MORITA EQUIVALENCE FOR SEMIRINGS WITHOUT IDENTITY[1]

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Abstract. In this paper, we study the Morita theory for semirings with slu. We characterize the equivalent functors between the subcategories of right semimodules over semirings with slu. Also, we give an equivalent condition of the Morita context of semirings (not necessarily with slu) and study the corresponding results in semirings with slu settings. Finally, we apply the results to semirings with identity.

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1. Introduction

Morita equivalence theory gave a characterization of equivalences between two module categories over two rings with 1. The Hom functor and tensor product functor play an important role in studying Morita equivalence theory. The Morita equivalence theory has also been studied in many other algebraic structures, such as rings without 1, semigroups, semirings, etc. In [1], Abrams generalized the Morita equivalence theory to rings without 1. Banaschewski [2] and Knauer [13] got the Morita equivalence theory for monoids. Talwar [20] and Lawson [17] investigated the Morita equivalence theory for semigroups with local units. Laan and Marki [16] continued to study the Morita contexts in semigroup settings. On the other hand, there are very few papers related to these theory for semirings. Recently, Katsov and Nam [12] initially studied these problems for a semiring with 1. In [12], they give the Morita equivalence theory for semirings with 1. Motivated by the paper [1] and [20], it is a natural thing to consider the generalized Morita theorems for semirings without 1.

In this paper, we shall study the Morita theory for semirings without 1. The paper is constructed as follows: In Section 2, we recall some notions on semirings and semimodules; In Section 3, we give the Morita theory for semirings with slu; In Section 4, we study the Morita context for semirings (not necessarily with slu); we apply the results to semirings with 1 in Section 5.

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2. Preliminaries

A semiring is a set R with two binary operations " $+$ " and " $\cdot$ " satisfying the followings:

(1) $(R, +)$ is a commutative semigroup;

(2) $(R, \cdot)$ is a semigroup;

(3) For all $a, b, c \in R$, $a(b + c) = ab + ac, (b + c)a = ba + ca$.

If a semiring R has an element 1 such that $1r = r = r1$, for all $r \in R$, then 1 is called the identity of R.

Let R be a semiring. A commutative semigroup $(M, +)$ is called a right R-semimodule over R if there is a scalar multiplication $M \times R \to M$, denoted by $(m, r) \mapsto mr$, satisfying the followings:

(1) $m(rr^{'}) = (mr)r^{'}$;

(2) $(m + m^{'})r = mr + m^{'}r$;

(3) $m(r + r^{'}) = mr + mr^{'}$;

for all $r, r^{'} \in R$ and $m, m^{'} \in M$.

If $MR = M$, then M is called unital.

Let M and N be two R-semimodules. A map $f : M \to N$ is called R-semimodule homomorphism if f satisfies the followings:

(1) $f(m_1 + m_2) = f(m_1) + f(m_2)$,

(2) $f(mr) = f(m)r$,

for all $m, m_1, m_2 \in M,\ r \in R$.

The set of all R-semimodule homomorphisms from M to N is denoted by $\mathrm{Hom}_R(M, N)$. Let $\mathrm{End}_R(M)$ be the set of all R-semimodule homomorphisms from M to itself.

Similarly, we can define left semimodules. Let Mod-R and R-Mod be the categories of right and left semimodules, respectively.

Denote by Mod-UR (UR-Mod) the subcategory of unital right (left) R-semimodules.

Definition 1. [[11], Definition 3.1] For a right semimodule $M \in$ Mod-R and a left semimodule $N \in R$-Mod, let F be the free semigroup generated by the cartesian product $M \times N$. The tensor product $M \otimes N$ is the factor semigroup $F/\sigma^{\sharp}$, where the congruence $\sigma^{\sharp}$ is generated by the relation σ of the form

$$< (m_1 + m_2, n), (m_1, n) + (m_2, n) >,\ < (m, n_1 + n_2), (m, n_1) + (m, n_2) >,$$

and

$$< (mr, n), (m, rn) >,$$

with $m_1, m_2 \in M$, $n_1, n_2 \in N$ and $r \in R$.

Let R and S be two semirings. A commutative semigroup M is called an R-S-bisemimodule if it is both a left R-semimodule and a right S-semimodule and satisfies $(rm)s = r(ms)$, for all $r \in R, m \in M, s \in S$, denoted by ${}_RM_S$. For ${}_RM_S$ and ${}_RN_S$, if a semigroup homomorphism $f : M \to N$ satisfies $rf(m)s = f(rms)$, then f is called bisemimodule homomorphism. Let ${}_R\mathrm{Hom}_S(M, N)$ be the set of all bisemimodule homomorphisms from M to N. Denote by R-Mod-S the category of all R-S-bisemimodules together with bisemimodule homomorphisms.

Let M be an S-R-bisemimodule and let N be a left R-semimodule. The tensor product $M \otimes_R N$ is a left S-semimodule with action

$$s \cdot \left(\sum_{i=1}^{n} m_i \otimes n_i \right) = \sum_{i=1}^{n} s \cdot m_i \otimes n_i;$$

Similarly, let M be a right R-semimodule and N be an R-S-bisemimodule, then $M \otimes_R N$ is a right S-semimodule with action

$$\left(\sum_{i=1}^{n} m_i \otimes n_i \right) \cdot s = \sum_{i=1}^{n} m_i \otimes n_i \cdot s,$$

where $s \in S, m_i \in M, n_i \in N$.

