [10], Al-Hawary, Amourah and Darus [1], Aljarah and Darus [3], Bulboaca [6] considered certain classes of first-order differential superordinations as well as superordination-preserving integral operators [7]. Ali et al. [2] have used the results of Bulboaca [6] to obtain sufficient conditions for normalized analytic functions $f\in\mathcal{A}(1)$ to satisfy:

$$q_1(z)\prec\frac{zf'(z)}{f(z)}\prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in $\mathbb{U}$ with $q_1(0)=q_2(0)=1$.

Also, Tuneski [14] obtained a sufficient condition for starlikeness of $f\in\mathcal{A}(1)$ in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$.

Recently, Shanmugam et al. [12, 11] and [13] obtained sufficient conditions for the normalized analytic function $f\in\mathcal{A}(1)$ to satisfy:

$$q_1(z)\prec\frac{f(z)}{zf'(z)}\prec q_2(z),$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z).$$

Shanmugam et al. [11] also obtained the so-called sandwich results for certain classes of analytic functions.

For the function $f \in \mathcal{A}(p)$, Faisal and Darus ([8]) defined the following differential operator:

$$A^0_{\lambda,p}(\alpha,\beta,\mu)f(z) = f(z),$$

$$A^1_{\lambda,p}(\alpha,\beta,\mu)f(z) = \left(\frac{\alpha+\beta-p(\mu+\lambda)}{\alpha+\beta}\right) f(z) + \left(\frac{\mu+\lambda}{\alpha+\beta}\right) zf'(z),$$

and for $n = 2, 3, \ldots,$

$$\begin{aligned} A^n_{\lambda,p}(\alpha,\beta,\mu)f(z) &= A(A^{n-1}_{\lambda,p}(\alpha,\beta,\mu)f(z)), \\ &= z^p + \sum_{k=p+1}^{\infty} \left[\frac{\alpha+(\mu+\lambda)(k-p)+\beta}{\alpha+\beta}\right]^n a_k z^k, \end{aligned} \tag{1.5}$$

for $f \in \mathcal{A}(p)$, $\alpha,\beta,\mu,\lambda \geq 0, \alpha+\beta \neq 0$ and $\mu+\lambda \neq 0$, $p \in \mathbb{N}$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. It is easily verified from (1.5) that

$$\begin{aligned} \frac{\mu+\lambda}{\alpha+\beta} z(A^n_{\lambda,p}(\alpha,\beta,\mu)f(z))' &= A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z) \\ &- \left(1 - \frac{p}{\alpha+\beta}(\mu+\lambda)\right) A^n_{\lambda,p}(\alpha,\beta,\mu)f(z). \end{aligned} \tag{1.6}$$

Also, for the function $f \in \mathcal{A}(p)$, Aouf et al. ([5]) defined the integral operator:

$$F^m_p(\rho,\vartheta)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\vartheta}{p+\vartheta+\rho(k-p)}\right]^m a_k z^k, \tag{1.7}$$

for $\rho > 0$, $\vartheta \geq 0$, $p \in \mathbb{N}$ and $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. It is easily verified from (1.7) that

$$\begin{aligned} \rho z(F^{m+1}_p(\rho,\vartheta)f(z))' &= (\vartheta+p)(F^m_p(\rho,\vartheta)f(z)) \\ &- [\vartheta+p(1-\rho)](F^{m+1}_p(\rho,\vartheta)f(z)). \end{aligned} \tag{1.8}$$

In this paper, we derive several subordination results, superordination results and sandwich results involving the differential operator $A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)$ given by (1.5) and the integral operator $F^m_p(\rho,\vartheta)f(z)$ given by (1.7).

2. Definitions and preliminaries

In order to prove our subordinations and superordinations results, we need the following definition and lemmas.

Definition 2.1 [10] Denote by Q, the set of all functions f that are analytic and injective on $\overline{\mathbb{U}}\backslash E(f)$, where

$$E(f) = \left\{\zeta \in \partial\mathbb{U} \lim_{z\to\zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(f)$.

