APPLICATION OF NEW GENERALIZED (G'/G)-EXPANSION METHOD TO THE $(3+1)$-DIMENSIONAL KADOMTSEV-PETVIASHVILI EQUATION

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Abstract. In this research article, seeking parameters dependent exact solutions, we implement the new generalized (G'/G)-expansion to the (3+1)-dimensional Kadomtsev-Petviashvili equation. The traveling wave solutions are expressed in terms of the hyperbolic functions, trigonometric functions, as well as rational functions. Herein, established is therefore the fact that the new generalized (G'/G)-expansion method offers an efficient and influential mathematical tool for constructing exact solutions of nonlinear evolution equations (NLEEs). In mathematical physics, finding the exact solutions of NLEEs reveals the salient features of the inner mechanism of possibly hidden complex physical phenomena, modeled by the given equations. In consequence to our current work and setup, not only does the new method appear to be straightforward and user-friendly, but also, it turns out easily implementable by computer programmed and symbolic algebra packages, yielding fast, albeit accurate results.

Keywords: homogeneous balance; new generalized (G'/G)-expansion method; nonlinear evolution equation; (3+1)-dimensional Kadomtsev-Petviashvili equation; solitary wave solutions; traveling wave solutions.
Mathematics Subject Classification: 35C07, 35C08, 35P99.

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1. Introduction

In recent years, scholars and researchers became highly interested in obtaining exact solutions for nonlinear partial differential equations (NLEEs). NLEEs are mathematical models of complex physical phenomena that may arise in engineering, applied mathematics, chemistry, biology, mechanics, physics, the exact solutions of which, when found, reveal the salient features of the hidden nonlinear dynamics. Therefore, developing means to crack such models and extract the exact solutions of NLEEs has become of utmost importance.

For the past three decades, searching for methods to solve NLEEs explicitly has been a central targets of numerical mathematics. Many reputed such methods have been developed. The list of methods include homogeneous balance [1], [2], hyperbolic tangent expansion [3], [4], trial function [5], nonlinear transform [6], theta function [7]–[9], inverse scattering transform [10], $\exp(-\varphi(\xi))$-expansion [11]–[14], Exp-function [15], [16], Hirota bilinear [17], Painleve expansion [18], (G'/G)-expansion [19]–[25, 36], improved (G'/G)-expansion [26], [27], new generalized (G'/G)-expansion [28]–[35], and Sumudu transform method [37]–[46]. For purpose, this paper aims to innovatively solve the (3+1)-dimensional Kadomtsev-Petviashvili equation by using the new generalized (G'/G)-expansion to show its suitability. The remainder of the paper is organized as follows: Section 2 is set for the new expansion method description while Section 3 is reserved for the method application to the (3+1)-dimensional Kadomtsev–Petviashvili equation. Results comparisons discussions are provided in Section 4, followed by the conclusion in Section 5. After acknowledgments, the paper culminates into a rich, albeit not exhaustive, set of scholarly references for current and future interest.

2. The new generalized (G'/G)-expansion method

Let us consider a general nonlinear PDE in the form,

$$P(u,\, u_t,\, u_x,\, u_{tt},\, u_{tx},\, u_{xx}, \ldots) = 0, \tag{1}$$

where $u = u(x,t)$is an unknown, P is a polynomial in $u(x,\,t)$and its derivatives including nonlinear terms are involved and the subscripts stand for the partial derivatives.

Step 1: We combine the real variables x and t by a complex variable, ξ,

$$u(x,\,t) = u(\xi), \xi = x \pm V\,t, \tag{2}$$

where, V is the speed of the traveling wave. The traveling wave transformation (2) converts Eq.(1)into an ODE, for $u = u(\xi)$, and with and superscripts indicating differentiation with respect to ξ, we obtain the following polynomial Q of u and it derivatives,

$$Q(u,\, u',\, u'',\, u''', \ldots) = 0. \tag{3}$$

Step 2: Accordingly, Eq. (3) can be integrated term by term once or more times, to yield constant(s) of integration. The integral constant (s) may be set to zero for simplicity.

