REPRODUCING KERNEL HILBERT SPACE METHOD FOR SOLVING FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract. This paper presents a computational technique for solving linear and nonlinear Fredholm integro-differential equations of fractional order. In addition, examples that illustrate the pertinent features of this method are presented, and the results of the study are discussed. Results have revealed that the RKHSM yields efficiently a good approximation to the exact solution.

Keywords: Reproducing Kernel Hilbert Space Method (RKHSM), fractional integro-differential equations and numerical solution.

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1. Introduction

One of the most important topics in applied mathematics is the integro-differential equations. This topic has many applications in engineering, economics, physics, chemistry, astronomy, biology, potential theory and electronics [24].

Solving integro-differential equations becomes one of the most interesting problems to many mathematicians. Numerous methods have been used to solve such problems. For instance, Adomian decomposition method [14], [15], variational iteration method [16], [17], generalized differential transform method [18], [19], nonstandard difference method [8], Adams-Bashforth-Moulton method [7], [22] and homotopy analysis method [9], [20], [23] have been used for this purpose.

The Fredholm integro-differential equations pop up when the differential equations are converted into integro-differential equations. The Reproducing Kernel Hilbert Space Method (RKHSM), which has been first introduced by Aronszajn [1], has played a major role in solving differential and integro-differential equations [2], [3], [4], [5], [6], [11], [12], [25].

This paper is organized as follows. In Section 2, we introduce some basic definitions about fractional calculus [13], [21] and reproducing kernel Hilbert spaces. The Reproducing Kernel Hilbert Space Method (RKHSM) is presented in Section 3. In Section 4, we examine the method by four numerical examples along with a comparison to the exact solutions. Finally, in Section 5, we give conclusions about the examples.

2. Mathematical preliminaries

Definition 2.1 The fractional derivative D^α of $f(t)$ in the Caputo's sense is defined as:

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{(m-\alpha-1)} f^{(m)}(\tau)\, d\tau,$$
$$m-1 < \alpha \leq m,\ m \in,\ t > 0,\ f(t) \in C_{-1}^m.$$

Definition 2.2 Let H be a Hilbert space of function $f : X \to F$ on a set X. A function $K : X \times X \to C$ is a reproducing kernel of H if the following conditions are satisfied:

1. $K(.,t) \in H$ for all $t \in X$.
2. $\langle f, K(.,t)\rangle = f(t)$ for all $f \in H$ for all $t \in X$.

Note that the reproducing kernel is unique, symmetrical and positively definite.

Definition 2.3 The space $W_2^m[a,b]$ is defined by:

$$W_2^m[a,b] = \{\ u : u^{(j)} \text{ are absolutely continuous functions,}$$
$$j = 1, 2, ..., m-1 \text{ and} u^{(m)} \in L^2[a,b]\}$$

with the inner product:

$$\langle u, v\rangle_{W_2^m} = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(t)v^{(m)}(t)\ dt$$

and the norm:

$$\|u\|_{W_2^m} = \sqrt{\langle u, u\rangle_{W_2^m}}.$$

3. Reproducing Kernel Hilbert Space Method (RKHSM)

In this article, we try to utilize the Reproducing Kernel Hilbert Space Method (RKHSM) to find a numerical solution of the following Fredholm integro-differential equation of fractional order:

$$D^\alpha u(x) = F(x, u(x), Tu(x)), a \le x \le b,\ m-1 < \alpha \le m, \tag{3.1}$$

with the initial condition

$$u^{(i)}(a) = \beta_i,\ i = 0, 1, ..., m-1, \tag{3.2}$$

where D^α denotes the Caputo fractional derivative of order α, $Tu(x)=\int_a^b h(x,t)u(t)dt$, β_i, $i = 0, 1, 2, ..., m-1$, are real finite constants, $h(x,t)$ is a known continuous function, $u(x)$ is an unknown function and $F(x, u(x), Tu(x))$ is a linear or nonlinear function. In particular, we present a numerical solution for Fredholm integro-differential equation of first and second fractional order.

The Reproducing Kernel Hilbert Space Method has the following steps:

Step(1): Homogenize the initial conditions by some kind of transformation.

Step(2): Construct a reproducing kernel space $W_2^{m+1}[a,b]$, in which every function satisfies the initial condition (3.2).

