GRADED SEMIPRIME AND GRADED WEAKLY SEMIPRIME IDEALS[1]

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Abstract. Let G be a group and R be a commutative G-graded ring with nonzero unity 1. In this article, we define the graded semiprime ideals and the graded weakly semiprime ideals and we introduce several results concerning such ideals.

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1. Introduction and preliminaries

Recall from [2] that a proper ideal I of R ($I \neq R$) is said to be weakly prime if $a, b \in R$ and $0 \neq ab \in I$ implies $a \in I$ or $b \in I$ and recall from [5] that a proper ideal I of R is said to be weakly semiprime if $a \in R$ and $0 \neq a^2 \in I$ implies $a \in I$. The purpose of this paper is to study a generalization of weakly prime (semiprime) ideals of commutative rings to the content of commutative G-graded rings.

Let G be a group with identity e. A ring R is said to be G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g and R_e (the identity component of R) is a subring of R and $1 \in R_e$. For $x \in R$, x can be written uniquely as $\sum_{g\in G} x_g$ where x_g is the component of x in R_g. Also we write $h(R) = \bigcup_{g\in G} R_g$ and $supp(R, G) = \{g \in G : R_g \neq 0\}$. For more details, see [8].

Let R be a G-graded ring and I be an ideal of R. Then I is called G-graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., if $x \in I$ and $x = \sum_{g\in G} x_g$, then $x_g \in I$ for all $g \in G$. An ideal of a G-graded ring need not be G-graded. To see this, consider

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$R = \mathbf{Z}[i]$ and $G = \mathbf{Z}_2$. Then R is G-graded by $R_0 = \mathbf{Z}$ and $R_1 = i\mathbf{Z}$. Now, $I = \langle 1+i \rangle$ is an ideal of R with $1+i \in I$. If I is G-graded, then $1 \in I$, so $1 = a(1+i)$ for some $a \in R$, i.e., $1 = (x+iy)(1+i)$ for some $x, y \in \mathbf{Z}$. Thus $1 = x - y$ and $0 = x + y$, i.e., $2x = 1$ and hence $x = \frac{1}{2}$ a contradiction. So, I is not G-graded.

A proper graded ideal I of R is said to be graded prime if whenever $a, b \in h(R)$ such that $ab \in I$, then either $a \in I$ or $b \in I$. Also, a proper graded ideal I of R is said to be graded weakly prime if whenever $a, b \in h(R)$ such that $0 \neq ab \in I$, then either $a \in I$ or $b \in I$. Recently, various generalizations of graded prime (graded weakly prime) ideals are studied in [1], [3], [7], [9].

In this article, we define a proper graded ideal I of R to be graded semiprime if whenever $a \in h(R)$ such that $a^2 \in I$, then $a \in I$ and we define a proper graded ideal I of R to be graded weakly semiprime if whenever $a \in h(R)$ such that $0 \neq a^2 \in I$, then $a \in I$.

In this article, we give an analogous study to that is given in [6].

2. Graded semiprime ideals

In this section, we define the graded semiprime ideals and we introduce several properties of such ideal.

Definition 2.1. Let R be a graded ring and I be a proper graded ideal of R. Then I is said to be graded semiprime if whenever $a \in h(R)$ such that $a^2 \in I$, then $a \in I$.

Clearly, every graded prime ideal is a graded semiprime ideal. However, the next example shows that the converse need not be true in general.

Example 2.2. Consider $R = \mathbf{Z}[i]$ and $G = \mathbf{Z}_2$. Then R is G-graded by $R_0 = \mathbf{Z}$ and $R_1 = i\mathbf{Z}$. Consider the graded ideal $I = \langle 6 \rangle \bigoplus \langle 0 \rangle$. Let $(a, b)^2 \in I$. Then $(a^2, b^2) \in I$ and then 6 divides a^2 and $b = 0$. Since 2 and 3 divide 6, 2 and 3 divide a^2 and since 2 and 3 are primes, 2 and 3 divide a and since 2 and 3 are relatively prime, 6 divides a and hence $(a, b) \in I$. So, I is graded semiprime. However, I is not graded prime since $(2, 0)(3, 0) \in I$ but $(2, 0) \notin I$ and $(3, 0) \notin I$.