Step 3: Suppose the traveling wave solution of Eq. (3) can be expressed as follows:

$$u(\xi)=\sum_{i=0}^{N}a_i(d+H)^i+\sum_{i=1}^{N}b_i(d+H)^{-i}, \tag{4}$$

where, either a_N or b_N may be zero, but both a_N and $b_N$could be zero at a time, a_i and b_i $(i=1,2,\dots,N)$ and d, are arbitrary constants to be determined, and $H(\xi)$ is given by

$$H(\xi)=(G'/G) \tag{5}$$

where $G=G(\xi)$ satisfies the following auxiliary nonlinear ordinary differential equation:

$$AG\,G''-BG\,G'-E\,G^2-C\;(G')^2=0 \tag{6}$$

where the prime stands for derivative with respect to ξ; A, B, C and E are real parameters.

Step 4: To determine the positive integer N, taking the homogeneous balance between the highest order nonlinear terms and the derivatives of the highest order appearing in Eq. (3).

Step 5: Substituting Eq. (4) and Eq. (6) and Eq. (5) into Eq. (3) with the value of N obtained in Step 4, we get polynomials in $(d+H)^N$and $(d+H)^{-N}$, $(N=0,1,2,...)$. We then collect coefficient a set of algebraic equations for d and V, a_i and b_i for $(i=0,1,2,...,N)$.

Step 6: Suppose that the value of the constants a_i and b_i $(i=1,2,...,N)$, d and V can be found. Since, the general solution of Eq. (6) is known to us, inserting the values of a_i and b_i $(i=1,2,...,N)$, d and V into Eq. (4), we obtain more general types and new exact traveling wave solutions of the NLEE (1). Using Eq.(6) solution, we get the solutions of Eq. (5):

When $B\neq 0$, $\psi=A-C$ and $\Omega=B^2+4E(A-C)>0$,

$$H(\xi)=\left(\frac{G'}{G}\right)=\frac{B}{2\psi}+\frac{\sqrt{\Omega}}{2\psi}\frac{C_1\sinh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)+C_2\cosh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)+C_2\sinh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)} \tag{7}$$

when $B\neq 0$, $\psi=A-C$ and $\Omega=B^2+4E(A-C)<0$,

$$H(\xi)=\left(\frac{G'}{G}\right)=\frac{B}{2\psi}+\frac{\sqrt{-\Omega}}{2\psi}\frac{-C_1\sin\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)+C_2\cos\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)}{C_1\cos\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)+C_2\sin\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)} \tag{8}$$

when $B\neq 0$, $\psi=A-C$ and $\Omega=B^2+4E(A-C)=0$,

$$H(\xi)=\left(\frac{G'}{G}\right)=\frac{B}{2\psi}+\frac{C_2}{C_1+C_2\xi} \tag{9}$$

when $B=0$, $\psi=A-C$ and $\Delta=\psi E>0$,

$$H(\xi)=\left(\frac{G'}{G}\right)=\frac{\sqrt{\Delta}}{\psi}\frac{C_1\sinh(\frac{\sqrt{\Delta}}{A}\xi)+C_2\cosh(\frac{\sqrt{\Delta}}{A}\xi)}{C_1\cosh(\frac{\sqrt{\Delta}}{A}\xi)+C_2\sinh(\frac{\sqrt{\Delta}}{A}\xi)} \tag{10}$$

when $B=0$, $\psi=A-C$ and $\Delta=\psi E<0$,

$$H(\xi)=\left(\frac{G'}{G}\right)=\frac{\sqrt{-\Delta}}{\psi}\frac{-C_1\sin(\frac{\sqrt{-\Delta}}{A}\xi)+C_2\cos(\frac{\sqrt{-\Delta}}{A}\xi)}{C_1\cos(\frac{\sqrt{-\Delta}}{A}\xi)+C_2\sin(\frac{\sqrt{-\Delta}}{A}\xi)} \tag{11}$$