- [5] The inner product space $W_2^1[a,b] = \{u|u$ is an absolutely continuous real valued function, $u' \in L^2[a,b]\}$ with the inner product $\langle u, v\rangle_{W_2^1} = \int_a^b (u(t)v(t) + u'(t)v'(t))\ dt$ is a complete reproducing kernel and its reproducing kernel is given by:

$$R(x,y) = \frac{1}{\sinh(b-a)}[\cosh(x+y-b-a) + \cosh(|x-y|-b-a)]$$

- [6] The inner product space $W_2^2[a,b] = \{u|u, u'$ are absolutely continuous real valued functions, $u'' \in L^2[a,b], u(a) = 0\}$ with the inner product $\langle u, v\rangle_{W_2^2} = u(a)v(a) + u'(a)v'(a) + \int_a^b u''(t)v''(t)\ dt$ is a complete reproducing kernel and its reproducing kernel is given by:

$$K(x,y) = \frac{-1}{6}\begin{cases} (a-y)(2a^2 - y^2 + 3x(2+y) - a(6+3x+y)) & \text{if } y \le x \\ (a-x)(2a^2 - x^2 + 3y(2+x) - a(6+3y+x)) & \text{if } y > x \end{cases}$$

- The inner product space $W_2^3[a,b] = \{u | u, u', u''$ are absolutely continuous real valued functions, $u''' \in L^2[a,b], u(a) = u'(a) = 0\}$ with the inner product $\langle u, v\rangle_{W_2^3} = u(a)v(a) + u'(a)v'(a) + u''(a)v''(a) + \int_a^b u'''(t)v'''(t)\, dt$ is a complete reproducing kernel and its reproducing kernel is given by:

$$S(x,y) = \frac{-1}{120}\begin{cases} h(x,y) & \text{if } y \le x \\ h(y,x) & \text{if } y > x \end{cases}$$

where; $h(x,y) = (a-y)^2(6a^3 + 5xy^2 - y^3 - 10x^2(3+y) - 3a^2(10+5x+y) + 2a(5x^2 - y^2 + 5x(6+y)))$

Step(3): Define the linear operator $L : W_2^{m+1}[a,b] \to W_2^1[a,b]$ such that $Lu(x) = D^\alpha u(x) = F(x, u(x), Tu(x))$.

It is clear that L is a bounded linear operator from $W_2^{m+1}[a,b]$ to $W_2^1[a,b]$.

Step(4): Construct an orthogonal function system of $W_2^{m+1}[a,b]$.

To do this, let $\{x_i\}_{i=1}^\infty$ be a countable dense set in $[a,b]$. Let $\varphi_i(x) = R(x_i, x)$ and $\psi_i(x) = L^*\varphi_i(x)$; where L^* is the self-adjoint operator of L.

The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^\infty$ of the space $W_2^{m+1}[a,b]$ can be determined by the Gram–Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^\infty$, $\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x)$, $i = 1, 2, ...,$ where β_{ik} are the orthogonalization coefficients.

Theorem 3.1 *Assume that the inverse operator L^{-1} exists. Then if $\{x_i\}_{i=1}^\infty$ is dense on $[a,b]$, then $\{\psi_i(x)\}_{i=1}^\infty$ is a complete system of $W_2^{m+1}[a,b]$.*

Proof. Let $u(x) \in W_2^{m+1}[a,b]$ such that $\langle u(x), \psi_i(x)\rangle_{W_2^{m+1}} = 0, \ \forall i = 1, 2, ...$ Then

$$\begin{aligned} 0 &= \langle u(x), \psi_i(x)\rangle_{W_2^{m+1}} = \langle u(x), L^*\varphi_i(x)\rangle_{W_2^{m+1}} \\ &= \langle Lu(x), \varphi_i(x)\rangle_{W_2^{m+1}} = \langle Lu(x), R(x_i, x)\rangle_{W_2^{m+1}} = Lu(x_i), \end{aligned}$$

and $\{x_i\}_{i=1}^\infty$ is dense on $[a,b]$, then $Lu(x) = 0 \Rightarrow u \equiv 0$ since L^{-1} exists.

Theorem 3.2 *If $\{x_i\}_{i=1}^\infty$ is dense on $[a,b]$ and the solution of equation* (3.1) *is unique, then the solution is given by:*

$$u(x) = \sum_{i=1}^{\infty} \beta_{ik} A_i \overline{\psi}_i(x), \ \textit{where } A_i = \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k)).$$

Proof. $\{\overline{\psi}_i(x)\}_{i=1}^\infty$ is a complete orthonormal basis of $W_2^{m+1}[a,b]$.

