NUMERICAL SOLUTION OF FRACTIONAL RELAXATION–OSCILLATION EQUATION USING CUBIC B-SPLINE WAVELET COLLOCATION METHOD

Raghvendra S. Chandel

Department of Mathematics
Govt. Geetanjali Girls College
Bhopal, M.P. – 462038
India
e-mail: rs_chandel2009@yahoo.co.in

Amardeep Singh

Department of Mathematics
Govt. Motilal Vigyan Mahavidyalaya
Bhopal, M.P. – 462008
India
e-mail: deepbeeb60@gmail.com

Devendra Chouhan

Department of Mathematics
IES, IPS Academy
Indore, M.P. – 452012
India
e-mail: dchouhan100@gmail.com

Abstract. A relaxation oscillator is a kind of oscillator based on a behavior of physical system's return to equilibrium after being disturbed. The relaxation-oscillation equation is the primary equation of relaxation and oscillation processes. The relaxation-oscillation equation is a fractional differential equation with initial conditions. In this paper, the approximate solutions of relaxation-oscillation equation are obtained by developing the wavelet collocation method to fractional differential equations using cubic B-spline wavelet. Analytical expressions of fractional derivatives in caputo sense for cubic B-spline functions are presented. The main advantage of the proposed method is that it transforms such problems into a system of algebraic equations which is suitable for computer programming. The reliability and efficiency of the proposed method are demonstrated in the numerical examples.

Keywords: fractional relaxation-oscillation equation; fractional differential equation; cubic B-spline function; wavelet collocation method.

2010 AMS Subject Classification: 42C40, 34C26, 34A08.

1. Introduction

The concept of fractional or non-integer order derivative and integration can be traced back to the genesis of integer order calculus itself [13]. The fractional differential equations have received considerable interest in recent years. Fractional differential equations have shown to be adequate models for various physical phenomena in areas like damping laws, diffusion processes etc. Most fractional differential equations do not have analytical solutions so we need approximate approach. Solution techniques for fractional differential equations have been studied extensively by many researchers such as Collocation method [8], [17], Adomian decomposition method [7], [18], Operational matrix method [9], [15], Variational iteration method [4], Tau method [16].

A relaxation-oscillator is a kind of oscillator based on a behavior of physical system's return back to equilibrium after being disturbed. There are many relaxation-oscillation models such as positive fractional derivative, fractal derivative and fractional derivative [11], [12], [21]. The relaxation-oscillation equation is the primary equation of relaxation and oscillation processes. The standard relaxation equation is

$$\frac{dy}{dx} + By = f(x)$$

where B denotes the Elc, E is the elastic modulus, $f(x)$ denotes E multiplying the strain rate. When $f(x) = 0$, we have the analytic solution $y(x) = Ce^{-Bx}$, where C is a constant determined by the initial condition.

The standard oscillation equation is

$$\frac{d^2y}{dx^2} + By = f(x)$$

where B equals $\frac{k}{m} = \omega^2$, k is the stiffness coefficient, m is the mass, ω the angular frequency. When $f(x) = 0$, we have the analytic solution $y(x) = C\cos\sqrt{B}x + D\sin\sqrt{B}x$, where C and D are constants determined by the initial conditions.

The fractional derivatives are employed in the relaxation and oscillation models to represent slow relaxation and damped oscillation [11], [12]. Fractional relaxation-oscillation model can be depicted as

$$\begin{aligned} &D_x^{\beta}y(x) + Ay(x) = f(x), \quad x > 0 \\ &y(0) = a \ \text{ if } 0 < \beta \le 1 \end{aligned}$$

or

$$y(0) = \lambda, \ \ y^{'}(0) = \mu \ \text{ if } \ 1 < \beta \le 2$$

where A is a positive constant. For $0 < \beta \le 2$, this equation is called the fractional relaxation-oscillation equation. When $0 < \beta < 1$, the model describes the relaxation with the power law attenuation. When $1 < \beta < 2$, the model depicts the damped oscillation with viscoelastic intrinsic damping of oscillator [3], [19].

This model has been applied in electrical model of the heart [20], signal processing [3], modeling cardiac pacemakers [5], Predator–Prey system [1], Spruce-budworm interactions [14] etc.

In the present paper, we intend to extend the cubic B-spline wavelet collocation method to solve fractional relaxation-oscillation equation. Expanding the unknown function as a linear combination of wavelet basis functions with unknown coefficients, the method transforms the differential equation into a system of algebraic equations. Whether the method can be extended to fractional differential equation depends on the calculation of fractional derivatives for all wavelet basis functions. In this paper, analytical expressions of fractional derivatives in caputo sense for wavelet basis functions are presented, which can save memory space and reduce computational complexity.