3. Application of the method

In this section, we use the new generalized (G'/G)-expansion method to look for the solitary wave solutions to the (3+1)-dimensional Kadomtsev–Petviashvili equation,

$$(u_t+6\,u\,u_x+u\,u_{xxx})_x-3\,u_{yy}-3\,u_{zz}=0. \tag{12}$$

Using the variable, $u(\xi)=u(x,y,z,t)$, $\xi=x+y+z-Vt$, we consider the resulting ODE,

$$(-V\,u'+6\,u\,u'+u''')'-6\,u''=0. \tag{13}$$

Then, integrating twice, and for K, the constant of integration to be determined, we obtain,

$$K-Vu-6u+3\,u^2+u''=0. \tag{14}$$

Taking the homogeneous balance between u^2 and u'' in Eq. (14), we get$N=2$. Therefore, with, a_0,a_1,a_2,b_1,b_2 and d constants to be determined, the solution of Eq. (14) has the form,

$$u(\xi)=a_0+a_1(d+H)+a_2(d+H)^2+b_1(d+H)^{-1}+b_2(d+H)^{-2}, \tag{15}$$

Substituting Eqs. (15), (5) and (6) into Eq. (14), the left-hand side is converted into polynomials in $(d+H)^N$ and $(d+H)^{-N}(N=1,2,\ldots)$. We collect each coefficient of these resulting polynomials to zero, yields a set of simultaneous algebraic equations, (for simplicity which are not presented here) to be Maple solved for a_0, a_1, a_2, b_1, b_2, d, K and V.

Case 1: $a_0=a_0$, $a_1=0$, $a_2=0$, $d=d$, $b_1=\frac{2}{A^2}(-2Ed\psi-EB+B^2d+2d^3\psi^2-3Bd^2\psi)$, $b_2=-\frac{2}{A^2}(d^4\psi^2-2Ed^2\psi+E^2+2Bd^3\psi+B^2d^2-2BdE)$,

$$V=\frac{1}{A^2}(6a_0A^2-6A^2+12d^2\psi^2+12Bd\psi-8E\psi+B^2), \tag{16}$$

$$\begin{aligned}K=\tfrac{1}{A^4}(&48Cd^2EA^2-72Bd^3CA^2-16BdEA^2+2B^3Ad-48C^3Ad^4+16C^3Ed^2\\&+14C^2B^2d^2-24C^3Bd^3-2B^3Cd-24a_0A^3Cd^2+12a_0A^3Bd^2+12a_0A^2C^2d^2\\&+8a_0A^2EC+2CB^2E-2EB^2A+4E^2C^2+4A^2E^2-12a_0A^2BdC+12a_0A^4d^2\\&-8a_0A^3E+a_0A^2B^2+32BAECd-28B^2Ad^2C+72BAC^2d^3-48C^2Ad^2E\\&-16C^2BdE+3a_0A^4+12d^4A^4+12C^4d^4-8AE^2C+72C^2d^4A^2+24Bd^3A^3\\&-16Ed^2A^3-48Cd^4A^3+14B^2d^2A^2),\end{aligned}$$

where $\psi = A - C$, a_0, A, B, C and E are free parameters.

Case 2: $a_1 = \frac{2}{A^2}(B\psi + 2d\psi^2)$, $a_2 = -\frac{2\psi^2}{A^2}$, $b_1 = 0$, $b_2 = 0$, $d = d$, $a_0 = a_0$,

$$K = \frac{1}{A^4}(48Cd^2EA^2 - 72Bd^3CA^2 - 16BdEA^2 + 2B^3Ad - 48C^3Ad^4 + 16C^3Ed^2 + 14C^2B^2d^2 - 24C^3Bd^3 - 2B^3Cd - 24a_0A^3Cd^2 + 12a_0A^3Bd^2 + 12a_0A^2C^2d^2 + 8a_0A^2EC + 2CB^2E - 2EB^2A + 4E^2C^2 + 4A^2E^2 - 12a_0A^2BdC + 12a_0A^4d^2 - 8a_0A^3E + a_0A^2B^2 + 32BAECd - 28B^2Ad^2C + 72BAC^2d^3 - 48C^2Ad^2E - 16C^2BdE + 3a_0A^4 + 12d^4A^4 + 12C^4d^4 - 8AE^2C + 72C^2d^4A^2 + 24Bd^3A^3 - 16Ed^2A^3 - 48Cd^4A^3 + 14B^2d^2A^2),$$

$$V = \frac{1}{A^2}(6a_0A^2 - 6A^2 + 12d^2\psi^2 + 12Bd\psi - 8E\psi + B^2). \tag{17}$$

where $\psi = A - C$, a_0, A, B, C and E are free parameters.