Theorem 2.3. *If I is a graded semiprime ideal of a graded ring R, then I_e is a semiprime ideal of R_e.*

Proof. Let $a \in R_e$ such that $a^2 \in I_e$. Since $R_e \subseteq h(R)$ and $I_e \subseteq I$, $a \in h(R)$ such that $a^2 \in I$ and since I is graded semiprime, $a \in I$ and then $a \in I \bigcap R_e = I_e$. Hence I_e is a semiprime ideal of R_e. ∎

Let R be a graded ring, I be a graded ideal of R and $a \in h(R)$. Then one can prove that $(I : a) = \{r \in R : ra \in I\}$ is an ideal of R. Clearly, if $a \in I$, then $(I : a) = R$. Moreover, we introduce the following result. Before that, we need to remember that $a \in R$ is said to be idempotent if $a^2 = a$.

Theorem 2.4. *Let R be a graded ring, I be a graded semiprime ideal of R and $a \in h(R)$. If a is an idempotent and $a \notin I$, then $(I : a)$ is a graded semiprime ideal of R.*

Proof. Since $a \notin I$, $(I : a)$ is a proper ideal of R. Firstly, we prove that $(I : a)$ is a graded ideal of R. Let $r \in (I : a)$. Then $r \in R$ such that $ra \in I$. Since R is graded,

$$r = \sum_{g \in G} r_g, \text{ where } r_g \in R_g$$

and then

$$\sum_{g \in G} r_g a = \left(\sum_{g \in G} r_g \right) a = ra \in I.$$

Since I is graded, $r_g a \in I$ for all $g \in G$ and then $r_g \in (I : a)$ for all $g \in G$ and hence $(I : a)$ is a graded ideal of R. Now, let $r \in h(R)$ such that $r^2 \in (I : a)$. Then $r^2 a \in I$ and since a is idempotent, $(ra)^2 = r^2 a^2 = r^2 a \in I$. Since $r, a \in h(R)$, $r \in R_g$ and $a \in R_h$ for some $g, h \in G$ and then $ra \in R_g R_h \subseteq R_{gh} \subseteq h(R)$ such that $(ra)^2 \in I$. Since I is graded semiprime, $ra \in I$ and then $r \in (I : a)$. Hence, $(I : a)$ is a graded semiprime ideal of R. ■

Theorem 2.5. *Let R be a graded ring and J be a graded ideal of R. If I is a graded semiprime ideal of R such that $J^2 \subseteq I$, then $J \subseteq I$.*

Proof. Let $a \in J$. Then since J is graded, $a = \sum_{g \in G} a_g$ with $a_g \in J$ for all $g \in G$ and then $a_g \in h(R)$ such that $a_g^2 \in J^2 \subseteq I$ for all $g \in G$. Since I is graded semiprime, $a_g \in I$ for all $g \in G$ and then $a = \sum_{g \in G} a_g \in I$. Hence, $J \subseteq I$. ■

S.E. Atani and F. Farzalipour in [4] define a proper graded ideal I of a graded ring R to be graded secondary if for every $r \in h(R)$, either $rI = I$ or $r^n I = 0$ for some integer n. We introduce the following:

Definition 2.6. *Let R be a graded ring and I be a proper graded ideal of R. Then I is said to be graded 2-secondary if for every idempotent $r \in h(R)$, either $rI = I$ or $rI = 0$.*

Theorem 2.7. *Let I be a graded 2-secondary ideal of a graded ring R and J be a graded semiprime subideal of I. Then J is a graded 2-secondary ideal of R.*

Proof. Let $r \in h(R)$ be an idempotent. Since I is graded 2-secondary, either $rI = I$ or $rI = 0$. If $rI = 0$, then $rJ \subseteq rI = 0$ and then $rJ = 0$. Suppose $rI = I$. We show that $rJ = J$. Clearly, $rJ \subseteq J$. Let $a \in J$. Since J is graded, $a = \sum_{g \in G} a_g$ with $a_g \in J$ for all $g \in G$ and then $a_g \in I = rI$ for all $g \in G$. So, for every $g \in G$, $a_g = rx_h$ for some $x_h \in I = rI$ and then $x_h = ry_k$ for some $y_k \in I$. So, $a_g = r^2 y_k = ry_k$ and then $(ry_k)^2 = a_g^2 \in J^2 \subseteq J$. Since J is graded semiprime, $ry_k \in J$ that is $x_h \in J$ and then $a_g = rx_h \in rJ$ for all $g \in G$ and hence $a = \sum_{g \in G} a_g \in rJ$. Thus $rJ = J$ and so J is a graded 2-secondary ideal of R. ∎

The proof of the next result is straightforward by Theorem 2.7.