2. Cubic spline basis functions on $H^2(I)$

Let $I = [0, L]$ be an interval with $4 < L$ and $H^2(I)$ be a Sobolev space which contains functions with square integrable second derivatives and the homogeneous Sobolev space $H_0^2(I)$ can be defined by

$$H_0^2(I) = \left\{ f(t) \in H^2(I), \;\; f(0) = f^{'}(0) = f(L) = f^{'}(L) = 0 \right\}$$

which is a Hilbert space equipped with inner product

$$\langle f, g \rangle = \int_I f^{''}(t) g^{''}(t) dt$$

Cai and Wang [2] gave a multi-resolution analysis (MRA) and a wavelet decomposition for $H_0^2(I)$ by constructing scaling spline functions

$$\varphi(t) = \frac{1}{6} \sum_{l=0}^{4} \binom{4}{l} (-1)^l (t-l)_+^3$$

$$\varphi_b(t) = \frac{3}{2} t_+^2 - \frac{11}{12} t_+^3 + \frac{3}{2}(t-1)_+^3 - \frac{3}{4}(t-2)_+^3 + \frac{1}{6}(t-3)_+^3$$

and wavelet functions

$$\psi(t) = -\frac{3}{7}\varphi(2t) + \frac{12}{7}\varphi(2t-1) - \frac{3}{7}\varphi(2t-2)$$

$$\psi_b(t) = \frac{24}{13}\varphi_b(2t) - \frac{6}{13}\varphi(2t)$$

which have compact supports, where

$$t_+^n = \begin{cases} t^n, & t > 0 \\ 0, & t \le 0 \end{cases}$$

Then the scaling spaces $V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_\infty = H_0^2(I)$ of MRA and a wavelet decomposition $H_0^2(I) = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1} \oplus \cdots$ corresponding to MRA and obtained by defining

$$V_j = span\left\{ \varphi_{j,k}(t), \varphi_{bj}(t), \varphi_{bj}(L-t), 0 \le k \le n_j - 4 \right\}$$

$$W_j = span\left\{ \psi_{j,k}(t), k = -1, 0, ..., n_j - 2 \right\}$$

in which $n_j = 2^j L$

$$\varphi_{j,k}(t) = \varphi(2^j t - k), \ \ \varphi_{bj}(t) = \varphi_b(2^j t), \ \ 0 \le k \le n_j - 4$$

$$\psi_{j,k}(t) = \psi(2^j t - k), \ \ 0 \le j, \ \ k = 0, 1, ..., n_j - 3$$

$$\psi_{j,-1}(t) = \psi_b(2^j t), \ \ \psi_{j,n_j-2}(t) = \psi_b(2^j (L - t))$$

For convenience, we set $\psi_{-1,k}(t) = \varphi_{0,k}(t)$, $0 \le k \le L-4$, $\psi_{-1,-1}(t) = \varphi_b(t)$, $\psi_{-1,L-3}(t) = \varphi_b(L-t)$, $n_{-1} = L - 1$.

Let

$$B_j = \left\{ 2^{-3j/2} \psi_{j,k(t)} \right\}_{k=-1}^{n_j-2}, \ -1 \le j \le \infty.$$

Wang [22] proved $B = \bigcup_{j=-1}^{\infty} B_j$ is an unconditional basis of $H_0^2(I)$, which turns out to be a basis of continuous space $C_0(I)$. For non-homogeneity Sobolev space $H^2(I)$, Cai and Wang [2] introduced boundary spline functions,

$$\eta_1(t) = (1-t)_+^3$$

$$\eta_2(t) = 2t_+ - 3t_+^2 + \frac{7}{6} t_+^3 - \frac{4}{3}(t-3)_+^3 + \frac{1}{6}(t-2)_+^3$$

to deal with the values of functions at boundary points.

2.1 The function approximation and convergence

Any function $f(t) \in H_0^2(I)$ can be uniquely expanded into cubic spline wavelet series by

$$f(t) = \sum_{j=-1}^{\infty} \sum_{k=1}^{n_j} d_{j,k} \psi_{j,k-2}(t) \tag{1}$$

with

$$d_{j,k} = \int_I f''(t) \left(\psi_{j,k-2}^* \right)''(t) dt$$

where $\psi_{j,k}^*(t)$ are dual functions of $\psi_{j,k}$. Truncating the infinite series (1) at $J-1$, we get

$$f_J(t) = \sum_{j=-1}^{J-1} \sum_{k=1}^{n_j} d_{j,k} \psi_{j,k-2}(t) \tag{2}$$

From Wang [22], we have

$$\| f(t) - f_J(t) \|_{H_0^2}^2 \to 0 \ \text{ as } \ J \to \infty$$

Hence any function $f(t) \in H_0^2(I)$ can be approximated by $f_J(t)$ defined in (2) and any function $f(t) \in H^2(I)$ can be approximated by,

$$f_J(t) = I_{b,J} f(t) + \sum_{j=-1}^{J-1} \sum_{k=1}^{n_j} d_{j,k} \psi_{j,k-2}(t) \tag{3}$$

and the approximation order is $O(2^{-4J})$ if $f(t)$ is sufficiently smooth [6], [10], [23], where

$$I_{b,J}f(t) = a_1\eta_1(2^Jt) + a_2\eta_2(2^Jt) + a_3\eta_2(2^J(L-t)) + a_4\eta_1(2^J(L-t))$$

