QUASI-FREDHOLM, SAPHAR SPECTRA FOR C_0 SEMIGROUPS GENERATORS

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Abstract. In this work, we show that the spectral inclusion of semigroups hold for Saphar, essentially Saphar and quasi-Fredholm spectra. Some stabilities results are also established.

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1. Introduction and preliminaries

Throughout, X denotes a complex Banach space, let A be a closed linear operator on X with domain $D(A)$, we denote by A^*, $R(A)$, $N(A)$, $R^\infty(A) = \bigcap_{n\geq 0} R(A^n)$, $\sigma(A)$, respectively the adjoint, the range, the null space, the hyper-range, the spectrum of A.

Recall that a closed operator A is said to be Kato operator or semi-regular if $R(A)$ is closed and $N(A) \subseteq R^\infty(A)$. Denote by $\rho_K(A)$:

$\rho_K(A) = \{\lambda \in \mathbb{C} : A-\lambda I \text{ is Kato }\}$ the Kato resolvent and $\sigma_K(A) = \mathbb{C}\backslash\rho_K(A)$ the Kato spectrum of A. It is well known that $\rho_K(A)$ is an open subset of $\mathbb{C}$.

For subspaces M, N of X we write $M \subseteq^e N$ (M is essentially contained in N) if there exists a finite-dimensional subspace $F \subset X$ such that $M \subseteq N + F$.

A closed operator S is called a generalized inverse of A if $R(A) \subseteq D(S)$, $R(S) \subseteq D(A)$, $ASA = A$ on $D(A)$ and $SAS = S$ on $D(S)$, wich equivalent to the fact that $R(A) \subseteq D(S)$, $R(S) \subseteq D(A)$, $ASA = A$ on $D(A)$.

A closed operator A is called a Saphar operator if A has a generalized inverse and $N(A) \subseteq R^\infty(A)$, which equivalent to the fact that A is Kato operator and has a generalized inverse, see [6].

If we assume in the definition above that $N(A) \subseteq^e R^\infty(A)$, A is said to be a essentially Saphar operator. The Saphar and essentially Saphar spectra are defined by

$$\begin{aligned}\sigma_{Sap}(A) &= \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Saphar}\}.\\ \sigma^e_{Sap}(A) &= \{\lambda \in \mathbb{C} : A - \lambda \text{ is not essentially Saphar}\}.\end{aligned}$$

$\sigma_{Sap}(A)$ is a compact non empty set of $\mathbb{C}$ and we have

$$\partial\sigma(A) \subseteq \sigma_K(A) \subseteq \sigma_{Sap}(A) \subseteq \sigma(A).$$

A one-parameter family $\{T(t)\}_{t\geq 0}$ of operators on X is called a C_0-semigroup of operators if:

1. $T(0) = I$.
2. $T(t+s) = T(t)T(s)$, $\forall t, s \geq 0$.
3. $\lim\limits_{t\to 0} T(t)x = x$, $\forall x \in X$.

$\{T(t)\}_{t\geq 0}$ has a unique infinitesimal generator A defined in the domain $D(A)$ by:

$$\begin{aligned}D(A) &= \left\{x \in X : \lim_{t\to 0}\frac{T(t)x - x}{t} \text{ exists}\right\} \text{ and}\\ Ax &= \lim_{t\to 0}\frac{T(t)x - x}{t}, \forall x \in D(A).\end{aligned}$$

Recall that for all $t \geq 0$, $T(t)$ is a bounded linear operator in X and A is a closed operator. Details for all this may be found in [7], [4].

Next, we introduce the following operator acting on X and depending on the parameters $\lambda \in \mathbb{C}$ and $t \geq 0$:

$$B_\lambda(t)x = \int_0^t e^{\lambda(t-s)}T(s)xds, x \in X.$$

It is well known that $B_\lambda(t)$ is a bounded linear operator on X [7, 4] and we have:

$$\begin{aligned}(e^{\lambda t} - T(t))^n x &= (\lambda - A)^n B^n_\lambda(t)x, \quad \forall x \in X,\ n \in \mathbb{N}\\ (e^{\lambda t} - T(t))^n x &= B^n_\lambda(t)(\lambda - A)^n x, \quad \forall x \in D(A^n),\ n \in \mathbb{N};\\ R^\infty(e^{\lambda t} - T(t)) &\subseteq R^\infty(\lambda - A);\\ N((\lambda - A)^n) &\subseteq N(e^{\lambda t} - T(t))^n.\end{aligned}$$

2. Spectral inclusions and stability for Saphar spectrum

Lemma 2.1 [8] *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. For $\lambda \in \mathbb{C}$ and $t \geq 0$, let $F_\lambda(t)x = \int_0^t e^{-\lambda s}B_\lambda(s)xds$, then:*