Case 3: $a_1 = 0$, $b_1 = 0$, $a_2 = -\frac{2\psi^2}{A^2}$, $d = -\frac{B}{2\psi}$, $b_2 = -\frac{1}{8A^2\psi^2}(16E^2\psi^2 + 8EB^2\psi + B^4)$, $V = \frac{2}{A^2}(3a_0A^2 - 3A^2 - B^2 - 4E\psi)$,

$$K = \frac{1}{A^4}(3a_0^2A^4 - 8a_0A^3E + 8a_0A^2EC - 2a_0A^2B^2 - 16A^2E^2 - 8EB^2A + 32AE^2C - B^4 - 16E^2C^2 + 8CB^2E). \tag{18}$$

where $\psi = A - C$, a_0, A, B, C and E are free parameters.

For Case 1, substituting Eq. (16) into Eq. (15), along with Eq. (7) and simplifying yields following travelling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{1_1}(\xi) = a_0 + b_1\left(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right)^{-1} + b_2\left(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right)^{-2}.$$

$$u_{1_2}(\xi) = a_0 + b_1\left(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right)^{-1} + b_2\left(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right)^{-2},$$

where $\xi = x - \left\{\frac{1}{A^2}(6a_0A^2 - 6A^2 + 12d^2\psi^2 + 12Bd\psi - 8E\psi + B^2)\right\}t$.

Substituting Eq. (16) into Eq. (15), along with Eq. (8) and simplifying, our exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{1_3}(\xi) = a_0 + b_1 \left(d + \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi}\cot\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right)^{-1} + b_2 \left(d + \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi}\cot\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right)^{-2}.$$

$$u_{1_4}(\xi) = a_0 + b_1 \left(d + \frac{B}{2\psi} - \frac{\sqrt{-\Omega}}{2\psi}\tan\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right)^{-1} + b_2 \left(d + \frac{B}{2\psi} - \frac{\sqrt{-\Omega}}{2\psi}\tan\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right)^{-2}$$

Substituting Eq. (16) into Eq. (15) together with Eq. (9) and simplifying, we obtain

$$u_{1_5}(\xi) = a_0 + b_1 \left(d + \frac{B}{2\psi} + \frac{C_2}{C_1 + C_2\xi}\right)^{-1} + b_2 \left(d + \frac{B}{2\psi} + \frac{C_2}{C_1 + C_2\xi}\right)^{-2}.$$

Substituting Eq. (16) into Eq. (15), along with Eq. (10) and simplifying, we obtain following traveling wave but solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{1_6}(\xi) = a_0 + b_1 \left(d + \frac{\sqrt{\Delta}}{\psi}\coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-1} + b_2 \left(d + \frac{\sqrt{\Delta}}{\psi}\coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-2}.$$

$$u_{1_7}(\xi) = a_0 + b_1 \left(d + \frac{\sqrt{\Delta}}{\psi}\tanh\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-1} + b_2 \left(d + \frac{\sqrt{\Delta}}{\psi}\tanh\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-2}.$$

Substituting Eq. (16) into Eq. (15), together with Eq. (11) and simplifying, our obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{1_8}(\xi) = a_0 + b_1 \left(d + \frac{\sqrt{-\Delta}}{\psi}\cot\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-1} + b_2 \left((d + \frac{\sqrt{-\Delta}}{\psi}\cot\left(\frac{\sqrt{-\Delta}}{A}\xi)\right)\right)^{-2}.$$

$$u_{1_9}(\xi) = a_0 + b_1 \left(d - \frac{\sqrt{-\Delta}}{\psi}\tan\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-1} + b \left(d - \frac{\sqrt{-\Delta}}{\psi}\tan\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-2}.$$

Again for Case 2, substituting Eq. (17) into Eq. (15) along with Eq. (7) and simplifying, the traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively become:

$$u_{2_1}(\xi) = a_0 + \frac{1}{2A^2}\left\{B^2 - \Omega\coth^2\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + 4d\psi\,(B + d\psi)\right\}.$$

$$u_{2_2}(\xi) = a_0 + \frac{1}{2A^2}\left\{B^2 - \Omega\tanh^2\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + 4d\psi\,(B + d\psi)\right\},$$

where

$$\xi = x - \left\{ \frac{1}{A^2} \left(6a_0 A^2 - 6A^2 + 12d^2\psi^2 + 12Bd\psi - 8E\psi + B^2\right) \right\} t.$$