Suppose $N = 2^JL + 3$ and $\Omega_J(t)$ is a $1 \times N$ vector as

$$\begin{aligned}\Omega_J(t) = \Big[&\eta_1(2^Jt), \eta_2(2^Jt), \eta_2(2^J(L-t)), \eta_1(2^J(Lt)), \psi_{-1,-1}(t), \psi_{-1,0}(t), ..., \psi_{-1,n_{-1}-2}(t),\\ &\psi_{0,-1}(t), \psi_{0,0}(t), \psi_{0,1}(t), ..., \psi_{0,L-3}(t), \psi_{0,n_0-2}(t)\\ &\psi_{1,-1}(t), \psi_{1,0}(t), \psi_{1,1}(t), ..., \psi_{1,2L-3}(t), \psi_{1,n_1-2}(t)\\ &...\\ &\psi_{J-1,-1}(t), \psi_{J-1,0}(t), \psi_{J-1,1}(t), ..., \psi_{J-1,n_J-3}(t), \psi_{J-1,n_J-2}(t)\Big]\\ &\triangleq \Big[\omega_1(t), \omega_2(t), ..., \omega_N(t)\Big]\end{aligned}$$

f_J (t) defined in (3) can be rewritten as

$$f_J(t) = \sum_{k=1}^{N} \widehat{f}_k\omega_k(t) = \Omega_J(t)\hat{f} \tag{4}$$

where $\hat{f} = [\hat{f}_1, \hat{f}_2, ..., \hat{f}_N]^T$ are the wavelet expansion coefficients, which can be determined by interpolating at collocation points,

$$t_{-1}^{(-1)} = \frac{1}{2^{J+1}}, \quad t_{L+1}^{(-1)} = L - \tfrac{1}{2^{J+1}},\ t_k^{(-1)} = k,\ k = 0, 1, ..., L \tag{5}$$

$$t_k^{(j)} = \frac{k+1.5}{2^j}, \quad -1 \le k \le 2^jL - 2,\ 0 \le j \le J-1. \tag{6}$$

The point value vanishing property of the wavelet function $\psi_{j,k}(t)$ and the compact supports of scaling spline functions $\varphi(t)$, $\varphi_b(t)$ can be used to reduce computational complexity [2].

Since all the wavelet basis functions are composed by one-sided power functions $(at-b)_+^k$ and $(b-at)_+^k$, $a > 0$, $b \ge 0$, $k = 1, 2, 3$, if the analytical expressions of $D^\alpha(at-b)_+^k$ and $D^\alpha(b-at)_+^k$, $a > 0$, $b \ge 0$, $k = 1, 2, 3$ are obtained, those of $D^\alpha\Omega_J(t)$ can be naturally achieved.

For one-sided power function $(at-b)_+^k$ and $(b-at)_+^k$, $a > 0$, $b \ge 0$, $k = 1, 2, 3$, analytical expressions of their fractional derivative can be obtained by the properties of Laplace transform and fractional derivatives.

For $(at-b)_+^k$, $a > 0$, $b > 0$ the results are as follows:

Theorem 2.1. *For* $m < \alpha \le m+1$, $m \in \mathbb{N}, k > 0, t > 0, a > 0, b \ge 0$, *if* $\alpha \le k$ *or* $\alpha \notin \mathbb{N}$, *we have*

$$D^\alpha(at-b)_+^k = a^\alpha\frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)}(at-b)_+^{k-\alpha}. \tag{7}$$

Proof. Let

$$f(t) = t_+^k, \quad g(t) = f(at-b) = (at-b)_+^k.$$

Then

$$F(s) = \int_0^\infty e^{-st} f(t)dt = \Gamma(1+k)s^{-1-k}. \tag{8}$$

According to the property of the Laplace transform, we have

$$G(s) = \mathcal{L}\big[f(at-b)\big](s) = \frac{1}{a}e^{-\left(\frac{b}{a}\right)s}F\Big(\frac{s}{a}\Big). \tag{9}$$

Now, from the property of fractional derivative, we can obtain

$$\begin{aligned}
\mathcal{L}\big[D^\alpha (at-b)_+^k\big](s) &= \mathcal{L}\big[D^\alpha g(t)\big](s) \\
&= s^\alpha G(s) - \sum_{n=0}^{m} s^{\alpha-n-1} g^{(n)}(0) \\
&= a^k \frac{\Gamma(1+k)}{\Gamma(1+k-a)} \mathcal{L}\left[\Big(t-\frac{b}{a}\Big)_+^{k-\alpha}\right] \\
&= a^\alpha \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)} \mathcal{L}\big[(at-b)_+^{k-\alpha}\big].
\end{aligned}$$

From the uniqueness of Laplace transform, we get

$$D^\alpha (at-b)_+^k = a^\alpha \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)} (at-b)_+^{k-\alpha}.$$

Similarly we can derive the analytical expressions of $D^\alpha (b-at)_+^k, \;\; 0 < a, 0 < b$.

Theorem 2.2. *For $m < \alpha \le m+1, \; m \in \mathbb{N}, \; k \in \mathbb{N}, \; 0 < t, \; 0 < a, \; 0 \le b$, if $0 < \alpha \le k$, we have*

$$D^\alpha (b-at)_+^k = \frac{(-1)^{k-1} a^\alpha k!}{\gamma_{k+1-\alpha}} (at-b)_+^{k-\alpha} + \sum_{l=m+1}^{k} \binom{k}{l} (-1)^l a^k b^{k-l} \frac{l!}{\gamma_{l+1-\alpha}} t_+^{l-\alpha}.$$

If $k < \alpha$ and $\alpha \notin \mathbb{N}$, we have

$$D^\alpha (b-at)_+^k = \frac{(-1)^{k-1} a^\alpha k!}{\gamma_{k+1-\alpha}} (at-b)_+^{k-\alpha},$$

where $\gamma_n = \Gamma(n-\alpha), \; n \in \mathbb{N}$.