1. *There exist a* $M \geq 1$ *and* $\omega > Re(\lambda)$ *such that*

$$F_\lambda(t) \leq \frac{M}{(\omega - Re(\lambda))^2} e^{(\omega - Re(\lambda))t}.$$

2. $\forall x \in X$, $F_\lambda(t)x \in D(A)$ *and* $(\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = tI$ *with*

$$G_\lambda(t) = e^{-\lambda t} I.$$

3. *The operators* $F_\lambda(t)$, $G_\lambda(t)$ *and* $B_\lambda(t)$ *are pairwise commute and for all* $x \in D(A)$*:*

$$\begin{aligned}(\lambda - A)F_\lambda(t)x &= F_\lambda(t)(\lambda - A)x\\ (\lambda - A)G_\lambda(t)x &= G_\lambda(t)(\lambda - A)x\\ (\lambda - A)B_\lambda(t)x &= B_\lambda(t)(\lambda - A)x\end{aligned}$$

Lemma 2.2 *Let* $\{T(t)\}_{t\geq 0}$ *a* C_0*-semigroup on* X *with infinitesimal generator* A*. Then for all* $t > 0$ *we have:*

$$e^{\lambda t} - T(t) \text{ has a generalized inverse} \Longrightarrow \lambda - A \text{ has a generalized inverse.}$$

Proof. Suppose that $e^{\lambda t} - T(t)$ has a generalized inverse then there exists a $S \in \mathcal{B}(X)$ such that:

$$(e^{\lambda t} - T(t))S(e^{\lambda t} - T(t)) = e^{\lambda t} - T(t)$$

According to Lemma 2.1, we have $(\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = tI$. Then:

$$\begin{aligned}t(\lambda - A) &= (\lambda - A)F_\lambda(t)(\lambda - A) + B_\lambda(t)G_\lambda(t)(\lambda - A)\\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (\lambda - A)B_\lambda(t)G_\lambda(t)\\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (e^{\lambda t} - T(t))G_\lambda(t)\\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (e^{\lambda t} - T(t))S(e^{\lambda t} - T(t))G_\lambda(t)\\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (\lambda - A)B_\lambda(t)S(\lambda - A)B_\lambda(t)G_\lambda(t)\\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (\lambda - A)B_\lambda(t)SB_\lambda(t)G_\lambda(t)(\lambda - A)\\ &= (\lambda - A)[F_\lambda(t) + B_\lambda(t)SB_\lambda(t)G_\lambda(t)](\lambda - A).\end{aligned}$$

Hence $\lambda - A$ has a generalized inverse. ∎

Theorem 2.1 *Let* $\{T(t)\}_{t\geq 0}$ *be a* C_0*-semigroup on* X *with infinitesimal generator* A*. Then for all* $t > 0$*:*

$$e^{t\sigma_{Sap}(A)} \subseteq \sigma_{Sap}(T(t)), \;\; e^{t\sigma^e_{Sap}(A)} \subseteq \sigma^e_{Sap}(T(t)).$$

Proof. Assume that $e^{\lambda t} - T(t)$ is a Saphar operator, then $e^{\lambda t} - T(t)$ has a generalized inverse and $N(e^{\lambda t} - T(t)) \subseteq R(e^{\lambda t} - T(t))$.

By Lemma 2.2, $\lambda - A$ has a generalized inverse, and we have:

$$N(\lambda - A) \subseteq N(e^{\lambda t} - T(t)) \subseteq R(e^{\lambda t} - T(t))^{\infty} \subseteq R(\lambda - A)^{\infty}.$$

Therefore, $\lambda - A$ is a Saphar operator.

Let M a finite dimensional subspace of X. We have

$$N(\lambda - A) \subseteq N(e^{\lambda t} - T(t)) \subseteq R(e^{\lambda t} - T(t))^{\infty} + M \subseteq R(\lambda - A)^{\infty} + M.$$

Hence $e^{\lambda t} - T(t)$ is an essentially Saphar operator implying that $\lambda - A$. ∎

Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. $\{T(t)\}_{t\geq 0}$ is said to be strongly stable if $\lim\limits_{t\to\infty} ||T(t)x|| = 0$ for all $x \in X$. We say that $\{T(t)\}_{t\geq 0}$ is uniformly stable if $\lim\limits_{t\to\infty} ||T(t)|| = 0$.

In [2], A. Elkoutri and M.A. Taoudi showed that $\{T(t)\}_{t\geq 0}$ is strongly stable if $\sigma_K(A) \cap i\mathbb{R} = \emptyset$.