Lemma 2.2 [10] *Let $q(z)$ be univalent in $\mathbb{U}$ and let θ and φ be analytic in a domain D containing $q(\mathbb{U})$ with $\varphi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set*

$$\psi(z) = zq'(z)\varphi(q(z)) \quad and \quad h(z) = \theta(q(z)) + \psi(z).$$

Suppose that

(i) *ψ is starlike univalent in $\mathbb{U}$,*

(ii) $\mathrm{Re}\left\{\dfrac{zh'(z)}{\psi(z)}\right\} > 0$ *for $z \in \mathbb{U}$.*

If $p(z)$ is a analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.1}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant of (2.1).

Lemma 2.3 [6] *Let $q(z)$ be convex univalent in $\mathbb{U}$ and let ϑ and ϕ be analytic in a domain D containing $q(\mathbb{U})$. Suppose that*

(i) $Re\left\{\dfrac{\vartheta'(q(z))}{\phi(q(z))}\right\} > 0$ *for $z \in \mathbb{U}$,*

(ii) $\Psi(z) = zq'(z)\phi(q(z))$ *is starlike univalent in $\mathbb{U}$.*

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathbb{U}) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\phi(p(z))$ is univalent in $\mathbb{U}$ and

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z)), \tag{2.2}$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant of (2.2).

3. Subordinations and superordinations results for $A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)$

Unless otherwise mentioned, we assume throughout this paper that $\epsilon, \eta \in \mathbb{C}$. We begin with the following result involving differential subordination between analytic functions.

Theorem 3.1 *Let $q(z)$ be univalent in $\mathbb{U}$ with $q(0)=1$. Further, assume that*

$$\operatorname{Re}\left\{\frac{2(\delta+\kappa)q(z)}{\delta}+1+\frac{zq''(z)}{q'(z)}\right\}>0. \ (\delta\in\mathbb{C}^*=\mathbb{C}-\{0\},\kappa\geq 0) \tag{3.1}$$

If $f\in\mathcal{A}(p)$ satisfies the following subordination condition:

$$\zeta(n,\lambda,p,\beta,\delta,\alpha;z)\prec \delta zq'(z)+(\delta+\kappa)\left(q(z)\right)^2, \tag{3.2}$$

where

$$\begin{aligned}&\zeta(n,\lambda,p,\beta,\delta,\alpha;z)\\ &=\frac{\delta(\alpha+\beta)}{(\mu+\lambda)}\left[\left(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^{1-\epsilon}-\epsilon\frac{\left(A^{n+2}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)\left(A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)}{\left(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^{1+\epsilon}}\right]\\ &+\delta\left(\epsilon-1\right)\left(\frac{\alpha+\beta}{\mu+\lambda}-p\right)\frac{A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^{\epsilon}}+(\delta+\kappa)\frac{\left(A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^2}{\left(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^{2\epsilon}},\end{aligned}$$

then

$$\frac{A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}}\prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define a function $p(z)$ by

$$p(z)=\frac{A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}},\quad (z\in\mathbb{U}). \tag{3.3}$$

Then the function $p(z)$ is analytic in $\mathbb{U}$ and $p(0)=1$. Differentiating (3.3) logarithmically with respect to z and using the identity (1.6) in the resulting equation, we get

$$\begin{aligned}\frac{p'(z)}{p(z)}&=\frac{A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)-(1-\frac{p}{\alpha+\beta}(\mu+\lambda))A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\frac{\mu+\lambda}{\alpha+\beta}zA^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}\\ &\quad-\epsilon\frac{A^{n+2}_{\lambda,p}(\alpha,\beta,\mu)f(z)-(1-\frac{p}{\alpha+\beta}(\mu+\lambda))A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\frac{\mu+\lambda}{\alpha+\beta}zA^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)},\end{aligned}$$