Using the expressions of fractional derivative for one-sided power functions, we can get analytical expressions of fractional derivative for the wavelet basis functions as follows, if $\alpha \notin \mathbb{N}$

$$
\begin{aligned}
D^{\alpha}\varphi(t) &= \frac{1}{\gamma_4}\sum_{l=0}^{4}\binom{4}{l}(-1)^l(t-l)_+^{3-\alpha} \triangleq \Phi(t)\\
D^{\alpha}\varphi(L-t) &= \frac{1}{\gamma_4}\sum_{l=0}^{4}\binom{4}{l}(-1)^l(t-L+l)_+^{3-\alpha} \triangleq \Phi_r(t,L)\\
D^{\alpha}\varphi_b(t) &= \frac{3}{\gamma_3}t_+^{2-\alpha} - \frac{11}{2\gamma_4}t_+^{3-\alpha} + \frac{9}{\gamma_4}(t-1)_+^{3-\alpha} - \frac{9}{2\gamma_4}(t-2)_+^{3-\alpha}\\
&\quad + \frac{1}{\gamma_4}(t-3)_+^{3-\alpha} \triangleq \Phi_{bl}(t)\\
D^{\alpha}\varphi_b(L-t) &= -\frac{3}{\gamma_3}(t-L)_+^{2-\alpha} - \frac{11}{2\gamma_4}(t-L)_+^{3-\alpha} + \frac{9}{\gamma_4}(t-L+1)_+^{3-\alpha}\\
&\quad - \frac{9}{2\gamma_4}(t-L+2)_+^{3-\alpha} + \frac{1}{\gamma_4}(t-L+3)_+^{3-\alpha} \triangleq \Phi_{br}(t,L)\\
D^{\alpha}\varphi_{j,k}(t) &= 2^{j\alpha}\Phi(2^jt-k),\ 0\le j, k=0,1,...,2^jL-4,\\
D^{\alpha}\psi(t) &= 2^{\alpha}\left(-\frac{3}{7}\Phi(2t)+\frac{12}{7}\Phi(2t-1)-\frac{3}{7}\Phi(2t-2)\right) \triangleq \Psi(t)\\
D^{\alpha}\psi_b(t) &= 2^{\alpha}\left(\frac{24}{13}\Phi_{bl}(2t)-\frac{6}{13}\Phi(2t)\right) \triangleq \Psi_{bl}(t)\\
D^{\alpha}\psi_b(L-t) &= 2^{\alpha}\left(\frac{24}{13}\Phi_{br}(2t,2L)-\frac{6}{13}\Phi_r(2t,2L)\right) \triangleq \Psi_{br}(t,L)\\
D^{\alpha}\psi_{j,k}(t) &= 2^{j\alpha}\Psi(2^jt-k),\ 0\le j,\ k=0,1,...,2^jL-3\\
D^{\alpha}\psi_b(2^jt) &= 2^{j\alpha}\Psi_{bl}(2^jt),\ \ D^{\alpha}\Psi_b(2^j(L-t)) = 2^{j\alpha}\Psi_{br}(2^jt,2^jL),\\
D^{\alpha}\eta_1(x) &= \frac{6}{\gamma_4}\begin{cases}(x-1)_+^{3-\alpha}-x_+^{3-\alpha}+\frac{\gamma_4}{\gamma_3}x_+^{2-\alpha}-\frac{\gamma_4}{2\gamma_2}x_+^{1-\alpha}, & 0<\alpha\le 1\\ (x-1)_+^{3-\alpha}-x_+^{3-\alpha}+\frac{\gamma_4}{\gamma_3}x_+^{2-\alpha}, & 1<\alpha\le 2\\ (x-1)_+^{3-\alpha}-x_+^{3-\alpha}, & 2<\alpha\le 3\\ (x-1)_+^{3-\alpha}, & 3<\alpha \text{ and } \alpha\notin\mathbb{N}\end{cases}\\
D^{\alpha}\eta_2(t) &= \frac{2}{\gamma_2}t_+^{1-\alpha} - \frac{6}{\gamma_3}t_+^{2-\alpha} + \frac{7}{\gamma_4}t_+^{3-\alpha} - \frac{8}{\gamma_4}(t-1)_+^{3-\alpha} + \frac{1}{\gamma_4}(t-2)_+^{3-\alpha}\\
D^{\alpha}\eta_2(L-t) &= \frac{2}{\gamma_2}(t-L)_+^{1-\alpha} + \frac{6}{\gamma_3}(t-L)_+^{2-\alpha} + \frac{7}{\gamma_4}(t-L)_+^{3-\alpha} - \frac{8}{\gamma_4}(t-L+1)_+^{3-\alpha}\\
&\quad + \frac{1}{\gamma_4}(t-L+2)_+^{3-\alpha}\\
D^{\alpha}\eta_1(L-t) &= \frac{6}{\gamma_4}(t-L+1)_+^{3-\alpha}
\end{aligned}
$$

where $\gamma_n = \Gamma(n-\alpha),\ n\in\mathbb{N}$.