In the following, we give a stability result for strongly continuous semigroups using the Saphar spectrum:

Proposition 2.1 *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. The following assertions are equivalent:*

1. $\sigma(A) \cap i\mathbb{R} = \emptyset$,
2. $\sigma_K(A) \cap i\mathbb{R} = \emptyset$,
3. $\sigma_{Sap}(A) \cap i\mathbb{R} = \emptyset$.

If the infinitesimal generator A verified one of this properties, $\{T(t)\}_{t\geq 0}$ is strongly stable.

Proof. Since $\sigma_K(A) \subset \sigma_{Sap}(A) \subset \sigma(A)$, the result is immediately comes from [2, Corollary 2.1]. ∎

Recall that a C_0-semigroup $\{T(t)\}_{t\geq 0}$ is said uniformly exponentially stable if there exists $\epsilon > 0$ such that

$$\lim_{t\to\infty} e^{\epsilon t} ||T(t)|| = 0.$$

Proposition 2.2 *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A, Γ is the unit circle in $\mathbb{C}$. The following assertions are equivalent:*

1. *$\{T(t)\}_{t\geq 0}$ is uniformly exponentially stable,*
2. *there exists $t_0 > 0$ such that $\sigma_K(T(t_0)) \cap \Gamma = \emptyset$,*
3. *there exists $t_0 > 0$ such that $\sigma_{Sap}(T(t_0)) \cap \Gamma = \emptyset$.*

Proof. 1) $\Longleftrightarrow$ 2): see [2, Corollary 2.2].

1) $\Longrightarrow$ 3) is obvious, see [4].

3) $\Longrightarrow$ 1) Suppose that there exists $t_0 > 0$ such that $\sigma_{Sap}(T(t_0)) \cap \Gamma = \emptyset$. Since $\sigma_K(T(t_0)) \subseteq \sigma_{Sap}(T(t_0))$, hence $\sigma_K(T(t_0)) \cap \Gamma = \emptyset$. By [2, Corollary 2.2] $\{T(t)\}_{t\geq 0}$ is uniformly exponentially stable. ∎

3. Spectral inclusion for quasi-Fredholm spectrum

Recall from [5] some definitions:

Definition 3.1 Let T a closed linear operator on X and let

$$\Delta(T) = \{n \in \mathbb{N}, \forall m \geq n, R(T^n) \cap N(T) = R(T^m) \cap N(T)\}$$

The degree of stable iteration $dis(T)$ of T is defined as $dis(T) = \inf \Delta(T)$, where $dis(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 3.2 Let T a closed linear operator on X. T is called a quasi-Fredholm operator of degree d if there exists an integer $d \in \mathbb{N}$ such that:

1. $dis(T) = d$;
2. $R(T^n)$ is closed in X for all $n \geq d$;
3. $R(T) + N(T^n)$ is closed in X for all $n \geq d$.

The quasi-Fredholm spectrum is defined by:

$$\sigma_{qF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a quasi-Fredholm}\}.$$

Proposition 3.3 *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. Then:*

$$dis(\lambda - A) \leq dis(e^{\lambda t} - T(t)).$$

Proof. If $dis(e^{\lambda t} - T(t)) = +\infty$ obvious.

If $dis(e^{\lambda t} - T(t)) = d \in \mathbb{N}^*$ then for all $n \geq d$, we have:

$$R((e^{\lambda t} - T(t))^n) \cap N((e^{\lambda t} - T(t))) = R((e^{\lambda t} - T(t))^d) \cap N(e^{\lambda t} - T(t)).$$

Then, for all $n \geq d$, we have:

$$R((\lambda - A)^n) \cap N(\lambda - A) = R((\lambda - A)^d) \cap N(\lambda - A).$$

Indeed, let $y \in R((\lambda - A)^d) \cap N(\lambda - A)$. Then there exists $x \in X$ such that $y = (\lambda - A)^d x$ and, by Lemma 2.1, $(\lambda - A)^d F_d(t) + B_\lambda^d(t) G_d(t) = I$ implies that $(\lambda - A)^d x = (\lambda - A)^{2d} F_d(t) x + (e^{\lambda t} - T(t))^d G_d(t) x$, since $y \in N(\lambda - A)$,

$y = (\lambda - A)^d x = (e^{\lambda t} - T(t))^n z = (\lambda - A)^n B_\lambda^n(t) z \in R((\lambda - A)^n) \cap N(\lambda - A)$ therefore, $dis(\lambda - A) \leq d$.