For an R-S-bisemimodule M and a left R-semimodule$N$, we have that $\mathrm{Hom}_R(M, N)$ is a left S-semimodule with action $(s \cdot f)(m) = f(m \cdot s)$; For an S-R-bisemimodule M and a right R-semimodule N, we have that $\mathrm{Hom}_R(M, N)$ is a right S-semimodule with action $(f \cdot s)(m) = f(s \cdot m)$; For a right R-semimodule and an S-R-bisemimodule N, we have that $\mathrm{Hom}_R(M, N)$ is a left S-semimodule with action $(s \cdot f)(m) = sf(m)$; For a left R-semimodule M and a R-S-bisemimodule N, we have that $\mathrm{Hom}_R(M, N)$ is a right S-semimodule with action $(f \cdot s)(m) = f(m)s$, for all $s \in S, f \in \mathrm{Hom}_R(M, N), m \in M$.

3. Morita equivalence for semirings with slu

In paper [10], Katsov proved that the tensor product functor $- \otimes_R B$ and Hom functor $\mathrm{Hom}_S(B, -)$ are adjoint pair, where R and S are semirings with identity and $B \in R$-Mod-S. In the following, we will prove that the statement also holds when R and S are semirings without identity.

Let $\mathcal{R}^\sharp$ be the smallest congruence on a semigroup S containing the relation $\mathcal{R}$. For $a, b \in S$, if there exist $u, v \in S \bigcup \{1\}$ such that

$$a = ucv, \ b = udv,$$

where $(c, d) \in \mathcal{R}$ or $(d, c) \in \mathcal{R}$, we say that a is connected to b by an elementary $\mathcal{R}$-transition [8].

Proposition 1. [8] *Let S be a semigroup and let $\mathcal{R}$ be a relation on S. For $a, b \in S$, we have $(a, b) \in \mathcal{R}^\sharp \Leftrightarrow$ one of the two conditions holds:*

(1) $a = b$.

(2) *there exists a sequence*

$$a = c_1 \to c_2 \to \cdots \to c_n = b$$

of elementary $\mathcal{R}$-transitions connecting a to b, where n is a positive integer.

Lemma 1. *Let R and S be two semirings. Assume $g : A \to Hom_S(B, C)$ is an R-semimodule homomorphism, where $A \in R$-Mod, $B \in S$-Mod-R and $C \in S$-Mod. For all $a \in A$, we write $g(a) = g_a$. Then*

(1) *If (b, a) is connected to $(b^{'}, a^{'})$ by an elementary σ-transition, where σ is the relation defined in Definition 1, we have $g_a(b) = g_{a'}(b^{'})$.*

(2) *We can define an S-semimodule homomorphism $g^{'} : B \otimes_R A \to C$ by $g^{'}\left(\sum_{i=1}^{n} b_i \otimes a_i\right) = \sum_{i=1}^{n} g_{a_i}(b_i)$.*

Proof. (1) Suppose (b, a) and $(b^{'}, a^{'})$ is connected by by an elementary σ-transition. Let $\mathcal{S}(B \times A)$ be the commutative free semigroup generated by $B \times A$. There exists (x, y) belongs to $\mathcal{S}(B \times A) \bigcup \{(0, 0)\}$ such that

$$\begin{aligned} (b, a) &= (x, y) + (c, d), \\ (b', a') &= (x, y) + (c', d'), \end{aligned}$$

where $((c, d),(c', d')) \in \sigma$ or $((c', d'),(c, d)) \in \sigma$.

Since $B \times A$ is the generating set and the elements in $B \times A$ are independent, we have $(x, y) = (0, 0)$. This implies that $((b, a),(b', a')) \in \sigma$ or $((b', a'),(b, a)) \in \sigma$. Without loss of generality, we assume $((b, a),(b', a')) \in \sigma$. Note that (b, a) and (b', a') are independent, there exists $r \in R$ such that $b = b'r, ra = a'$. Since g is an R-semimodule homomorphism, we have $r \cdot g(a) = g(r \cdot a)$, for all $a \in A$. Then, we get $r \cdot g_a = g_{ra}$. Hence, we have

$$g_a(b) = g_a(b'r) = (rg_a)(b') = g_{ra}(b') = g_{a'}(b').$$

(2) If $b_1 \otimes a_1 = b_2 \otimes a_2$, then $(b_1, a_1) = (b_2, a_2)$ or (b_1, a_1) is connected to (b_2, a_2) by an elementary σ-transition by Proposition 1. By part (1), we have $g_{a_1}(b_1) = g_{a_2}(b_2)$. On the other hand, if $b \otimes a = \sum_{i=1}^{n} b_i \otimes a$, then we have $\left((b, a), \left(\sum_{i=1}^{n} b_i, a\right)\right) \in \sigma$ and $b = \sum_{i=1}^{n} b_i$. This concludes that $g_a(b) = \sum_{i=1}^{n} g_a(b_i)$. Similarly, if $b \otimes a = \sum_{i=1}^{n} b \otimes a_i$, we get $g_a(b) = \sum_{i=1}^{n} g_{a_i}(b)$. Then we prove that $g^{'}$ is well-defined.