3. Solving fractional relaxation-oscillation equation by cubic B-spline wavelet collocation method

We consider the linear fractional differential equation

$$
\left\{ \begin{array}{rcll} D^{\alpha}y(t)+y(t) & = & f(t), & 0<t\leq L \\ y^{(n)}(0) & = & y_0^{(n)}, & n=0,1,2,...,m \end{array} \right. \tag{10}
$$

where $m<\alpha\leq m+1,\ m\in\mathbb{N}$ and D^{α} denotes the Caputo fractional derivative of order α. To solve problem (10), we approximate $y(t)$ by

$$
y(t)\approx\sum_{k=1}^{N}\hat{y}_k\omega_k(t)=\Omega_J(t)\hat{y}\triangleq y_J(t) \tag{11}
$$

where vector $\hat{y}=[\hat{y}_1,...,\hat{y}_N]^T$ is unknown. The α order derivative of $y(t)$ is approximated by

$$
D^{\alpha}y(t)\approx\sum_{k=1}^{N}\hat{y}_kD^{\alpha}\omega_k(t)=D^{\alpha}\Omega_J(t)\hat{y}\triangleq D^{\alpha}y_J(t) \tag{12}
$$

Denote all collocation points defined in (5) and (6) in an order set $\{t_i\}_{i=1}^{N}$, where $t_1=0$. The expansion coefficients $\hat{y}$ can be determined by interpolating conditions at all collocation points, namely

$$
\begin{array}{rcll} y_J(t_k) & = & y(t_k), & k=1,2,...,N \\ D^{\alpha}y_J(t_k) & = & D^{\alpha}y(t_k), & k=2,3,...,N \end{array}
$$

Consequently, interpolating the fractional differential equation (10) by $y_J(t)$ at all collocation points, we obtain

$$
\left. \begin{array}{rcll} D^{\alpha}y_J(t_k) & = & -y_J(t_k)+f(t_k), & 2\leq k\leq N \\ y_J^{(n)}(0) & = & y^{(n)}(0), & n=0,1,2,...,m \end{array} \right\} \tag{13}
$$

where $m<\alpha\leq m+1,\ m\in\mathbb{N}$.

Denote

$$
B_1 = \begin{pmatrix} \omega_1(t_2) & \omega_2(t_2) & \cdots & \omega_N(t_2) \\ \dots & \dots & \ddots & \dots \\ \omega_1(t_N) & \omega_2(t_N) & \cdots & \omega_N(t_N) \end{pmatrix}
$$

$$
B_2 = \begin{pmatrix} D^{\alpha}\omega_1(t_2) & D^{\alpha}\omega_2(t_2) & \cdots & D^{\alpha}\omega_N(t_2) \\ \dots & \dots & \ddots & \dots \\ D^{\alpha}\omega_1(t_N) & D^{\alpha}\omega_2(t_N) & \cdots & D^{\alpha}\omega_N(t_N) \end{pmatrix}
$$

where B_1 and B_2 are obtained by analytical method. Then, according to equations (11), (12), equation (13) can be represented as,

$$\left\{ \begin{array}{l} A\hat{y} = b \\ c_n\hat{y} = y_0^{(n)}, \quad n = 0, ..., m, \end{array} \right. \tag{14}$$

where $A = B_1 + B_2$ and

$$b = \Big(f(t_2), ..., f(t_N)\Big)^T,$$
$$c_n = \Big(\omega_1^{(n)}(0), \omega_2^{(n)}(0), ..., \omega_N^{(n)}(0)\Big), \ n = 0, 1, ..., m.$$

Let

$$A_m^T = \Big(A^T, c_0^T, c_1^T, ..., c_m^T\Big), \quad b_m^T = \Big(b^T, y_0^{(0)}, y_0^{(1)}, ..., y_0^{(m)}\Big).$$

Then (14) can be written as,

$$A_m\hat{y} = b_m. \tag{15}$$

Consequently, the wavelet expansion coefficient $\hat{y}$ can be obtained by solving linear equations (15), the approximated solution $y_J(t)$ in (10) can be effectively constructed by discrete wavelet transform technique.

If $0 < \alpha < 1$, the coefficient matrix A_0 is nonsingular, so equation (15) have and only have one solution. If $1 < \alpha$, equations (15) are over-determined, coefficient matrix A_m is column full rank and least squares solution can be regarded as the approximate solution.

4. Numerical Examples

In order to illustrate the effectiveness of the proposed method, some numerical examples are given in this section. The examples presented have exact solutions and also have been solved by other numerical methods. This allows us to compare the numerical results obtained by proposed method with the analytical solutions or those obtained by the other methods. Absolute errors between approximate solutions y_N and the corresponding exact solutions y, i.e., $N_e = |y_N - y|$ are considered.

Example 1. Let us consider the fractional relaxation-oscillation equation [8], [13]

$$D_a^{3/2}y(x) = -y(x)$$

with initial conditions $y(0) = 1, \ \ y^{'}(0) = 0$.