Substituting Eq. (17) into Eq. (15), along with Eq. (8) and simplifying, yields exact solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{2_3} = a_0 + \frac{1}{2A^2} \left\{ B^2 + \Omega \cot^2 \left(\frac{\sqrt{-\Omega}}{2A}\xi \right) + 4d\psi\ (B + d\psi) \right\}.$$

$$u_{2_4}(\xi) = a_0 + \frac{1}{2A^2} \left\{ B^2 + \Omega \tan^2 \left(\frac{\sqrt{-\Omega}}{2A}\xi \right) + 4d\psi\ (B + d\psi) \right\}.$$

Substituting Eq. (17) into Eq. (15), along with Eq. (9) and simplifying, our obtained solution becomes:

$$u_{2_5}(\xi) = a_0 + \frac{1}{2A^2} \left\{ B^2 - \left(\frac{2\,\psi\, C_2}{C_1 + C_2\xi} \right)^2 + 4\, d\psi\, (B + d\psi) \right\},$$

Substituting Eq. (17) into Eq. (15), together with Eq. (10) and simplifying, yields following traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{2_6}(\xi) = a_0 + \frac{2}{A^2} \left\{ d\psi(B + d\psi) + \sqrt{\Delta} \left(B \coth \left(\frac{\sqrt{\Delta}}{A}\xi \right) - \sqrt{\Delta} \coth^2 \left(\frac{\sqrt{\Delta}}{A}\xi \right) \right) \right\}.$$

$$u_{2_7}(\xi) = a_0 + \frac{2}{A^2} \left\{ d\psi\,(B + d\psi) + \sqrt{\Delta} \left(B \tanh \left(\frac{\sqrt{\Delta}}{A}\xi \right) - \sqrt{\Delta} \tanh^2 \left(\frac{\sqrt{\Delta}}{A}\xi \right) \right) \right\}.$$

Substituting Eq. (17) into Eq. (15), along with Eq. (11) and simplifying, our exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{2_8}(\xi) = a_0 + \frac{2}{A^2} \left\{ d\psi\,(B + d\psi) + \sqrt{\Delta} \left(iB \cot \left(\frac{\sqrt{-\Delta}}{A}\xi \right) + \sqrt{\Delta} \cot^2 \left(\frac{\sqrt{-\Delta}}{A}\xi \right) \right) \right\}.$$

$$u_{2_9}(\xi) = a_0 + \frac{2}{A^2} \left\{ d\psi(B + d\psi) - \sqrt{\Delta} \left(iB \tan \left(\frac{\sqrt{-\Delta}}{A}\xi \right) - \sqrt{\Delta} \tan^2 \left(\frac{\sqrt{-\Delta}}{A}\xi \right) \right) \right\},$$

Finally, for Case 3, substituting Eq. (18) into Eq. (15), together with Eq. (7) and simplifying, yields following traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{3_1}(\xi) = a_0 - \frac{\Omega}{2A^2} \coth^2 \left(\frac{\sqrt{\Omega}}{2A}\xi \right) + \frac{4b_2\psi^2}{\Omega} \tanh^2 \left(\frac{\sqrt{\Omega}}{2A}\xi \right).$$

$$u_{3_2}(\xi) = a_0 - \frac{\Omega}{2A^2} \tanh^2 \left(\frac{\sqrt{\Omega}}{2A}\xi \right) + \frac{4b_2\psi^2}{\Omega} \coth^2 \left(\frac{\sqrt{\Omega}}{2A}\xi \right).$$

where

$$\xi = x - \left\{ \frac{2}{A^2}(3a_0A^2 - 3A^2 - B^2 - 4E\psi) \right\} t.$$

Substituting Eq. (18) into Eq. (15), along with Eq. (8) and simplifying, we obtain following solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{3_3}(\xi) = a_0 + \frac{\Omega}{2A^2} \cot^2 \left(\frac{\sqrt{-\Omega}}{2A} \xi \right) - \frac{4b_2\psi^2}{\Omega} \tan^2 \left(\frac{\sqrt{-\Omega}}{2A} \xi \right).$$

$$u_{3_4}(\xi) = a_0 + \frac{\Omega}{2A^2} \tan^2 \left(\frac{\sqrt{-\Omega}}{2A} \xi \right) - \frac{4b_2\psi^2}{\Omega} \cot^2 \left(\frac{\sqrt{-\Omega}}{2A} \xi \right).$$