If $d = 0$, for all $n \geq d$, we have:

$$R((e^{\lambda t} - T(t))^n) \cap N((e^{\lambda t} - T(t))) = N(e^{\lambda t} - T(t)).$$

Then, $\forall n \in \mathbb{N}$,

$$N((e^{\lambda t} - T(t))) \subset R((e^{\lambda t} - T(t))^n).$$

Then

$$N(\lambda - A) \subset N((e^{\lambda t} - T(t))) \subset R((e^{\lambda t} - T(t))^n) \subset R((\lambda - A)^n).$$

Hence

$$N(\lambda - A) \cap R((\lambda - A)^n) = N(\lambda - A) \cap R((\lambda - A)^0).$$

Therefore, $dis(\lambda - A) = 0$. ■

Proposition 3.4 *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. If $R((e^{\lambda t} - T(t))^n)$ is closed for all $n \geq d$, then $R((\lambda - A^n))$ is closed for all $n \geq d$.*

Proof. Let $y_n = (\lambda - A)^n x_n \to y$, as $n \to \infty$, we show that $y \in R((\lambda - A)^n)$. According to Lemma 2.1 there exists $F_d(t)$ and $G_d(t)$ to bounded linear operators such that

$$(\lambda - A)^d F_d(t) + B_\lambda^d(t) G_d(t) = I. \quad (1)$$

$B_\lambda^n(t) y_n = B_\lambda^n(t)(\lambda - A)^n x_n = (e^{\lambda t} - T(t))^n x_n \in R((e^{\lambda t} - T(t))^n)$.
Since $B_\lambda^n(t) y_n \to B_\lambda^n(t) y$, then there exists $z \in X$ such that

$$B_\lambda^n(t) y = (e^{\lambda t} - T(t))^n z.$$

By (1):

$$(\lambda - A)^n x_n = (\lambda - A)^{2n} F_n(t) x_n + (e^{\lambda t} - T(t))^n G_n(t) x_n \text{ as } n \to \infty.$$

We have:

$$\begin{aligned} y &= (\lambda - A)^n F_n(t) y + (e^{\lambda t} - T(t))^n G_n(t) z \\ &= (\lambda - A)^n [F_n(t) y + B_\lambda^n(t) G_n(t) z] \in R((\lambda - A)^n). \end{aligned}$$ ■

Proposition 3.5 *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A and $d \in \mathbb{N}$. If $R(e^{\lambda t} - T(t)) + N((e^{\lambda t} - T(t))^d)$ is closed in X, then $R(\lambda - A) + N((\lambda - A)^d)$ is closed.*

Proof. Suppose that $R(e^{\lambda t}-T(t))+N((e^{\lambda t}-T(t))^d)$ is closed in X.

Let $y_n=(\lambda-A)x_n+z_n\to y$, as $n\to\infty$, $z_n\in N((\lambda-A)^d)$.

$B_\lambda^d(t)y_n=B_\lambda^d(t)(\lambda-A)x_n+B_\lambda^d(t)z_n\in R(e^{\lambda t}-T(t))+N((e^{\lambda t}-T(t))^d)$, then

$$B_\lambda^d(t)y\in R(e^{\lambda t}-T(t))+N((e^{\lambda t}-T(t))^d)$$
$$B_\lambda^d(t)y=((e^{\lambda t}-T(t))x+z \text{ with } z\in N((e^{\lambda t}-T(t)^d))$$
$$B_\lambda^{2d}(t)y=B_\lambda^d(t)(e^{\lambda t}-T(t))x+B_\lambda^d(t)z \text{ and } B_\lambda^d(t)z\in N((\lambda-A)^d).$$

Then

$$\begin{aligned} y &= (\lambda-A)^{2d}F_d(t)y+B_\lambda^{2d}(t)G_d(t)y\\ &= (\lambda-A)^{2d}F_d(t)y+G_d(t)[B_\lambda^d(t)(e^{\lambda t}-T(t))x+B_\lambda^d(t)z]\\ &= (\lambda-A)[(\lambda-A)^{2d-1}F_d(t)y+G_d(t)B_\lambda^{d-1}(t)]\\ &\quad + G_d(t)B_\lambda^d(t)z\in R(\lambda-A)+N((\lambda-A)^d). \end{aligned}$$

■

Corollary 3.1 *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. If $R(e^{\lambda t}-T(t))$ is closed in X, then $R(\lambda-A)$ is closed.*

Theorem 3.2 *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X with infinitesimal generator A. Then, for all $t>0$,*

$$e^{t\sigma_{qF}(A)}\subseteq\sigma_{qF}(T(t)).$$

Proof. Direct consequence of the three last proposition. ■

If $dis(T)=0$, T to be a Kato operator and by using the Corollary 3.1 and Proposition 3.3 we have the result of A. Elkoutri and M.A. Taoudi [2]:

Corollary 3.2 *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X with infinitesimal generator A. Then for all $t>0$:*

$$e^{t\sigma_K(A)}\subseteq\sigma_K(T(t)).$$

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