For all $s \in S$, we have

$$g^{'}(s \sum_{i=1}^{n} b_i \otimes a_i) = g^{'}\left(\sum_{i=1}^{n} sb_i \otimes a_i\right) = \sum_{i=1}^{n} g_{a_i}(sb_i) = s \sum_{i=1}^{n} g_{a_i}(b_i) = sg^{'}\left(\sum_{i=1}^{n} b_i \otimes a_i\right).$$

Therefore, $g^{'}$ is an S-semimodule homomorphism. ■

Theorem 1. *Let R and S be two semirings. For $A \in R$-Mod, $B \in S$-Mod-R and $C \in S$-Mod, there is a semigroup isomorphism*

$$\Delta : Hom_S(B \otimes_R A, C) \cong Hom_R(A, Hom_S(B, C)).$$

Similarly, for $A \in$ Mod-R, $B \in R$-Mod-S and $C \in$ Mod-S, there is a semigroup isomorphism

$$\Omega : Hom_S(A \otimes_R B, C) \cong Hom_R(A, Hom_S(B, C)).$$

Proof. We only prove the first isomorphism. If $f \in \mathrm{Hom}_S(B \otimes_R A, C)$, for all $a \in A$, we can define a map $f_a : B \to C$ by $f_a(b) = f(b \otimes a)$. Then we can check that the map $\bar{f} : A \to \mathrm{Hom}_S(B, C)$ by putting $\bar{f}(a) = f_a$ is an S-semimodule homomorphism. It is a routine procedure to check that $\Delta : f \mapsto \bar{f}$ is a homomorphism.

Now, we exhibit the inverse of Δ. Define $\Delta^{-1} : \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \to \mathrm{Hom}_S(B \otimes_R A, C)$ by putting $\Delta(g) = g^{'}$, where $g^{'}$ is defined as in Lemma 1. Hence, Δ is a semigroup isomorphism. ∎

By Theorem 1, we have that $(- \otimes_R B, \mathrm{Hom}_S(B, -))$ is an adjoint pair. We write $\mathcal{L} = - \otimes_R B$ and $\mathcal{K} = \mathrm{Hom}_S(B, -)$. Hence, we have the following canonical transformations

$$\upsilon : \mathcal{LK} \to Id_{\text{Mod-}S} \quad \text{and} \quad \eta : Id_{\text{Mod-}R} \to \mathcal{KL}$$

such that

$$\upsilon_C(\varphi \otimes b) = \varphi(b) \quad \text{and} \quad \eta_A(a)(b) = a \otimes b,$$

where $\varphi \in HB, b \in B$ and $a \in A$.

In particular, we take $A = B = S = R$, then we have $\upsilon_C : \mathrm{Hom}_R(R, C) \otimes_R R \to R$ by putting $\upsilon_C\left(\sum_{i=1}^{n} \varphi_i \otimes r_i\right) = \sum_{i=1}^{n} \varphi_i(r_i)$, where $C \in$ Mod-R.

Let R be a semiring and $C \in$ Mod-R. For $c \in C$, we define $\lambda_c : R \to C$ by putting $\lambda_c(r) = cr$. Then $\lambda_c \in \mathrm{Hom}_R(R, C)$. For all $\varphi \in \mathrm{Hom}_R(R, C)$, $r \in R$, since

$$(\varphi \cdot r)(a) = \varphi(ra) = \varphi(r)a = \lambda_{\varphi(r)}(a),$$

where $a \in R$, we have $\varphi \cdot r = \lambda_{\varphi(r)}$.

Definition 2. Let R be a semiring and E be an idempotents set of R. If for any finite number of elements $r_1, ..., r_n \in R$, there exists $e \in E$ such that

$$er_1 = r_1 = r_1e, ..., er_n = r_n = r_ne,$$

then E is called a set of local units of R (abbreviated slu). In this case, we say R is a semiring with slu.

Obviously, if R is a semiring with 1, then R is a semiring with slu. If R is an idempotent commutative (multiplication) semiring, by induction, we can prove that R is a semiring with slu.

Theorem 2. *Let R be a semiring with local units. Then we have $End_R(R) \otimes_R R \cong R$ as R-semimodules.*

Proof. For all $r \in R$, there exists $e \in E$ such that $er = re = r$. Define $\mu_R : R \to \mathrm{End}_R(R) \otimes_R R$ by putting $\mu_R(r) = \lambda_r \otimes e$. Suppose $f \in E$ with $rf = fr = r$. Since $\lambda_r = \lambda_e \cdot r$, we have

$$\lambda_r \otimes e = \lambda_e \cdot r \otimes e = \lambda_e \otimes re = \lambda_e \otimes r = \lambda_e \otimes rf = \lambda_e \cdot r \otimes f = \lambda_r \otimes f.$$