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i(x)\rangle_{W_2^{m+1}} \overline{\psi}_i(x) = \sum_{i=1}^{\infty} \langle u(x), \sum_{k=1}^{i} \beta_{ik}\psi_k(x)\rangle_{W_2^{m+1}} \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle u(x), \psi_k(x)\rangle_{W_2^{m+1}} \overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle u(x), L^*\varphi_k(x)\rangle_{W_2^{m+1}} \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle Lu(x), \varphi_k(x)\rangle_{W_2^1} \overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle Lu(x), R(x_k, x)\rangle_{W_2^1} \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} Lu(x_k)\overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k)) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x) \end{aligned}$$

The approximate solution $u_n(x)$ can be obtained by taking finitely many terms in the series representation of $u(x)$ and so $u_n(x) = \sum_{i=1}^{n} A_i \overline{\psi}_i(x)$. Note that, since $W_2^{m+1}[a,b]$ is a Hilbert space, then $\sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k),\ Tu(x_k))\overline{\psi}_i(x) < \infty$. Also, note that, if $u(x) \in W_2^{m+1}[a,b]$, then $\exists c_i > 0$ such that $|u_n^{(i)}(x)| \leq c_i ||u^{(i)}(x)||$, $i = 0, 1, 2, ...$

Theorem 3.3 *The approximate solution $u_n(x)$ and its derivatives $u_n^{(i)}(x)$ are uniformly convergent to $u(x)$ and $u^{(i)}(x)$, respectively.*

Proof.

$$\begin{aligned} |u_n(x) - u(x)| &= |\langle u_n(x) - u(x), K(x,y)\rangle_{W_2^{m+1}}| \\ &\leq ||K(x,y)||_{W_2^{m+1}} || ||u_n(x) - u(x)||_{W_2^{m+1}} || \\ &\leq c_0 ||u_n(x) - u(x)||_{W_2^{m+1}} \end{aligned}$$

and

$$\begin{aligned} |u_n^{(i)}(x) - u^{(i)}(x)| &= |\langle u_n^{(i)}(x) - u^{(i)}(x), K^{(i)}(x,y)\rangle_{W_2^{m+1}}| \\ &\leq ||K^{(i)}(x,y)||_{W_2^{m+1}} || ||u_n^{(i)}(x) - u^{(i)}(x)||_{W_2^{m+1}} || \\ &\leq c_i ||u_n^{(i)}(x) - u^{(i)}(x)||_{W_2^{m+1}},\ i = 1, 2, ... \end{aligned}$$

Thus $u_n(x)$ and $u_n^{(i)}(x)$ converge uniformly to $u(x)$ and $u^{(i)}(x)$ respectively.

4. Numerical examples

In this section, we propose four numerical examples to illustrate the accuracy and efficiency of the Reproducing Kernel Hilbert Space Method. The computations are performed by Mathematica 9.0.We compare the results by this method with the exact solution of each example.

Example 4.1 [10] Consider the following Fredholm integro-differential equation:

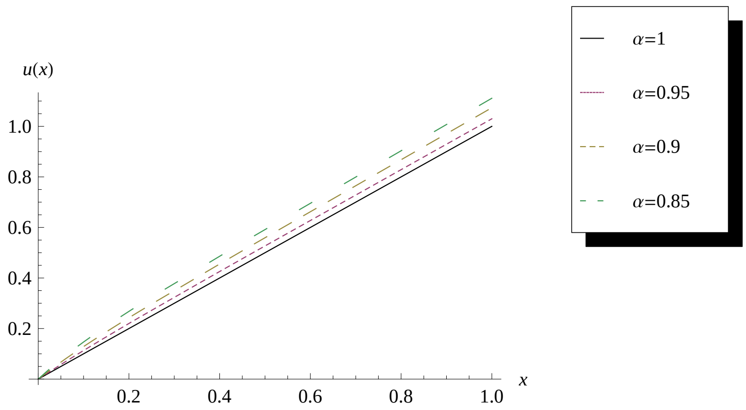
$$\begin{aligned} D^{\alpha}u(x) &= \frac{159}{160} - \frac{x^2}{64} + \int_0^{0.5} (x^2 + t)(u(t))^2\, dt,\ 0 \leq x \leq 0.5,\ 0 < \alpha \leq 1, \\ u(0) &= 0. \end{aligned}$$

The exact solution for $\alpha = 1$ is $u(x) = x$.

Solution: Using the RKHSM, taking $x_i = \frac{i}{n}$, $i = 1, 2, ..., 100$, the numerical results are given in Table 1 and the graphs of the approximate solutions for different values of α are given in Figure 1.