Corollary 2.8. *Let I be a graded 2-secondary ideal of a graded ring R and J be a graded semiprime ideal of R. Then $I \bigcap J$ is a graded 2-secondary ideal of R.*

If R is a graded ring and I is an ideal of R, then R/I is a graded ring by $(R/I)_g = (R_g + I)/I$. $a \in R$ is said to be simple nilpotent if $a^2 = 0$.

Theorem 2.9. *Let I be a graded ideal of a graded ring R. Then I is a graded semiprime ideal of R if and only if R/I has no nonzero homogeneous simple nilpotent elements.*

Proof. Suppose I is a graded semiprime ideal of R. Let $a + I \in h(R/I)$ be a simple nilpotent. Then $(a + I)^2 = 0 + I$ and then $a^2 + I = 0 + I$ that is $a^2 \in I$. Since $a + I \in h(R/I)$, $a + I \in (R/I)_g = (R_g + I)/I$ for some $g \in G$ and then $a \in R_g \subseteq h(R)$. So, $a \in h(R)$ such that $a^2 \in I$ and since I is graded semiprime, $a \in I$ that is $a + I = 0 + I$. Hence, R/I has no nonzero homogeneous simple nilpotent elements. Conversely, let $a \in h(R)$ such that $a^2 \in I$. Then $(a + I)^2 = a^2 + I = 0 + I$ and by assumption, $a + I = 0 + I$ that is $a \in I$. Hence, I is a graded semiprime ideal of R. ∎

Theorem 2.10. *Let $I \subseteq J$ be proper graded ideals of a graded ring R. Then J is a graded semiprime ideal of R if and only if J/I is a graded semiprime ideal of R/I.*

Proof. Suppose J is a graded semiprime ideal of R. Firstly, we prove that J/I is graded. Let $r + I \in J/I$. Then $r + I \in R/I$ and since R/I is graded, $r+I = \sum_{g \in G}(r+I)_g$ where $(r+I)_g \in (R/I)_g = (R_g+I)/I$ and then $\left(\sum_{g \in G} r_g\right) + I = \sum_{g \in G}(r_g + I) = \sum_{g \in G}(r + I)_g = r + I \in J/I$ and hence $\sum_{g \in G} r_g \in J$. Since J is graded, $r_g \in J$ for all $g \in G$ and then $(r + I)_g = r_g + I \in J/I$ for all $g \in G$ and hence J/I is graded ideal of R/I. Now, let $a + I \in h(R/I)$ such that $(a + I)^2 \in J/I$. Then $a \in h(R)$ such that $a^2 \in J$ and since J is graded semiprime, $a \in J$ and then $a + I \in J/I$. Hence, J/I is a graded semiprime ideal of R/I. Conversely, let $a \in h(R)$ such that $a^2 \in J$. Then $a+I \in h(R/I)$ such that $(a+I)^2 = a^2+I \in J/I$ and since J/I is graded semiprime, $a+I \in J/I$ that is $a \in J$. Hence, J is a graded semiprime ideal of R. ∎

If R and R' are two G-graded rings, then $R \times R'$ is a G-graded ring by $(R \times R')_g = R_g \times R'_g$.