$$\begin{aligned}\Rightarrow p'(z)&=p(z)\frac{\alpha+\beta}{\mu+\lambda}\left[\frac{A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{zA^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}-\frac{(1-\frac{p}{\alpha+\beta}(\mu+\lambda))}{z}\right.\\ &\quad\left.-\epsilon\frac{A^{n+2}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{zA^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)}+\epsilon\frac{(1-\frac{p}{\alpha+\beta}(\mu+\lambda))}{z}\right],\\ &=\frac{(\alpha+\beta)}{z(\mu+\lambda)}\left[(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z))^{1-\epsilon}-\epsilon\frac{(A^{n+2}_{\lambda,p}(\alpha,\beta,\mu)f(z))(A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z))}{(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z))^{1+\epsilon}}\right]\\ &\quad+\frac{(\epsilon-1)}{z}(\frac{\alpha+\beta}{\mu+\lambda}-p)\frac{A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z))^{\epsilon}}.\end{aligned}$$

$$\Rightarrow \delta zp'(z) + (\delta+\kappa)(p(z))^2$$
$$= \frac{\delta(\alpha+\beta)}{(\mu+\lambda)}\left[\left(A_{\lambda,p}^{n+1}(\alpha,\beta,\mu)f(z)\right)^{1-\epsilon} - \epsilon\frac{\left(A_{\lambda,p}^{n+2}(\alpha,\beta,\mu)f(z)\right)\left(A_{\lambda,p}^{n}(\alpha,\beta,\mu)f(z)\right)}{\left(A_{\lambda,p}^{n+1}(\alpha,\beta,\mu)f(z)\right)^{1+\epsilon}}\right]$$
$$+\,\delta(\epsilon-1)\left(\frac{\alpha+\beta}{\mu+\lambda}-p\right)\frac{A_{\lambda,p}^{n}(\alpha,\beta,\mu)f(z)}{\left(A_{\lambda,p}^{n+1}(\alpha,\beta,\mu)f(z)\right)^{\epsilon}} + (\delta+\kappa)\frac{\left(A_{\lambda,p}^{n}(\alpha,\beta,\mu)f(z)\right)^2}{\left(A_{\lambda,p}^{n+1}(\alpha,\beta,\mu)f(z)\right)^{2\epsilon}}.$$

Using (3.2), we obtain

$$\delta zp'(z) + (\delta+\kappa)(p(z))^2 \prec \delta zq'(z) + (\delta+\kappa)(q(z))^2.$$

Therefore, now Theorem 3.1 follows by applying Lemma 2.2 by setting

$$\theta(w) = (\delta+\kappa)w^2 \quad \text{and} \quad \varphi(w) = \delta. \qquad \blacksquare$$

Corollary 3.2 *Let* $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ *in Theorem* 3.1. *Further, assume that* (3.1) *holds.*

If $f \in \mathcal{A}(p)$ *satisfies the following subordination condition:*

$$\zeta(n,\lambda,p,\beta,\delta,\alpha;z) \prec \frac{\delta(A-B)z}{(1+Bz)^2} + (\delta+\kappa)\left(\frac{1+Az}{1+Bz}\right)^2,$$

then

$$\frac{A_{\lambda,p}^{n}(\alpha,\beta,\mu)f(z)}{\left[A_{\lambda,p}^{n+1}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \prec \frac{1+Az}{1+Bz},$$

and the function $\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Also, let $q(z) = \dfrac{1+z}{1-z}$. *Then for* $f \in \mathcal{A}(p)$ *we have*

$$\zeta(n,\lambda,p,\beta,\delta,\alpha;z) \prec \frac{2\delta z}{(1-z)^2} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^2,$$

then

$$\frac{A_{\lambda,p}^{n}(\alpha,\beta,\mu)f(z)}{\left[A_{\lambda,p}^{n+1}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \prec \frac{1+z}{1-z},$$

and the function $\dfrac{1+z}{1-z}$ *is the best dominant.*

Finally, by taking $q(z) = \left(\dfrac{1+z}{1-z}\right)^{\mu}$, $(0 < \mu \le 1)$, *then for* $f \in \mathcal{A}(p)$ *we have,*

$$\zeta(n,\lambda,p,\beta,\delta,\alpha;z) \prec \frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^{2\mu},$$

then

$$\frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^\epsilon} \prec \left(\frac{1+z}{1-z}\right)^\mu,$$

and the function $\left(\frac{1+z}{1-z}\right)^\mu$ *is the best dominant.*

Next, by appealing to Lemma 2.3, we prove the following.