Substituting Eq. (18) into Eq. (15), along with Eq. (9) and simplifying, we get

$$u_{3_5}(\xi) = a_0 - \frac{2\psi^2}{A^2} \left(\frac{C_2}{C_1 + C_2\xi} \right)^2 + b_2 \left(\frac{C_2}{C_1 + C_2\xi} \right)^{-2}.$$

Substituting Eq. (18) into Eq. (15), along with Eq. (10) and simplifying, yields following exact traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{3_6}(\xi) = a_0 - \frac{2\psi^2}{A^2} \left(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi} \coth(\frac{\sqrt{\Delta}}{A}\xi) \right)^2 + b_2 \left(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi} \coth(\frac{\sqrt{\Delta}}{A}\xi) \right)^{-2}.$$

$$u_{3_7}(\xi) = a_0 - \frac{2\psi^2}{A^2} \left(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi} \tanh(\frac{\sqrt{\Delta}}{A}\xi) \right)^2 + b_2 \left(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi} \tanh(\frac{\sqrt{\Delta}}{A}\xi) \right)^{-2}.$$

Substituting Eq. (18) into Eq. (15), along with Eq. (11) and simplifying, our obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{3_8}(\xi) = a_0 - \frac{2\psi^2}{A^2} \left(\frac{-B}{2\psi} + \frac{\sqrt{-\Delta}}{\psi} \cot \left(\frac{\sqrt{-\Delta}}{A} \xi \right) \right)^2 + b_2 \left(\frac{-B}{2\psi} + \frac{\sqrt{-\Delta}}{\psi} \cot \left(\frac{\sqrt{-\Delta}}{A} \xi \right) \right)^{-2}.$$

$$u_{3_9}(\xi) = a_0 - \frac{2\psi^2}{A^2} \left(\frac{-B}{2\psi} - \frac{\sqrt{-\Delta}}{\psi} \tan \left(\frac{\sqrt{-\Delta}}{A} \xi \right) \right)^2 + b \left(\frac{-B}{2\psi} - \frac{\sqrt{-\Delta}}{\psi} \tan \left(\frac{\sqrt{-\Delta}}{A} \xi \right) \right)^{-2}.$$

4. Discussion

Comparison of the *new* with the *basic* (G'/G)-expansion is given below followed by the advantages. In Ref. [24], Song and Ge used the linear ordinary differential equation as auxiliary equation and traveling wave solutions presented in the form $u(\xi) = \sum_{i=0}^{m} a_i(G'/G)^i$, where $a_m \neq 0$. It is notable to point out that several of our solutions are coincided with already published results, if parameters are given particular values which verify our solutions. Furthermore, in Ref. [24], Song and Ge investigated the (3+1)-dimensional Kadomtsev–Petviashvili equation to find exact solutions via the basic (G'/G)-expansion method and achieved only three solutions (A1)-(A6) in the Appendix. In contrast we give a set of twenty seven solutions for the (3+1)-dimensional Kadomtsev–Petviashvili equation.

This reveals the immediate advantage of the *new approach* over the *basic* (G'/G)-expansion method, since it provides a large quantity of new exact solutions with some free parameters. The extra solutions of course yield more significance into the understanding of the physical phenomena studied. On top of the enriched physical significance, the NLEEs closed form solutions assist the numerical solvers to compare the correctness of their results and help them in convergence and stability analysis studies.

5. Conclusion and future work

The new generalized (G'/G)-expansion method is innovatively and lucratively used to establish traveling wave solutions of the (3+1)-dimensional Kadomtsev–Petviashvili equation. Comparing with the other methods in the literature, the new generalized (G'/G)-expansion method appears to be easier and faster, by means of the symbolic algebra packages. This article confirms that the method is direct, brief and effective. The method can be used for treating many other NLEEs of mathematical physics. Treading in this direction we plan to use the Sumudu transform to study the equations and compare its solutions to the those already obtained herein and in the literature. We plan to use a numerico-analytical hybrid approach based on the new generalized expansion, and Sumudu transform methods [37]–[46].