Theorem 2.11. *Let R and R' be two G-graded rings and I be a graded ideal of R. Then I is a graded semiprime ideal of R if and only if $I \times R'$ is a graded semiprime ideal of $R \times R'$.*

Proof. Suppose I is a graded semiprime ideal of R. Firstly, we prove that $I \times R'$ is graded. Let $(x, y) \in I \times R'$. Then $(x, y) \in R \times R'$ and since $R \times R'$ is graded, $(x, y) = \sum_{g \in G} (x, y)_g$ where $(x, y)_g \in (R \times R')_g = R_g \times R'_g$ and then $\left(\sum_{g \in G} x_g, \sum_{g \in G} y_g\right) = \sum_{g \in G} (x_g, y_g) = \sum_{g \in G} (x, y)_g = (x, y) \in I \times R'$ and hence $\sum_{g \in G} x_g \in I$. Since I is graded, $x_g \in I$ for all $g \in G$ and then $(x, y)_g = (x_g, y_g) \in I \times R'$ for all $g \in G$ and hence $I \times R'$ is a graded ideal of $R \times R'$. Now, let $(a, b) \in h(R \times R')$ such that $(a, b)^2 \in I \times R'$. Then $a \in h(R)$ such that $a^2 \in I$ and since I is graded semiprime, $a \in I$ and then $(a, b) \in I \times R'$. Hence, $I \times R'$ is a graded semiprime ideal of $R \times R'$. Conversely, let $a \in h(R)$ such that $a^2 \in I$. Then $(a, 1') \in h(R) \times R'_e \subseteq h(R) \times h(R') = h(R \times R')$ such that $(a, 1')^2 \in I \times R'$ and since $I \times R'$ is graded semiprime, $(a, 1') \in I \times R'$ that is $a \in I$. Hence, I is a graded semiprime ideal of R. ∎

Similarly, we can prove the following:

Theorem 2.12. *Let R and R' be two G-graded rings and J be a graded ideal of R'. Then J is a graded semiprime ideal of R' if and only if $R \times J$ is a graded semiprime ideal of $R \times R'$.*

3. Graded weakly semiprime ideals

In this section, we define the graded weakly semiprime ideals and we introduce several properties of such ideal.

Definition 3.1. Let R be a graded ring and I be a proper graded ideal of R. Then I is said to be graded weakly semiprime if whenever $a \in h(R)$ such that $0 \neq a^2 \in I$, then $a \in I$.

Clearly, every graded semiprime ideal is a graded weakly semiprime ideal. However, the next example shows that the converse need not be true in general.

Example 3.2. Consider $R = M_2(K)$ (the ring of all 2×2 matrices with entries from a field K) and $G = \mathbf{Z_4}$. Then R is G-graded by

$$R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, R_2 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \text{ and } R_1 = R_3 = 0.$$

Consider $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. Then I is a graded weakly semiprime ideal of R. However, I is not graded semiprime since $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R_2 \subseteq h(R)$ such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in I$ but $A \notin I$.

Theorem 3.3. *If I is a graded weakly semiprime ideal of a graded ring R, then I_e is a weakly semiprime ideal of R_e.*

Proof. Let $a \in R_e$ such that $0 \neq a^2 \in I_e$. Since $R_e \subseteq h(R)$ and $I_e \subseteq I$, $a \in h(R)$ such that $0 \neq a^2 \in I$ and since I is graded semiprime, $a \in I$ and then $a \in I \bigcap R_e = I_e$. Hence I_e is a semiprime ideal of R_e. ■

Let R be a graded ring, I be a graded ideal of R and $a \in h(R)$. Then we define the graded ideal $(I :_{h(R)} a) = \{r \in h(R) : ra \in I\}$.

Theorem 3.4. *Let I be a graded ideal of a graded ring R. If $(I :_{h(R)} a^2) = (0 :_{h(R)} a^2) \bigcup (I :_{h(R)} a)$ for all $a \in h(R)$, then I is a graded weakly semiprime ideal of R.*

Proof. Let $a \in h(R)$ such that $0 \neq a^2 \in I$. Then $1 \in R_e \subseteq h(R)$ such that $1.a^2 \in I$ that is $1 \in (I :_{h(R)} a^2)$. By assumption and since $a^2 \neq 0$, $1 \in (I :_{h(R)} a)$ and then $a = 1.a \in I$. Hence, I is a graded weakly semiprime ideal of R. ■

Theorem 3.5. *Let $I \subseteq J$ be proper graded ideals of a graded ring R. If J is a graded weakly semiprime ideal of R, then J/I is a graded weakly semiprime ideal of R/I.*