This proves that μ_R is independent of the choice of the idempotent. Suppose $r_1, ..., r_n \in R$. There exists $e \in E$ such that

$$er_1 = r_1, ..., er_n = r_n.$$

Then

$$\mu_R\left(\sum_{i=1}^{n} r_i\right) = \lambda_{\sum\limits_{i=1}^{n} r_i} \otimes e = \sum_{i=1}^{n} \lambda_{r_i} \otimes e = \sum_{i=1}^{n} \mu_R(r_i).$$

For all $r \in R, a \in R$, there exists $e \in E$ such that $re = er = r,\ ea = ae = a$. Then

$$\mu_R(ar) = \lambda_{ar} \otimes e = \lambda_a \cdot r \otimes e = \lambda_a \otimes re = \lambda_a \otimes er = (\lambda_a \otimes e)r = \mu_R(a)r.$$

This proves that μ_R is an R-semimodule homomorphism. Also, we can easily check that μ_R is the inverse of the map υ_R. So we have that υ_R is an isomorphism. That is, $\mathrm{End}_R(R) \otimes_R R \cong R$ as R-semimodule. ∎

Let R be a semiring with slu and $M \in$ Mod-UR. We denote by

$$\text{Mod-}FR = \{M \in \text{Mod-}UR \mid \upsilon_M \text{ is an isomorphism}\}.$$

Definition 3. Let R and S be two semirings with slu. We say that R and S are Morita equivalent, if the two subcategories Mod-FR and Mod-FS are equivalent.

Analogous to Theorem 6.1 in [20], we have the following.

Theorem 3. *Let R and S be two semirings with slu and G : Mod-$FR \rightleftharpoons$ Mod-FS : H are equivalent functors. Set $U = H(S)$ and $V = G(R)$. Then the following conditions hold:*

(1) $G \cong Hom_R(U, -) \otimes_S S$, $\ H \cong Hom_S(V, -) \otimes_R R$;

(2) $U \cong Hom_S(V, S) \otimes_R R, V \cong Hom_R(U, R) \otimes_S S$ as bisemimodules.

Proof. For $r \in R$, we have $\lambda_r \in \mathrm{End}_R(R)$. Then $G(\lambda_r) \in \mathrm{End}_S(G(R)) = \mathrm{End}_S(V)$. For $r \in R, v \in V$, defining $r \cdot v = G(\lambda_r)(v)$. It is easy to check that V is an R-S-bisemimodule. Similarly, we have that U is an S-R-bisemimodule.

(1) Since G and H are equivalent functors, we have

$$\mathrm{Hom}_S(S, G(M)) \cong \mathrm{Hom}_R(H(S), M) = \mathrm{Hom}_R(U, M),$$

For all $M \in$ Mod-FR. We obviously have $G(M) \in$ Mod-FS. By the definition of Mod-FS, we have

$$G(M) \cong \mathrm{Hom}_S(S, G(M)) \otimes_S S.$$

Then $G(M) \cong \mathrm{Hom}_R(U, M) \otimes_S S$. This proves that $G \cong \mathrm{Hom}_R(U, -) \otimes_S S$. Similarly, we have $H \cong \mathrm{Hom}_S(V, -) \otimes_R R$.

(2) Let $\Gamma : G \longrightarrow \mathrm{Hom}_R(U, -) \otimes_S S$ be the natural isomorphism. We know that Γ_R is a right S-isomorphism. There exists a commutative diagram:

$$\begin{array}{ccc} G(R)(=V) & \xrightarrow{G(\lambda_r)} & G(R)(=V) \\ \downarrow{\Gamma_R} & & \downarrow{\Gamma_R} \\ \mathrm{Hom}_R(U,R) \otimes_S S & \xrightarrow[\mathrm{Hom}_{R(U,\lambda_r) \otimes_S Id_S}]{} & \mathrm{Hom}_R(U,R) \otimes_S S, \end{array}$$

where

$$(\mathrm{Hom}_R(U, \lambda_r) \otimes_S Id_S)(\varphi \otimes t) = \lambda_r \varphi \otimes t.$$

Since $\lambda_r \varphi(u) = r\varphi(u) = (r \cdot \varphi)(u)$, for all $u \in U$, we have $\lambda_r \varphi = r \cdot \varphi$. Hence, we have

$$(\mathrm{Hom}_R(U, \lambda_r) \otimes_S Id_S)(\varphi \otimes t) = \lambda_r \varphi \otimes t = r \cdot \varphi \otimes t = r \cdot (\varphi \otimes t).$$

This concludes that $(\mathrm{Hom}_R(U, \lambda_r) \otimes_S Id_S)\Gamma_R(v) = r \cdot \Gamma_R(v)$, for all $v \in V$. Using the commutative diagram, we have

$$\Gamma_R(rv) = \Gamma_R(G(\lambda_r)(v)) = (\mathrm{Hom}_R(U, \lambda_r) \otimes_S Id_S)\Gamma_R(v) = r(\Gamma_R(v)).$$

It follows that Γ_R is a left R-semimodule homomorphism. This proves that $V \cong \mathrm{Hom}_R(U, R) \otimes_S S$ as S-R-bisemimodules. Similarly, we have $U \cong \mathrm{Hom}_S(V, S) \otimes_R R$ as R-S-bisemimodules. ∎

4. Morita context for semirings

In the following, we study the Morita context in semiring settings.