Appendix: Song and Ge's solutions [24]

Song and Ge [24] obtained the following exact (3+1)-dimensional Kadomtsev–Petviashvili solutions by using the *basic* (G'/G)-expansion method:

When $\lambda^2 - 4\mu > 0$,

$$(A.1) \qquad u_1 = -\frac{1}{2}(\lambda^2 - 4\mu)\left(\frac{C_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}\right)^2 + \frac{\lambda^2}{2} - 2\mu,$$

and

$$(A.2) \quad u_2 = -\frac{1}{2}(\lambda^2-4\mu)\left(\frac{C_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}\right)^2 - \frac{1}{6}(2\lambda^2+4\mu) + \frac{\lambda^2}{2},$$

where $\xi = x + y + z - (-6 + \lambda^2 - 4\mu)t$ and C_1, C_2 are arbitrary constants.

For $C_2 \neq 0$, $C_1^2 < C_2^2$, the above solutions (A.1) turns into

$$u_1 = -\frac{1}{2}(\lambda^2 - 4\mu)\tanh^2\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2} + \xi_0\right) + \frac{\lambda^2}{2} - 2\mu.$$

and the solution (A.2) turns into

$$u_2 = -\tfrac{1}{2}(\lambda^2 - 4\mu)\tanh^2\left(\tfrac{\sqrt{\lambda^2-4\mu}\xi}{2} + \xi_0\right) - \tfrac{1}{6}(2\lambda^2 + 4\mu) + \tfrac{\lambda^2}{2}, \xi_0 = \tanh^{-1}\left(\tfrac{C_1}{C_2}\right),$$

when $\lambda^2 - 4\mu < 0$,

$$(A.3) \qquad u_3 = -\tfrac{1}{2}(4\mu - \lambda^2)\left(\tfrac{-C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)+C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)}{C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)+C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)}\right)^2 + \tfrac{\lambda^2}{2} - 2\mu,$$

and

$$(A.4) \quad u_4 = -\tfrac{1}{2}(4\mu-\lambda^2)\left(\tfrac{-C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)+C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)}{C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)+C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)}\right)^2 - \tfrac{1}{6}(2\lambda^2+4\mu) + \tfrac{\lambda^2}{2},$$

where $\xi = x + y + z - (-6 - \lambda^2 + 4\mu)t$ and C_1, C_2 are arbitrary constants.

For $C_2 \neq 0$, $C_1^2 < C_2^2$, solutions (A.3) and (A.4), respectively, turn into

$$u_3 = -\tfrac{1}{2}(4\mu - \lambda^2)\cot^2\left(\tfrac{\sqrt{\lambda^2-4\mu}\xi}{2} + \xi_0\right) + \tfrac{\lambda^2}{2} - 2\mu,$$

and

$$u_4 = -\tfrac{1}{2}(4\mu - \lambda^2)\cot^2\left(\tfrac{\sqrt{4\mu-\lambda^2}\xi}{2} + \xi_0\right) - \tfrac{1}{6}(2\lambda^2 + 4\mu) + \tfrac{\lambda^2}{2}, \xi_0 = \tanh^{-1}\left(\tfrac{C_1}{C_2}\right).$$

when $\lambda^2 - 4\mu = 0$,

$$(A.5) \qquad u_5 = \tfrac{-2C_2^2}{(C_1+C_2\xi)^2} + \tfrac{\lambda^2}{2} - 2\mu,$$

where $\xi = x + y + z + 6t$ and C_1, C_2 are arbitrary constants.

$$(A.6) \qquad u_6 = \tfrac{-2C_2^2}{(C_1+C_2\xi)^2} - \tfrac{1}{6}(2\lambda^2 + 4\mu) + \tfrac{\lambda^2}{2},$$

where $\xi = x + y + z + 6t$ and C_1, C_2 are arbitrary constants.

Acknowledgments. The Authors acknowledge and salute the IJPAM editorial board management, and thank the consequent anonymous reviewers diligent efforts and critiques that helped improve the flow, style and scientific value of this paper. Furthermore, Fethi Bin Muhammad Belgacem wishes to acknowledge the continued support of the Kuwait Public Authority for Applied Education and Training Research Department, (PAAET RD), under grant BE-13-09.

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Accepted: 09.01.2016