Table 1: Numerical solution for Example 4.1

x	Exact solution	Approximate solution	$\alpha = 0.95$	$\alpha = 0.9$	$\alpha = 0.85$	Absolute Error
0.1	0.1	0.099939991363	0.113016834790	0.128634865162	0.146699446320	$6.008636902 \times 10^{-5}$
0.2	0.2	0.199939610015	0.220010263397	0.242736648403	0.267905445031	$6.038998433 \times 10^{-5}$
0.3	0.3	0.299939275627	0.323254327854	0.349509106541	0.377709699677	$6.072437286 \times 10^{-5}$
0.4	0.4	0.399938895226	0.425587656134	0.455226839973	0.485948442440	$6.11047734 \times 10^{-5}$
0.5	0.5	0.499938453679	0.526740419380	0.559172708891	0.591464486819	$6.154632005 \times 10^{-5}$
0.6	0.6	0.599937976527	0.627358967434	0.662044915706	0.695387823420	$6.202347289 \times 10^{-5}$
0.7	0.7	0.699937499374	0.727977515489	0.764917122520	0.799311178222	$6.250062573 \times 10^{-5}$
0.8	0.8	0.799937022221	0.828596063543	0.867789329335	0.903234533023	$6.297777857 \times 10^{-5}$
0.9	0.9	0.899936545068	0.929214611597	0.970661536149	1.007157887824	$6.345493142 \times 10^{-5}$
1.0	1.0	0.999936067915	1.029833159652	1.073533742964	1.111081242626	$6.393208426 \times 10^{-5}$



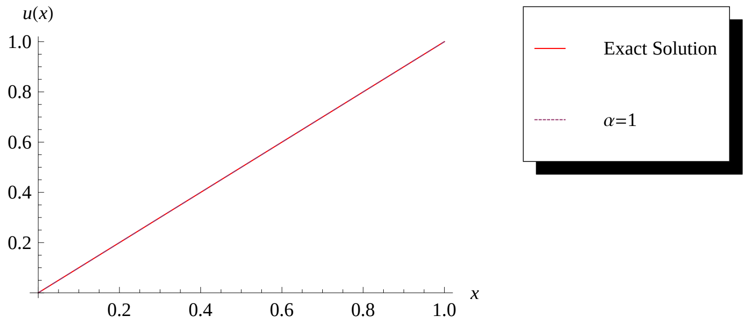


Figure 1: The comparison of approximate solution for $\alpha = 1,\ 0.95,\ 0.9,\ 0.85$ and the exact solution of Example 4.1

From the numerical results in Table 1 and Figure 1, it is clear that the approximate solutions are in good agreement with the exact solutions when $\alpha = 1$ and the solution continuously depends on the fractional derivative.

Example 4.2 Consider the following Fredholm integro-differential equation:

$$\begin{aligned} D^{\alpha}u(x) &= 1-e^{-1}+\int_0^1 e^{-u'(t)}\,dt,\ 0\le x\le 1,\ 0<\alpha\le 1 \\ u(0) &= 0. \end{aligned}$$

The exact solution for $\alpha=1$ is $u(x)=x$.

Solution: Using the RKHSM, taking $x_i=\frac{i}{n}$, $i=1,2,...,100$, the numerical results are given in Table 2 and the graphs of the approximate solutions for different values of α are given in Figure 2.

Table 2: Numerical solution for Example 4.2

x	Exact solution	Approximate solution	$\alpha=0.95$	$\alpha=0.9$	$\alpha=0.85$	Absolute Error
0.1	0.1	0.108054998483	0.156451786148	0.173281483420	0.191698426480	8.054998483×10^{-3}
0.2	0.2	0.207890498221	0.257439610084	0.278328548533	0.302058264862	7.890498222×10^{-3}
0.3	0.3	0.307726094251	0.358085456916	0.380495866370	0.405858362179	7.726094251×10^{-3}
0.4	0.4	0.407561690281	0.457995881326	0.480896758596	0.506474316490	7.561690282×10^{-3}
0.5	0.5	0.507397288668	0.557094440490	0.579691700902	0.604675994093	7.397288668×10^{-3}
0.6	0.6	0.607232887414	0.655462544381	0.677052169259	0.700787627295	7.232887415×10^{-3}
0.7	0.7	0.707068486161	0.753185454380	0.773140771156	0.795040013697	7.068486161×10^{-3}
0.8	0.8	0.806904084907	0.850329848517	0.868089303354	0.887619221137	6.904084908×10^{-3}
0.9	0.9	0.906739683654	0.946946541978	0.961999911229	0.978674307554	6.739683654×10^{-3}
1.0	1.0	1.006575282400	1.043073761178	1.054948132982	1.068315777047	6.575282401×10^{-3}