Definition 4. Let R and S be two semirings (not necessarily with slu). If there exist commutative semigroups U and V, such that

(1) U is an R-S-bisemimodule, V is an S-R-bisemimodule;

(2) there are bisemimodule homomorphisms $\tau : U \otimes_S V \to R$ and $\mu : V \otimes_R U \to S$ written corresponding as

$$\tau(u \otimes v) = \langle u, v \rangle, \quad \mu(v \otimes u) = [v, u]$$

such that

$$\langle u_1, v \rangle \cdot u_2 = u_1 \cdot [v, u_2], \quad [v_1, u] \cdot v_2 = v_1 \cdot \langle u, v_2 \rangle$$

for each $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$.

Then (R, S, U, V, τ, μ) is called a Morita context.

Analogous to Theorem 1 in [18], we get the following theorem.

Theorem 4. *Let U and V be two commutative semigroups. Then the following two conditions are equivalent:*

(1) *There exist two semirings R and S such that (R, S, U, V, τ, μ) is a Morita context.*

(2) *There exist semigroup homomorphisms $\Phi : U \otimes_Z V \otimes_Z U \to U$ and $\Psi : V \otimes_Z U \otimes_Z V \to V$ such that*

(I) $\Phi(\Phi(u_1 \otimes v_1 \otimes u_2) \otimes v_2 \otimes u_3) = \Phi(u_1 \otimes \Psi(v_1 \otimes u_2 \otimes v_2) \otimes u_3)$
$= \Phi(u_1 \otimes v_1 \otimes \Phi(u_2 \otimes v_2 \otimes u_3))$;

(II) $\Psi(\Psi(v_1 \otimes u_1 \otimes v_2) \otimes u_2 \otimes v_3) = \Psi(v_1 \otimes \Phi(u_1 \otimes v_2 \otimes u_2) \otimes v_3)$
$= \Psi(v_1 \otimes u_1 \otimes \Psi(v_2 \otimes u_2 \otimes v_3))$.

Proof. (1) $\Rightarrow$ (2) : Suppose that (R, S, U, V, τ, μ) is a Morita context. Define $\Phi : U \otimes_Z V \otimes_Z U \to U$ and $\Psi : V \otimes_Z U \otimes_Z V \to V$ by putting $\Phi(u_1 \otimes v_1 \otimes u_2) = \tau(u_1 \otimes v_1) \cdot u_2$ and $\Psi(v_1 \otimes u_1 \otimes v_2) = \mu(v_1 \otimes u_1) \cdot v_2$. We can easily check that Φ and Ψ satisfy the two conditions in 2).

(2) $\Rightarrow$ (1) : Define $F_a : U \to U$ by putting $F_a(u) = \Phi(a \otimes u)$ and define $G_b : V \to V$ by putting $G_b(v) = \Psi(b \otimes v)$, where $a \in U \otimes_Z V$ and $b \in V \otimes_Z U$. Then F_a and G_b are semigroup homomorphisms.

We write $\mathcal{F} = \{F_a | a \in U \otimes_Z V\}$ and $\mathcal{G} = \{R_b | b \in V \otimes_Z U\}$. For all $F_{u_1 \otimes v_1}, F_{u_2 \otimes v_2} \in \mathcal{F}$, for all $w \in U$, we have

$$\begin{aligned} F_{u_1 \otimes v_1} F_{u_2 \otimes v_2}(w) &= \Phi(u_1 \otimes v_1 \otimes \Phi(u_2 \otimes v_2 \otimes w)) \\ &= \Phi(\Phi(u_1 \otimes v_1 \otimes u_2) \otimes v_2 \otimes w) \\ &= F_{\Phi(u_1 \otimes v_1 \otimes u_2) \otimes v_2}(w). \end{aligned}$$

That is, $F_{u_1 \otimes v_1} F_{u_2 \otimes v_2} = F_{\Phi(u_1 \otimes v_1 \otimes u_2) \otimes v_2} \in \mathcal{F}$. It is easy to check that $F_a + F_b = F_{a+b} \in \mathcal{F}$ and multiplication distributes over addition from either side. Hence, $\mathcal{F}$ is a subsemiring of ${}_Z\mathrm{End}_Z(U)$. Similarly, we have that $\mathcal{G}$ is a a subsemiring of ${}_Z\mathrm{End}_Z(V)$.

For all $u \in U$, $F_a \in \mathcal{F}$, $G_b \in \mathcal{G}$, define $F_a \cdot u = \Phi(a \otimes u)$ and $u \cdot G_b = \Phi(u \otimes b)$. Then we can check that U is a $\mathcal{F}$-$\mathcal{G}$-bisemimodule. Similarly, for all $v \in V$, we can define $G_b \cdot v = \Psi(b \otimes v)$ and $v \cdot F_a = \Psi(v \otimes a)$. Then V is a $\mathcal{G}$-$\mathcal{F}$-bisemimodule.