Theorem 3.3 *Let $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0)=1$. Assume that*

$$\text{Re}\left\{\frac{2(\delta+\kappa)q(z)q'(z)}{\delta}\right\} > 0. \tag{3.4}$$

Let $f \in \mathcal{A}(p)$ such that $\frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^\epsilon} \in \mathcal{H}[q(0),1]\cap Q$, $\zeta(n,\lambda,p,\beta,\delta,\alpha;z)$ *is univalent in $\mathbb{U}$ and the following superordination condition*

$$(\delta+\kappa)\left(q(z)\right)^2 + \delta z q'(z) \prec \zeta(n,\lambda,p,\beta,\delta,\alpha;z), \tag{3.5}$$

holds, then

$$q(z) \prec \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^\epsilon}, \tag{3.6}$$

and $q(z)$ is the best subordinant.

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^\epsilon}.$$

Then, from the assumption of Theorem 3.3, the function $p(z)$ is analytic in $\mathbb{U}$ and (3.4) holds. Hence, the subordination (3.5) is equivalent to

$$(\delta+\kappa)\left(q(z)\right)^2 + \delta z q'(z) \prec (\delta+\kappa)\left(p(z)\right)^2 + \delta z p'(z).$$

The assertion (3.6) of Theorem 3.3 now follows by an application of Lemma 2.3. ∎

Corollary 3.4 *Let* $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ *in Theorem* 3.3. *Further, assume that* (3.4) *holds.*

If $f \in \mathcal{A}(p)$ such that $\frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^\epsilon} \in \mathcal{H}[q(0),1]\cap Q$, $\zeta(n,\lambda,p,\beta,\delta,\alpha;z)$ *is univalent in $\mathbb{U}$ and the following superordination condition*

$$\frac{\delta(A-B)z}{(1+Bz)^2} + (\delta+\kappa)\left(\frac{1+Az}{1+Bz}\right)^2 \prec \zeta(n,\lambda,p,\beta,\delta,\alpha;z),$$

holds, then

$$\frac{1+Az}{1+Bz} \prec \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best subordinant.*

Also, let $q(z) = \dfrac{1+z}{1-z}$, *then for* $f \in \mathcal{A}(p)$ *we have*

$$\frac{2\delta z}{(1-z)^2} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^2 \prec \zeta(n,\lambda,p,\beta,\delta,\alpha;z),$$

then

$$\frac{1+z}{1-z} \prec \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}}$$

and the function $\dfrac{1+z}{1-z}$ *is the best subordinant.*

Finally, by taking $q(z) = \left(\dfrac{1+z}{1-z}\right)^{\mu}$, $(0<\mu\leq 1)$, *then for* $f \in \mathcal{A}(p)$ *we have*

$$\frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^{2\mu} \prec \zeta(n,\lambda,p,\beta,\delta,\alpha;z),$$

then

$$\left(\frac{1+z}{1-z}\right)^{\mu} \prec \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}},$$

and the function $\left(\dfrac{1+z}{1-z}\right)^{\mu}$ *is the best subordinant.*

Combining Theorem 3.1 and Theorem 3.3, we get the following sandwich theorem.