For $0 < \alpha \leq 2$, the exact solution of this problem is $y(x) = E_\alpha(-x^\alpha)$. Here $E_\alpha(x)$ is called the Mittag-Leffler function,

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}.$$

The numerical results are obtained by proposed cubic B-spline wavelet collocation method for $L = 10$ and $J = 3, 4, 5$. Comparisons of numerical and exact solutions are shown in Table 1. Additionally, we tabulated the comparison results between proposed method for $J = 5$ and Taylor collocation method [8] in Table 2.

Table 1. Numerical results for Example 1.

x	Exact Solution	Present Method					
		$J = 3$	Ne for $J = 3$	$J = 4$	Ne for $J = 4$	$J = 5$	Ne for $J = 5$
0.0	1.0000000	1.0000000	$0.0000E-0$	1.0000000	$0.0000E-0$	1.0000000	$0.0000E-0$
0.1	0.9763777	0.9763799	$0.2200E-5$	0.9763777	$0.1000E-10$	0.9763777	$0.2000E-14$
0.2	0.9340362	0.9340468	$0.1064E-4$	0.9340362	$0.5810E-10$	0.9340362	$0.7000E-13$
0.3	0.8808084	0.8808726	$0.6425E-4$	0.8808084	$0.2528E-9$	0.8808084	$0.9025E-12$
0.4	0.8200563	0.8202624	$0.2061E-3$	0.8200563	$0.6709E-8$	0.8200563	$0.2500E-12$
0.5	0.7540488	0.7547919	$0.7431E-3$	0.7540488	$0.7232E-7$	0.7540488	$0.3251E-11$
0.6	0.6845298	0.6863748	$0.1845E-2$	0.6845298	$0.4312E-7$	0.6845298	$0.6219E-11$
0.7	0.6129215	0.6165955	$0.3674E-2$	0.6129217	$0.2279E-6$	0.6129215	$0.8174E-11$
0.8	0.5404169	0.5473109	$0.6894E-2$	0.5404165	$0.4852E-6$	0.5404169	$0.4549E-10$
0.9	0.4680306	0.4797706	$0.1174E-1$	0.4680322	$0.1658E-5$	0.4680306	$0.6731E-10$
1.0	0.3966293	0.4146093	$0.1798E-1$	0.3966272	$0.2127E-5$	0.3966293	$0.9272E-10$

Table 2. Comparisons of errors between present method and Taylor Collocation method [8]

x	Exact Solution	Present met. for $J = 5$	Taylor Col. Met. [8]
0.0	1.00000000	$0.00000E-0$	$0.00000E-0$
0.2	0.93403621	$0.52954E-15$	$0.44805E-7$
0.4	0.82232699	$0.12194E-14$	$0.60316E-7$
0.6	0.68452989	$0.37252E-11$	$0.67627E-7$
0.8	0.54041695	$0.21659E-10$	$0.69751E-7$
1.0	0.39662936	$0.75281E-10$	$0.66570E-7$

Figure 1 shows the comparison of approximate solutions and exact solutions. Figure 2 shows the comparison of error functions.

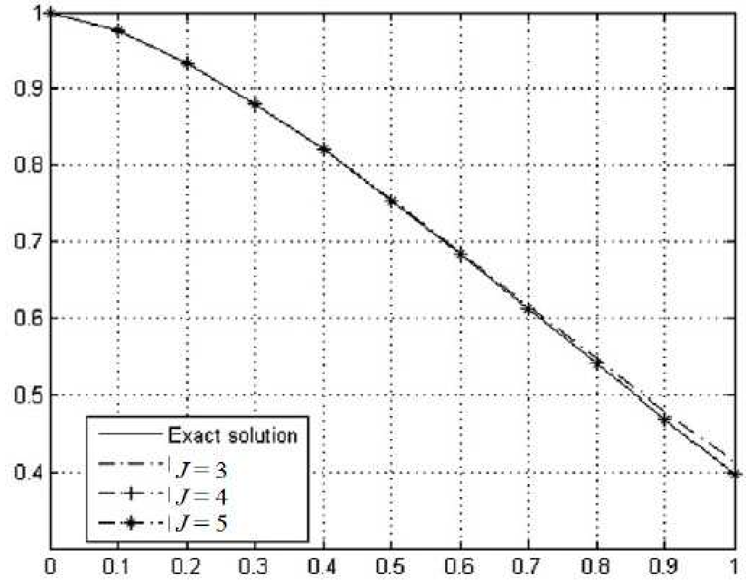

Figure 1. Comparison of approximate solutions and exact solutions.

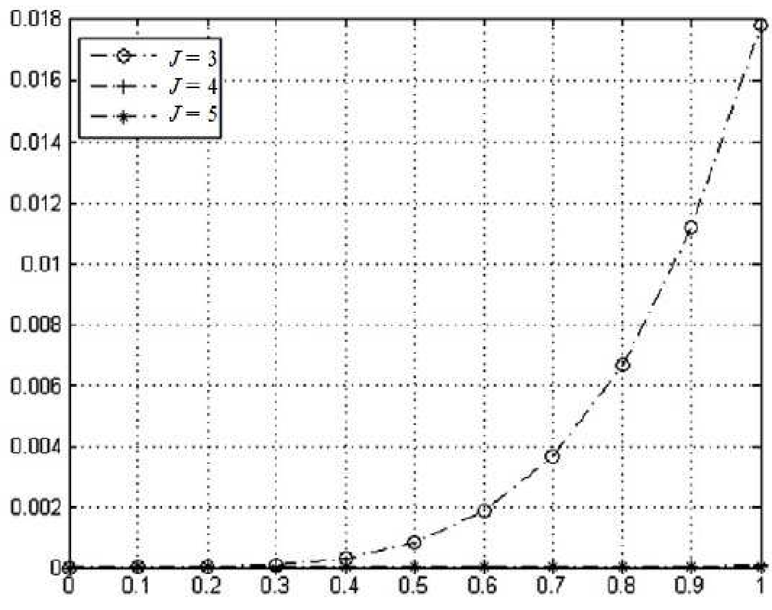

Figure 2. Comparison of errors at different levels.