Now, we define $\alpha : U \otimes_{\mathcal{G}} V \to \mathcal{F}$ and $\beta : V \otimes_{\mathcal{F}} U \to \mathcal{G}$ by putting $\alpha(u \otimes v) = F_{u \otimes v}$ and $\beta(v \otimes u) = G_{v \otimes u}$, where $u \in U$ and $v \in V$. It is easy to check that α and β are both bisemimodule homomorphisms. Then

$$\alpha(u_1 \otimes v) \cdot u_2 = F_{u_1 \otimes v} \cdot u_2 = \Phi(u_1 \otimes v \otimes u_2) = u_1 \cdot G_{v \otimes u_2} = u_1 \beta(v \otimes u_2).$$

Similarly, we have

$$\beta(v_1 \otimes u) v_2 = v_1 \alpha(u \otimes v_2).$$

Then $(\mathcal{F}, \mathcal{G}, U, V, \alpha, \beta)$ is a Morita context. ∎

Definition 5. Let R and S be two semirings with slu. A Morita context (R, S, U, V, τ, μ) is called unital, if U is a unital R-S-bisemimodule and V is a unital S-R-bisemimodule.

Lemma 2. *Let (R, S, U, V, τ, μ) be a unital Morita context. For any $u_1, ..., u_n \in U$ and $v_1, ..., v_m \in V$, then*

(1) *there exists idempotent $e \in R$ such that $eu_i = u_i$ and $v_j e = v_j$, for all $i = 1, \cdots, n; j = 1, ..., m$.*

(2) *there exists idempotent $f \in S$ such that $u_i f = u_i$ and $ev_j = v_j$, for all $i = 1, ..., n; j = 1, ..., m$.*

Proof. (1) Since U and V are unital, for every i and j, there exist $a_i \in U$, $b_j \in V$, $x_i \in R, y_j \in S$ such that

$$u_i = x_i a_i, \quad v_j = b_j y_j.$$

Since R is a semiring with slu, there exists an idempotent $e \in R$ such that $ex_i = x_i$ and $y_j e = y_j$. Then we can prove the result.

Similarly, we can prove (2). ■

We extend Lemma 3.15 in [17] to semiring settings.

Lemma 3. *Let (R, S, U, V, τ, μ) be a unital Morita context. If τ and μ are surjective, then τ and μ are isomorphisms.*

Proof. Firstly, we show that τ is injective. Suppose

$$\tau\Big(\sum_i u_i \otimes v_i\Big) = \tau\Big(\sum_j u'_j \otimes v'_j\Big), \text{ i.e., } \sum_i \langle u_i\, v_i\rangle = \sum_j \langle u'_j, v'_j\rangle .$$

There exist idempotents $e, f \in R$ such that $eu_i = u_i$ and $v'_j f = v'_j$ by Lemma 2. Since τ is surjective, we can assume that

$$\tau\Big(\sum_k a_k \otimes b_k\Big) = e \text{ and } \tau\Big(\sum_l c_l \otimes d_l\Big) = f.$$

Then we have

$$\begin{aligned}
\sum_i u_i \otimes v_i &= \tau\big(\sum_k a_k \otimes b_k\big) \sum_i u_i \otimes v_i = \sum_k \sum_i \langle a_k, b_k\rangle\, u_i \otimes v_i \\
&= \sum_k \sum_i a_k[b_k, u_i] \otimes v_i = \sum_k \sum_i a_k \otimes [b_k, u_i]v_i \\
&= \sum_k \sum_i a_k \otimes b_k \langle u_i, v_i\rangle = \sum_k a_k \otimes b_k \sum_i \langle u_i, v_i\rangle \\
&= \sum_k a_k \otimes b_k \sum_j \langle u'_j, v'_j\rangle = \sum_k \langle a_k, b_k\rangle \sum_j u'_j \otimes v'_j \\
&= e \sum_j u'_j \otimes v'_j.
\end{aligned}$$

Similarly, we have $\sum_j u'_j \otimes v'_j = \big(\sum_i u_i \otimes v_i\big) f$. So we have

$$\sum_i u_i \otimes v_i = e \sum_j u'_j \otimes v'_j = e\Big(\sum_i u_i \otimes v_i\Big) f = \Big(\sum_i u_i \otimes v_i\Big) f = \sum_j u'_j \otimes v'_j.$$

This proves that τ is injective. Similarly, we can prove that μ is injective. ∎

Let R be a semiring with slu. Define $\tau : R \otimes_R R \to R$ by $\tau(r_1 \otimes r_2) = r_1 r_2$, then (R, R, R, R, τ, τ) is a unital Morita context and τ is surjective. Hence, we have $R \otimes_R R \cong R$ by Lemma 3.

Theorem 5. *Let U and V be two commutative semigroups. Then the following two conditions are equivalent:*

(1) *There exist two semirings R and S with slu such that (R, S, U, V, τ, μ) is a unital Morita context and τ and μ are surjective.*

In this case, $-\otimes_R V$: Mod-$FR \rightleftharpoons$ Mod-FS : $-\otimes_S U$ are equivalent functors.

(2) *There exist surjective semigroup homomorphisms $\Phi : U \otimes V \otimes U \to U$ and $\Psi : V \otimes U \otimes V \to V$ satisfy the two conditions in part (2) of Theorem 4 and*

(III) *For any finite elements $x_k \in U$, $y_l \in V$, $k = 1, 2, ..., n$, $l = 1, 2, ..., m$, there exist finite elements $u_i \in U, v_i \in V$ such that*

(i) $\Phi(\sum_i (u_i \otimes v_i) \otimes x_k) = x_k$;

(ii) $\Psi(y_l \otimes \sum_i (v_i \otimes u_i)) = y_l$;

(iii) $\sum_i \Phi(\sum_i (u_i \otimes v_i) \otimes u_i) \otimes v_i = \sum_i (u_i \otimes v_i)$.