Proof. By the proof of Theorem 2.10, J/I is a graded ideal of R/I. Let $a + I \in h(R/I)$ such that $0 \neq (a + I)^2 \in J/I$. Then $a \in h(R)$ such that $0 \neq a^2 \in J$ and since J is graded weakly semiprime, $a \in J$ and then $a + I \in J/I$. Hence, J/I is a graded weakly semiprime ideal of R/I. ■

Theorem 3.6. *Let $I \subseteq J$ be proper graded ideals of a graded ring R. If I is a graded weakly semiprime ideal of R and J/I is a graded weakly semiprime ideal of R/I, then J is a graded weakly semiprime ideal of R.*

Proof. Let $a \in h(R)$ such that $0 \neq a^2 \in J$. Then $a + I \in h(R/I)$ such that $(a + I)^2 \in J/I$. If $a^2 \in I$, then since I is graded weakly semiprime, $a \in I \subseteq J$ and then it is done. Suppose $a^2 \notin I$. Then $0 \neq (a + I)^2 \in J/I$ and since J/I is graded weakly semiprime, $a + I \in J/I$ and then $a \in J$. Hence, J is a graded weakly semiprime ideal of R. ■

Theorem 3.7. *Let I be a graded 2-secondary ideal of a graded ring R and J be a graded weakly semiprime subideal of I. If J has no nonzero homogeneous simple nilpotent elements, then J is a graded 2-secondary ideal of R.*

Proof. Let $r \in h(R)$ be an idempotent. Since I is graded 2-secondary, either $rI = I$ or $rI = 0$. If $rI = 0$, then $rJ \subseteq rI = 0$ and then $rJ = 0$. Suppose $rI = I$. We show that $rJ = J$. Clearly, $rJ \subseteq J$. Let $a \in J$. If $a = 0$, then $a = r.0 \in rJ$. Suppose $a \neq 0$. Since J is graded, $a = \sum_{g \in H} a_g$ with $a_g \in J$ for all $g \in H$ where $H = supp(R, G)$ (i.e., $a_g \neq 0$) and then $a_g \in I = rI$ for all $g \in H$. So, for every $g \in H$, $a_g = rx_h$ for some $0 \neq x_h \in I = rI$ and then $x_h = ry_k$ for some $0 \neq y_k \in I$. So, $a_g = r^2 y_k = ry_k$ and then $(ry_k)^2 = a_g^2 \in J^2 \subseteq J$. If

$(ry_k)^2 = 0$, then $a_g^2 = 0$, i.e., a_g is a homogeneous simple nilpotent element in J and then by assumption, $a_g = 0$ a contradiction. So, $(ry_k)^2 \neq 0$. Since J is graded weakly semiprime, $ry_k \in J$ that is $x_h \in J$ and then $a_g = rx_h \in rJ$ for all $g \in H$ and hence $a = \sum_{g \in H} a_g \in rJ$. Thus $rJ = J$ and so J is a graded 2-secondary ideal of R. ∎

The proof of the next result is straightforward by Theorem 3.7.

Corollary 3.8. *Let I be a graded 2-secondary ideal of a graded ring R and J be a graded weakly semiprime ideal of R. If J has no nonzero homogeneous simple nilpotent elements, then $I \bigcap J$ is a graded 2-secondary ideal of R.*

Theorem 3.9. *Let R and R' be two G-graded rings and I be a graded ideal of R. If $I \times R'$ is a graded weakly semiprime ideal of $R \times R'$, then I is a graded weakly semiprime ideal of R.*

Proof. Let $a \in h(R)$ such that $0 \neq a^2 \in I$. Then $(a, 1') \in h(R) \times R'_e \subseteq h(R) \times h(R') = h(R \times R')$ such that $0 \neq (a, 1')^2 \in I \times R'$ and since $I \times R'$ is graded semiprime, $(a, 1') \in I \times R'$ that is $a \in I$. Hence, I is a graded semiprime ideal of R. ∎

Similarly, we can prove the following:

Theorem 3.10. *Let R and R' be two G-graded rings and J be a graded ideal of R'. If $R \times J$ is a graded weakly semiprime ideal of $R \times R'$, then J is a graded weakly semiprime ideal of R'.*

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