Example 2. Consider the following fractional relaxation-oscillation equation,

$$D_a^{1/2} y(x) = -y(x)$$

with initial condition $y(0) = 1$.

For $0 \leq x \leq 1$, the approximate solutions for Example 2 are obtained by using the proposed cubic B-spline wavelet collocation method for $L = 10$ and $J = 3, 4, 5$. Comparison of numerical results with the exact solution is shown in table 3 and plotted the numerical results in Figure 3 for various J.

Table 3. Numerical results for Example 2.

x	Exact Solution	Present Method					
		$J = 3$	Ne for $J = 3$	$J = 4$	Ne for $J = 4$	$J = 5$	Ne for $J = 5$
0.0	1.0000000	1.0000000	0.0000$E-0$	1.0000000	0.0000$E-0$	1.0000000	0.0000$E-0$
0.1	0.7235784	0.7235782	0.1618$E-6$	0.7235784	0.3174$E-10$	0.7235784	0.7416$E-13$
0.2	0.6437882	0.6437925	0.4382$E-5$	0.6437882	0.3472$E-10$	0.6437882	0.3095$E-12$
0.3	0.5920184	0.5920375	0.1916$E-4$	0.5920184	0.9437$E-9$	0.5920184	0.6723$E-12$
0.4	0.5536062	0.5539199	0.3137$E-3$	0.5536062	0.6128$E-8$	0.5536062	0.1846$E-11$
0.5	0.5231565	0.5303615	0.7205$E-2$	0.5231565	0.1215$E-7$	0.5231565	0.4251$E-11$
0.6	0.4980245	0.5088645	0.1084$E-1$	0.4980245	0.2189$E-7$	0.4980245	0.8736$E-11$
0.7	0.4767027	0.4950143	0.1834$E-1$	0.4767025	0.2891$E-6$	0.4767027	0.2519$E-10$
0.8	0.4582460	0.4929617	0.3471$E-1$	0.4582469	0.9954$E-6$	0.4582460	0.5291$E-10$
0.9	0.4420214	0.4879841	0.4596$E-1$	0.4420289	0.7534$E-5$	0.4420214	0.9254$E-10$
1.0	0.4275835	0.4801973	0.5261$E-1$	0.4275934	0.9915$E-5$	0.4275835	0.4518$E-9$

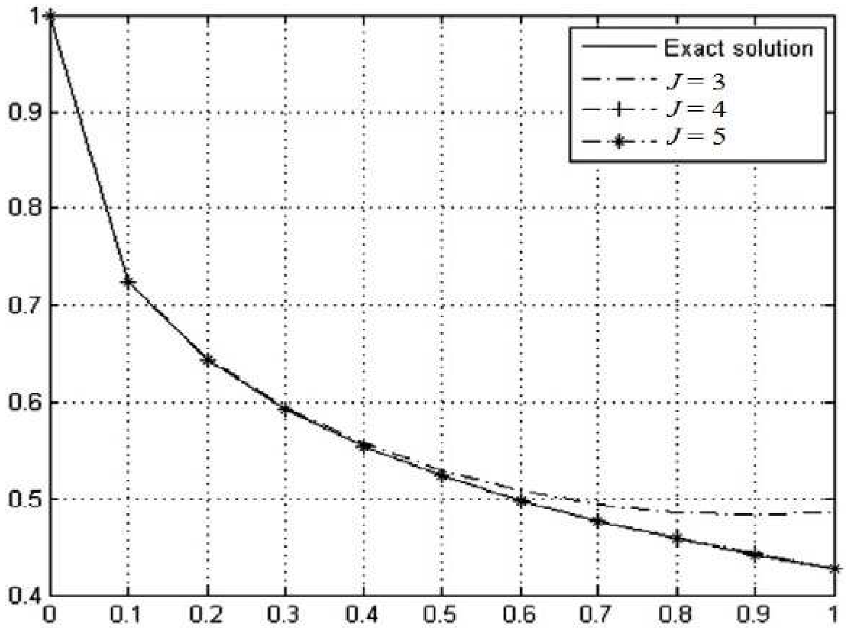

Figure 3. Comparison of approximate solutions and exact solutions.

Example 3. Consider the following fractional relaxation-oscillation model,

$$D_x^{\alpha} y(x) + Ay(x) = f(x), \quad x > 0,$$

with initial condition

$$y(0) = a, \quad \text{if } \; 0 < \alpha \leq 1$$

or

$$y(0) = \lambda \; \text{ and } \; y^{'}(0) = \mu, \quad \text{if } \; 1 < \alpha \leq 2$$

where A is a positive constant.