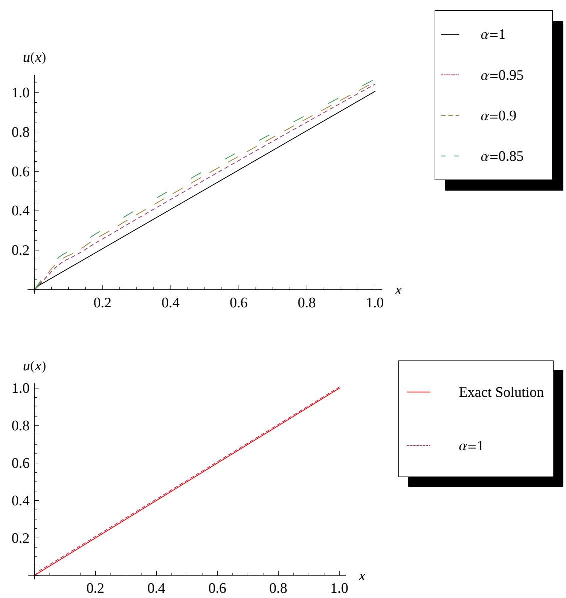


Figure 2: The comparison of approximate solution for $\alpha=1,\ 0.95,\ 0.9,\ 0.85$ and the exact solution of Example 4.2

Example 4.3 *Consider the following Fredholm integro-differential equation:*

$$\begin{aligned} D^{\alpha}u(x) &= -\sin x + \cos 1 - \sin 1 + \int_0^1 tu(t)\,dt,\ \ 0 \le x \le 1,\ \ 1 < \alpha \le 2 \\ u(0) &= 0,\ \ u^{'}(0) = 1. \end{aligned}$$

The exact solution for $\alpha = 2$ *is* $u(x) = \sin x$.

Solution: After homogenizing the initial condition, using the RKHSM, taking $x_i = \frac{i}{n}$, $i = 1, 2, ..., 100$, the numerical results are given in Table 3 and the graphs of the approximate solutions for different values of α are given in Figure 3.

Table 3: Numerical solution for Example 4.3

x	Exact solution	Approximate solution	$\alpha = 1.9$	$\alpha = 1.8$	$\alpha = 1.7$	Absolute Error
0.1	0.099833	0.099989718827	0.099870015390	0.099833619706	0.099776985988	1.563021804×10^{-4}
0.2	0.198669	0.199240887840	0.198769728794	0.198769728794	0.197772917339	5.715570458×10^{-4}
0.3	0.29552	0.296675726695	0.295414566297	0.295414566297	0.292143843212	$1.155520034\times1-^{-3}$
0.4	0.389418	0.391264329285	0.388662210215	0.388662210215	0.381667812369	1.845986977×10^{-3}
0.5	0.479426	0.482028377768	0.477490789153	0.472134288222	0.465372844980	2.602839165×10^{-3}
0.6	0.564642	0.568044521139	0.560976049482	0.552662840607	0.542447997876	3.402047744×10^{-3}
0.7	0.644518	0.648448284409	0.638283208666	0.626439778742	0.612213179934	4.230597172×10^{-3}
0.8	0.717356	0.722438475921	0.708664341922	0.692797709199	0.674107303611	5.082385022×10^{-3}
0.9	0.783327	0.789282034337	0.771458588879	0.751171033040	0.727681590487	5.95512471×10^{-3}
1.0	0.841471	0.848319221670	0.826093568986	0.801094676834	0.772596495739	6.848236863×10^{-3}