Theorem 3.5 *Let* q_1 *and* q_2 *be convex univalent in* $\mathbb{U}$ *with* $q_1(0) = q_2(0) = 1$ *and satisfy* (3.1) *and* (3.4) *respectively.*

If $f \in \mathcal{A}(p)$ *such that* $\dfrac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \in \mathcal{H}[q(0),1]\cap Q$, $\zeta(n,\lambda,p,\beta,\delta,\alpha;z)$ *is univalent in* $\mathbb{U}$ *and*

$$(\delta+\kappa)\left(q_1(z)\right)^2 + \delta z q_1'(z) \prec \zeta(n,\lambda,p,\beta,\delta,\alpha;z) \prec (\delta+\kappa)\left(q_2(z)\right)^2 + \delta z q_2'(z),$$

holds, then

$$q_1(z) \prec \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \prec q_2(z),$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Corollary 3.6 *Let* $q_i(z) = \dfrac{1+A_i z}{1+B_i z}$ $(i = 1, 2; -1 \leq B_2 < B_1 < A_1 \leq A_2 \leq 1)$ *in Theorem 3.5.*

If $f \in \mathcal{A}(p)$ *such that* $\dfrac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \in \mathcal{H}[q(0), 1] \cap Q$, $\zeta(n, \lambda, p, \beta, \delta, \alpha; z)$ *is univalent in* $\mathbb{U}$ *and*

$$\frac{\delta(A_1 - B_1)z}{(1+B_1 z)^2} + (\delta+\kappa)\left(\frac{1+A_1 z}{1+B_1 z}\right)^2 \prec \zeta(n, \lambda, p, \beta, \delta, \alpha; z)$$
$$\prec \frac{\delta(A_2 - B_2)z}{(1+B_2 z)^2} + (\delta+\kappa)\left(\frac{1+A_2 z}{1+B_2 z}\right)^2,$$

holds, then

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \prec \frac{1+A_2 z}{1+B_2 z},$$

and $\dfrac{1+A_1 z}{1+B_1 z}$ *and* $\dfrac{1+A_2 z}{1+B_2 z}$ *are, respectively, the best subordinant and the best dominant.*

4. Subordinations and superordinations results for $F^m_p(\rho,\vartheta)f(z)$

In this section, we derive subordination and superordination results for $p-$valent functions involving the integral operator $F^m_p(\rho,\vartheta)f(z)$ given by (1.7).

Theorem 4.1 *Let* $q(z)$ *be univalent in* $\mathbb{U}$ *with* $q(0) = 1$*. Further, assume that*

$$\text{(4.1)} \qquad \operatorname{Re}\left\{\frac{2(\delta+\kappa)q(z)}{\delta} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0. \ (\delta \in \mathbb{C}^* = \mathbb{C} - \{0\}, \kappa \geq 0),$$

If $f \in \mathcal{A}(p)$ *satisfies the following subordination condition:*

$$\text{(4.2)} \qquad \Lambda(p,\vartheta,\rho;z) \prec \delta z q'(z) + (\delta+\kappa)\left(q(z)\right)^2,$$

where

$$\Lambda(p,\vartheta,\rho;z) = \frac{\delta(\vartheta+p)}{\rho}\left[\frac{\left(F^{m-1}_p(\rho,\vartheta)f(z)\right)\left(F^{m+1}_p(\rho,\vartheta)f(z)\right) - \eta\left(F^m_p(\rho,\vartheta)f(z)\right)^2}{\left(F^{m+1}_p(\rho,\vartheta)f(z)\right)^{\eta+1}}\right]$$
$$+ \frac{\delta(\eta-1)[\vartheta+p(1-\rho)]}{\rho}\frac{\left(F^m_p(\rho,\vartheta)f(z)\right)}{\left(F^{m+1}_p(\rho,\vartheta)f(z)\right)^{\eta}} + (\delta+\kappa)\frac{\left(F^m_p(\rho,\vartheta)f(z)\right)^2}{\left(F^{m+1}_p(\rho,\vartheta)f(z)\right)^{2\eta}},$$

then

$$\frac{F^m_p(\rho,\vartheta)f(z)}{\left[F^{m+1}_p(\rho,\vartheta)f(z)\right]^{\eta}} \prec q(z),$$

and $q(z)$ *is the best dominant.*

Proof. Define a function $p(z)$ by

$$p(z) = \frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta}, \quad (z \in \mathbb{U}). \tag{4.3}$$