IV) *For any finite elements $x_k \in U$, $y_l \in V, k = 1, 2, ..., n$, $l = 1, 2, ..., m$, there exist finite elements $u'_i \in U, v'_i \in V$ such that*

(i) $\Psi(\sum_i (v'_i \otimes u'_i) \otimes y) = y$;

(ii) $\Phi(x \otimes \sum_i (u'_i \otimes v'_i)) = y$;

(iii) $\sum_i \Psi(\sum_i (v'_i \otimes u'_i) \otimes v'_i) \otimes u'_i = \sum_i (v'_i \otimes u'_i)$.

Proof. (1) $\Rightarrow$ (2) : For any finite elements $x_k \in U$, $y_l \in V$, $k = 1, 2, ..., n$, $l = 1, 2, ..., m$, by Lemma 2, there exists an idempotent $e \in R$ with $ex_k = x_k = x_k e$ and $ey_l = y_l = y_l e$. Since τ is surjective, there exist finite elements $u_i \in U, v_i \in V$ such that $\tau(\sum_i (u_i \otimes v_i)) = e$. Then we have

$$\Phi\Big(\sum_i (u_i \otimes v_i) \otimes x_k\Big) = \tau\Big(\sum_i u_i \otimes v_i\Big) \cdot x_k = ex_k = x_k$$

and

$$\Psi\Big(y_l \otimes \sum_i (v_i \otimes u_i)\Big) = \sum_i \mu(y_l \otimes v_i) \otimes u_i = \sum_i y_l \tau(v_i \otimes u_i) = y_l e = y_l.$$

Since $e = \tau(\sum_i u_i \otimes v_i)$ is an idempotent, we have

$$\tau\Big(e\Big(\sum_i u_i \otimes v_i\Big)\Big) = e\tau\Big(\sum_i u_i \otimes v_i\Big) = \tau\Big(\sum_i u_i \otimes v_i\Big).$$

As τ is an isomorphism, we get that $e(\sum_i u_i \otimes v_i) = \sum_i u_i \otimes v_i$. Then

$$\begin{aligned}\sum_i \Phi\Big(\sum_i (u_i \otimes v_i) \otimes u_i\Big) \otimes v_i &= \sum_i \Big(\tau\Big(\sum_i u_i \otimes v_i\Big) u_i\Big) \otimes v_i \\ &= \tau\Big(\sum_i u_i \otimes v_i\Big) \sum_i u_i \otimes v_i \\ &= e \sum_i u_i \otimes v_i = \sum_i u_i \otimes v_i.\end{aligned}$$

This proves that (III) holds. Similarly, we can prove that (IV) holds.

Suppose that (R, S, U, V, τ, μ) is a unital Morita context and τ and μ are surjective. For all $X \in$ Mod-FS, we have $\mathrm{Hom}_S(S, X) \otimes_S S \cong X$. Since S is a semirng with slu, we have $S \otimes_S S \cong S$. So we get $X \otimes_S S \cong X$. Then

$$(X \otimes_S V) \otimes_R U \cong X \otimes_S (V \otimes_R U) \cong X \otimes_S S \cong X.$$

Similarly, for all $Y \in$ Mod-FR, we have $(Y \otimes_R U) \otimes_R V \cong Y$.

Then $- \otimes_R V$: Mod-$FR \rightleftharpoons$ Mod-FS : $- \otimes_S U$ are equivalent functors.

(2) $\Rightarrow$ (1) : For all $F_{a_1 \otimes b_1}, \cdots, F_{a_n \otimes b_n} \in \mathcal{F}$, by condition (III), there exist finite elements $u_i \in U, v_i \in V$ such that $\Phi(\sum_i (u_i \otimes v_i) \otimes a_k) = a_k$, $\Psi(b_l \otimes \sum_i (v_i \otimes u_i)) = b_l$ and $\sum_i \Phi(\sum_i (u_i \otimes v_i) \otimes u_i) \otimes v_i = \sum_i (u_i \otimes v_i)$. Hence,

$$\begin{aligned} F_{\sum_i (u_i \otimes v_i)} F_{a_k \otimes b_l} &= F_{\Phi(\sum_i (u_i \otimes v_i) \otimes a_k) \otimes b_l} = F_{a_k \otimes b_l}; \\ F_{a_k \otimes b_l} F_{\sum_i (u_i \otimes v_i)} &= F_{\sum_i \Phi(a_k \otimes b_l \otimes u_i) \otimes v_i} = F_{a_k \otimes \Psi(b_l \otimes \sum_i (u_i \otimes v_i))} = F_{a_k \otimes b_l}; \\ F_{\sum_i (u_i \otimes v_i)} F_{\sum_i (u_i \otimes v_i)} &= F_{\sum_i \Phi(\sum_i (u_i \otimes v_i) \otimes u_i) \otimes v_i} = F_{\sum_i (u_i \otimes v_i)}. \end{aligned}$$

This shows that $\mathcal{F}$ is a semiring with slu. Similarly, we have that $\mathcal{G}$ is a semiring with slu.