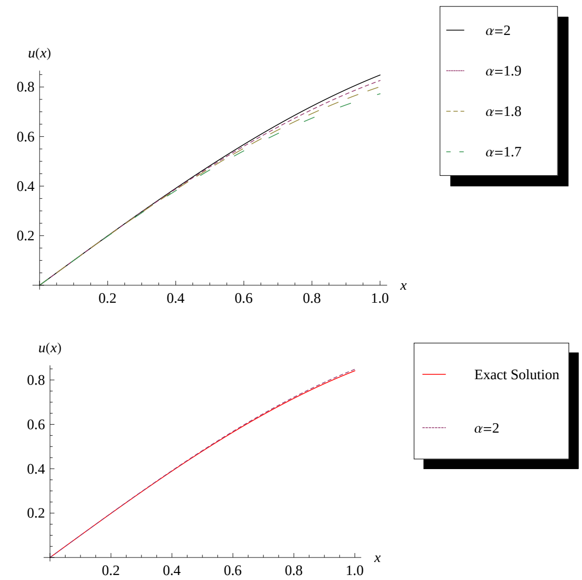


Figure 3: The comparison of approximate solution for $\alpha = 2,\ 1.9,\ 1.8,\ 1.7$ and the exact solution of Example 4.3

Example 4.4 Consider the following Fredholm integro-differential equation:

$$\begin{aligned} D^{\alpha}u(x) &= \frac{7}{3} - \int_0^1 u(t)\, dt,\ 0 \leq x \leq 1,\ 1 < \alpha \leq 2 \\ u(0) &= 0,\ u'(0) = 0. \end{aligned}$$

The exact solution for $\alpha = 2$ is $u(x) = x^2$.

Solution: After homogenizing the initial condition, using the RKHSM, taking $x_i = \frac{i}{n}$, $i = 1, 2, ..., 100$, the numerical results are given in Table 4 and the graphs of the approximate solutions for different values of α are given in Figure 4.

Table 4: Numerical solution for Example 4.4

x	Exact solution	Approximate Solution	$\alpha = 1.9$	$\alpha = 1.8$	$\alpha = 1.7$	Absolute Error
0.1	0.01	0.010405401644	0.014490416891	0.018809205338	0.024515140267	$4.054016446 \times 10^{-4}$
0.2	0.04	0.040822066202	0.052658527691	0.064101825247	0.077782662525	$8.220662025 \times 10^{-4}$
0.3	0.09	0.091219921199	0.112365258336	0.132306142532	0.155163543905	$1.219921199 \times 10^{-3}$
0.4	0.16	0.161598966635	0.196030155195	0.224104427911	0.257037814014	$1.598966635 \times 10^{-3}$
0.5	0.25	0.251959202510	0.300380543172	0.335078830520	0.376862841928	$1.95920251 \times 10^{-3}$
0.6	0.36	0.362300628824	0.423965990357	0.464756071741	0.514218183642	$2.300628825 \times 10^{-3}$
0.7	0.49	0.492623245669	0.566517177780	0.612556734097	0.668320944474	$2.623245669 \times 10^{-3}$
0.8	0.64	0.642927053202	0.727761383896	0.777968145385	0.838464560659	$2.927053202 \times 10^{-3}$
0.9	0.81	0.813212051427	0.907459460749	0.960544497758	1.024040283810	$3.212051427 \times 10^{-3}$
1.0	1.0	1.003478240343	1.105400019562	1.159895873357	1.224521962802	$3.478240344 \times 10^{-3}$

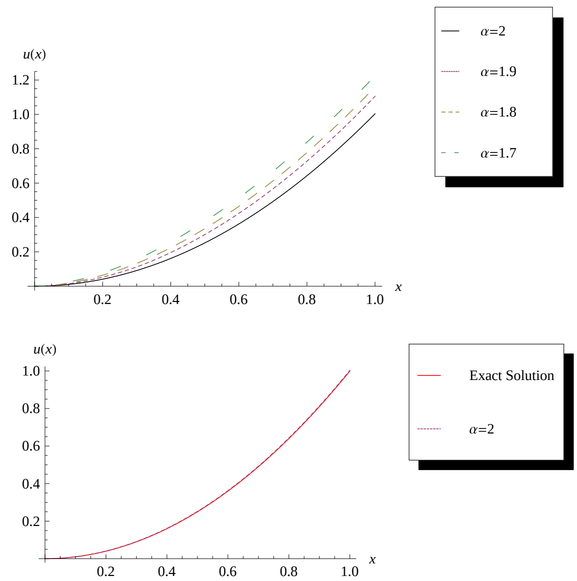


Figure 4: The comparison of approximate solution for $\alpha = 2,\ 1.9,\ 1.8,\ 1.7$ and the exact solution of Example 4.4

5. Conclusion

In this paper, the Reproducing Kernel Hilbert Space Method (RKHSM) has been successfully applied to find an approximate solution for Fredholm integro-differential equations of fractional order. The above examples show that the RKHSM yields a good approximation to the exact solution in an efficient way.

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