Then the function $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. Differentiating (4.3) logarithmically with respect to z and using the identity (1.8) in the resulting equation, we have

$$\begin{aligned} &= \frac{\delta(\vartheta+p)}{\rho}\left[\frac{\left(F_p^{m-1}(\rho,\vartheta)f(z)\right)\left(F_p^{m+1}(\rho,\vartheta)f(z)\right) - \eta\left(F_p^m(\rho,\vartheta)f(z)\right)^2}{\left(F_p^{m+1}(\rho,\vartheta)f(z)\right)^{\eta+1}}\right] \\ &\quad + \frac{\delta(\eta-1)\left[\vartheta+p(1-\rho)\right]}{\rho}\frac{\left(F_p^m(\rho,\vartheta)f(z)\right)}{\left(F_p^{m+1}(\rho,\vartheta)f(z)\right)^\eta} + (\delta+\kappa)\frac{\left(F_p^m(\rho,\vartheta)f(z)\right)^2}{\left(F_p^{m+1}(\rho,\vartheta)f(z)\right)^{2\eta}} \\ &= \delta z p'(z) + (\delta+\kappa)\left(p(z)\right)^2, \end{aligned}$$

that is,

$$\delta z p'(z) + (\delta+\kappa)\left(p(z)\right)^2 \prec \delta z q'(z) + (\delta+\kappa)\left(q(z)\right)^2.$$

Therefore, Theorem 4.1 now follows by applying Lemma 2.2 by setting

$$\theta(w) = (\delta+\kappa)w^2 \text{ and } \varphi(w) = \delta.$$

■

Corollary 4.2 *Let* $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ *in Theorem* 4.1. *Further, assume that* (4.1) *holds.*

If $f \in \mathcal{A}(p)$ *satisfies the following subordination condition:*

$$\Lambda(p,\vartheta,\rho;z) \prec \frac{\delta(A-B)z}{(1+Bz)^2} + (\delta+\kappa)\left(\frac{1+Az}{1+Bz}\right)^2,$$

then

$$\frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta} \prec \frac{1+Az}{1+Bz},$$

and the function $\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Also, let $q(z) = \dfrac{1+z}{1-z}$. *Then, for* $f \in \mathcal{A}(p)$, *we have*

$$\Lambda(p,\vartheta,\rho;z) \prec \frac{2\delta z}{(1-z)^2} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^2,$$

then

$$\frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta} \prec \frac{1+z}{1-z},$$

and the function $\frac{1+z}{1-z}$ *is the best dominant.*

Finally, by taking $q(z) = \left(\frac{1+z}{1-z}\right)^{\mu}$, $(0 < \mu \leq 1)$, *then for* $f \in \mathcal{A}(p)$ *we have,*

$$\Lambda(p,\vartheta,\rho;z) \prec \frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^{2\mu},$$

then

$$\frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^{\eta}} \prec \left(\frac{1+z}{1-z}\right)^{\mu},$$

and the function $\left(\frac{1+z}{1-z}\right)^{\mu}$ *is the best dominant.*

Next, by appealing to Lemma 2.3 we prove the following.

Theorem 4.3 *Let* $q(z)$ *be convex univalent in* $\mathbb{U}$ *with* $q(0) = 1$. *Assume that*

$$\operatorname{Re}\left\{\frac{2(\delta+\kappa)q(z)q'(z)}{\delta}\right\} > 0. \tag{4.4}$$

Let $f \in \mathcal{A}(p)$ *such that* $\frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^{\eta}} \in \mathcal{H}[q(0),1] \cap Q$, $\Lambda(p,\vartheta,\rho;z)$ *is univalent in* $\mathbb{U}$ *and the following superordination condition*

$$(\delta+\kappa)\left(q(z)\right)^2 + \delta z q'(z) \prec \Lambda(p,\vartheta,\rho;z), \tag{4.5}$$

holds, then

$$q(z) \prec \frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^{\eta}}, \tag{4.6}$$

and $q(z)$ *is the best subordinant.*

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^{\eta}}.$$

Then from the assumption of Theorem 4.3, the function $p(z)$ is analytic in $\mathbb{U}$ and (4.4) holds. Hence, the subordination (4.5) is equivalent to

$$(\delta+\kappa)\left(q(z)\right)^2 + \delta z q'(z) \prec (\delta+\kappa)\left(p(z)\right)^2 + \delta z p'(z),$$