Since Φ and Ψ are surjective, we obviously have that U and V are unital as bisemimodules and α and β are surjective. Hence, $(\mathcal{F}, \mathcal{G}, U, V, \alpha, \beta)$ is a unital Morita context. ∎

5. Applications to semirings with identity

Let R be a semiring with 1. For all $M \in$ Mod-R, we have $\mathrm{Hom}_R(R, M) \otimes_R R \cong \mathrm{Hom}_R(R, M) \cong M$. Hence, Mod-$FR$ = Mod-R. By Theorem 3, we have the following statement.

Theorem 6. *Let R and S be two semirings with identity and G : Mod-$R \rightleftharpoons$ Mod-S : H are equivalent functors. Let $U = H(S)$ and $V = G(R)$. Then we have the following:*

(1) $G \cong Hom_R(U, -)$, $H \cong Hom_S(V, -)$;

(2) $U \cong Hom_S(V, S), V \cong Hom_R(U, R)$ *as bisemimodules.*

Proof. (1) Since R is a semiring with identity and U is a unital semimodule, we have $\mathrm{Hom}_R(U, N)$ is also a unital semimodule, for all $N \in$ Mod-S. This implies that

$$G \cong \mathrm{Hom}_R(U, -) \otimes_S S \cong \mathrm{Hom}_R(U, -).$$

Similarly, we have $H \cong \mathrm{Hom}_S(V, -)$.

(2) is obvious. ∎

Using Theorem 5, we can get the following theorem.

Theorem 7. *Let U and V be two commutative semigroups. Then the following two conditions are equivalent:*

(1) *There exist two semirings with identity R and S such that (R, S, U, V, τ, μ) be a unital Morita context and τ and μ are surjective. In this case, $- \otimes_R V$: Mod-$R \rightleftharpoons$ Mod-S : $- \otimes_S U$ are equivalent functors.*

(2) *There exist surjective semigroup morphisms $\Phi : U \otimes_Z V \otimes_Z U \to U$ and $\Psi : V \otimes_Z U \otimes_Z V \to V$ satisfy the two conditions in part (2) of Theorem 4 and*

(III)′ *There exist finite elements $u_i \in U, v_i \in V$ such that*
$\Phi(\sum_i (u_i \otimes v_i) \otimes x) = x$ *and* $\Psi(y \otimes \sum_i (v_i \otimes u_i)) = y$, *for all* $x \in U, y \in V$.

(IV)′ *There exist finite elements $u_i' \in U, v_i' \in V$ such that*
$\Psi(\sum_i (v_i' \otimes u_i') \otimes y) = y$ *and* $\Phi(x \otimes \sum_i (u_i' \otimes v_i')) = x$, *for all* $x \in U, y \in V$.

Proof. (1) $\Rightarrow$ (2) : Suppose $1_R = \tau(\sum_i u_i \otimes v_i)$. Analogous to the proof of Theorem 5, we can get that condition (III)′ is valid. Suppose $1_S = \tau(\sum_i v_i' \otimes u_i')$. We can prove that condition (IV)′ is valid.

(2) $\Rightarrow$ (1) : For all $F_{u \otimes v} \in \mathcal{F}$, we have

$$F_{\sum_i (u_i \otimes v_i)} F_{u \otimes v} = F_{u \otimes v} F_{\sum_i (u_i \otimes v_i)} = F_{u \otimes v}.$$

This proves that $F_{\sum_i (u_i \otimes v_i)}$ is the identity of $\mathcal{F}$. Similarly, we can prove that $\mathcal{G}$ has identity. Using Theorem 5, we can prove the statement. ∎

The following theorem generalizes the corresponding result in semigroup theory to semiring theory.

Theorem 8. *Let R and S be two semirings with identity and the two semirings are equivalent as Theorem 6. We identify V with $U^* = Hom_R(U,R)$ and S with $End_R(U)$. Define $\tau : U^* \otimes_R U \to R$ given by $\tau(\varphi \otimes u) = \langle \varphi, u \rangle$ and $\mu : U \otimes_S U^* \to S$ given by $\mu(u \otimes \varphi) = [u,\varphi]$, where $[u,\varphi]u' = u\langle \varphi, u' \rangle$. Then (R,S,U,V,τ,μ) defines a Morita context.*

Proof. We can easily check that τ and μ are both bisemimodule homomorphisms. For all $\psi \in U^*, u, u' \in U$, then

$$\begin{aligned} \left\langle \psi[u,\varphi], u' \right\rangle &= \left\langle \psi, [u,\varphi]u' \right\rangle = \left\langle \psi, u\left\langle \varphi, u' \right\rangle \right\rangle \\ &= \left\langle \psi, u \right\rangle \left\langle \varphi, u' \right\rangle = \left\langle \left\langle \psi, u \right\rangle \varphi, u' \right\rangle. \end{aligned}$$

This proves that $\psi[u,\varphi] = \langle \psi, u \rangle \varphi$. Hence, (R,S,U,V,τ,μ) is a Morita context. ■

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