First, consider $A = 1$ and $f(x) = 0$.

For $L = 10, J = 5$ on applying the proposed method for $\alpha = 0.25, 0.5, 0.75$ and 1 with initial condition $y(0) = 1$, Figure 4 shows that the numerical results are consistent with the exact ones and as α approaches to 1, the corresponding solutions of Example 3, approach that of integer-order differential equation.

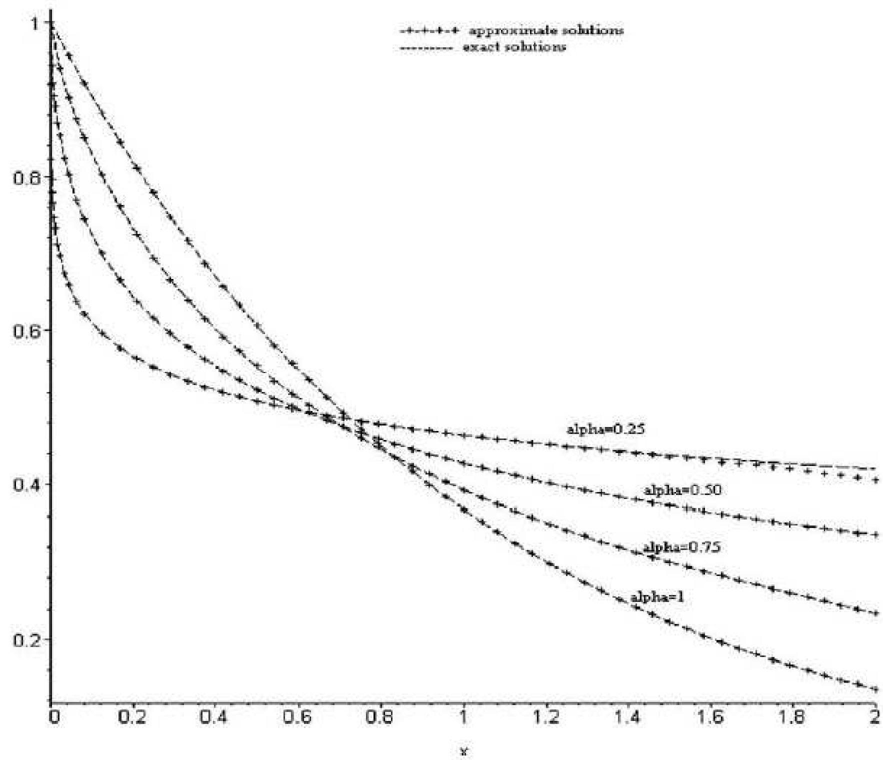

Figure 4. Comparison of approximate results with exact solutions for different values of α.

For $\lambda = 1$ and $\mu = 0$, Figure 5 illustrates the numerical solutions by proposed method and exact solution for $\alpha = 1.25, 1.5, 2$. Obviously, the numerical results are in good agreement with the exact ones. For $\alpha = 2$, the above fractional relaxation-oscillation equation given in Example 3, is the oscillation equation and the exact solution is $y(x) = cosx$.

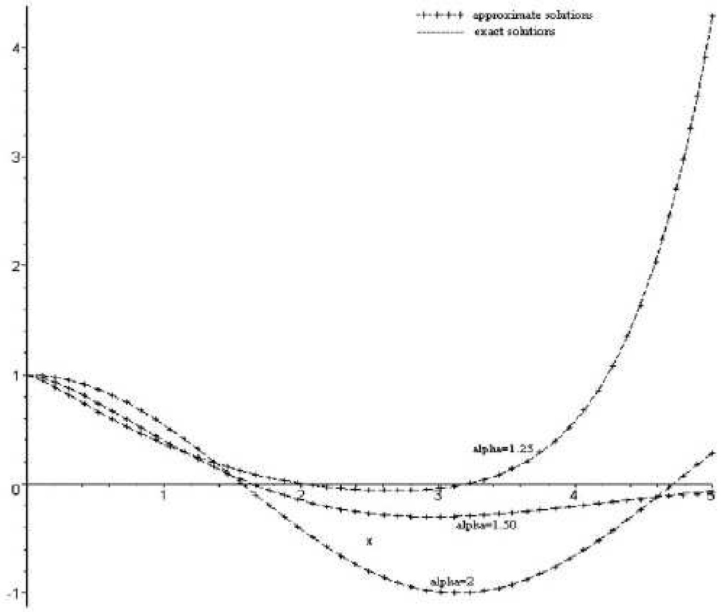

Figure 5. Comparison of approximate solutions with exact solutions for different values of α.

5. Conclusion

In this study, the cubic B-spline wavelet collocation method has been applied to obtain approximate solutions of fractional relaxation-oscillation equation. We have demonstrated the accuracy and efficiency of the proposed technique. The convergence of the method can be seen from the given figures. The better approximations may be obtained by increasing the Value of J. Table 2 shows that the present method is more accurate than Taylor collocation method. Numerical results obtained by the proposed method fairly match with exact solutions. The error of wavelet collocation method does not accumulate over time, so the proposed cubic B-spline wavelet collocation method is very simple, accurate and valid to solve fractional relaxation-oscillation equation.

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Accepted: 18.12.2015