The assertion (4.6) of Theorem 4.3 now follows by an application of Lemma 2.3. ∎

Corollary 4.4 *Let* $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ *in Theorem* 4.3. *Further, assume that* (4.4) *holds.*

If $f \in \mathcal{A}(p)$ *such that* $\dfrac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta} \in \mathcal{H}[q(0),1] \cap Q$, $\Lambda(p,\vartheta,\rho;z)$ *is univalent in* $\mathbb{U}$ *and the following superordination condition*

$$\frac{\delta(A-B)z}{(1+Bz)^2} + (\delta+\kappa)\left(\frac{1+Az}{1+Bz}\right)^2 \prec \Lambda(p,\vartheta,\rho;z),$$

holds, then

$$\frac{1+Az}{1+Bz} \prec \frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best subordinant.*

Also, let $q(z) = \dfrac{1+z}{1-z}$, *then for* $f \in \mathcal{A}(p)$ *we have*

$$\frac{2\delta z}{(1-z)^2} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^2 \prec \Lambda(p,\vartheta,\rho;z),$$

then

$$\frac{1+z}{1-z} \prec \frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta},$$

and the function $\dfrac{1+z}{1-z}$ *is the best subordinant.*

Finally, by taking $q(z) = \left(\dfrac{1+z}{1-z}\right)^\mu$, $(0 < \mu \leq 1)$, *then for* $f \in \mathcal{A}(p)$ *we have*

$$\frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^{2\mu} \prec \Lambda(p,\vartheta,\rho;z),$$

then

$$\left(\frac{1+z}{1-z}\right)^\mu \prec \frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta},$$

and the function $\left(\dfrac{1+z}{1-z}\right)^\mu$ *is the best subordinant.*

Combining Theorem 4.1 and Theorem 4.3, we get the following sandwich theorem.

Theorem 4.5 *Let* q_1 *and* q_2 *be convex univalent in* $\mathbb{U}$ *with* $q_1(0) = q_2(0) = 1$ *and satisfy* (4.1) *and* (4.4) *respectively.*

If $f \in \mathcal{A}(p)$ *such that* $\dfrac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta} \in \mathcal{H}[q(0),1] \cap Q$, $\Lambda(p,\vartheta,\rho;z)$ *is* *univalent in* $\mathbb{U}$ *and*

$$(\delta+\kappa)\left(q_1(z)\right)^2+\delta z q_1'(z) \prec \Lambda(p,\vartheta,\rho;z) \prec (\delta+\kappa)\left(q_2(z)\right)^2+\delta z q_2'(z),$$

holds, then

$$q_1(z) \prec \frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta} \prec q_2(z),$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Corollary 4.6 *Let* $q_i(z)=\dfrac{1+A_iz}{1+B_iz}$ $(i=1,2;-1\le B_2<B_1<A_1\le A_2\le 1)$ *in Theorem 4.5.*

If $f\in\mathcal{A}(p)$ *such that* $\dfrac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta}\in\mathcal{H}[q(0),1]\cap Q$, $\Lambda(p,\vartheta,\rho;z)$ *is univalent in* $\mathbb{U}$ *and*

$$\frac{\delta(A_1-B_1)z}{(1+B_1z)^2}+(\delta+\kappa)\left(\frac{1+A_1z}{1+B_1z}\right)^2 \prec \Lambda(p,\vartheta,\rho;z) \prec \frac{\delta(A_2-B_2)z}{(1+B_2z)^2}+(\delta+\kappa)\left(\frac{1+A_2z}{1+B_2z}\right)^2,$$

holds, then

$$\frac{1+A_1z}{1+B_1z} \prec \frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta} \prec \frac{1+A_2z}{1+B_2z},$$

and $\dfrac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *are, respectively, the best subordinant and the best dominant.*

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