PASSAGE OF PROPERTY (gw) FROM TWO OPERATORS TO THEIR TENSOR PRODUCT

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Abstract. A Banach space operator satisfies property (gw) if the complement of its B-Weyl essential approximate point spectrum in its approximate point spectrum is the set of isolated eigenvalues of the operator. We give necessary and/or sufficient conditions ensuring the passage of property (gw) from two Banach space operators A and B to their tensor product. In particular, we present a revised version of Theorem 2.3 in [20].

Keywords: property (gw), generalized a-Weyl's theorem, tensor product.

2000 Mathematics Subject Classification: Primary 47A80, 47A53; Secondary 47A10, 47A11.

1. Introduction and preliminary

Let $T \in \mathcal{L}(X)$ be a bounded linear operator acting on a Banach space X. We denote by T^*, $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ denote the adjoint, the null space, the range, the spectrum, the point spectrum and the approximate point spectrum of T respectively. If the range $R(T)$ of T is closed and $\alpha(T) := dimN(T) < \infty$ (resp. $\beta(T) := codimR(T) < \infty$) then T is called an upper (resp. a lower) semi-Fredholm operator. If T is either an upper or a lower semi-Fredholm operator, then T is called a semi-Fredholm operator, and the index of T is defined by

$$ind(T) = \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator. If T is Fredholm of index zero, then T is said to be a Weyl operator. The Weyl spectrum of T is defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and the *Weyl essential approximate point spectrum* is defined by

$$\sigma_{SF_-^+}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Fredholm with } ind(T-\lambda) \leq 0\}.$$

For each nonnegative integer n let $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some n, $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. lower) semi-Fredholm operator then T is called an *upper* (resp. *lower*) *semi-B-Fredholm* operator. A *semi-B-Fredholm* operator is an upper or lower semi-B-Fredholm operator. If, moreover, $T_{[n]}$ is a Fredholm operator then T is called a *B-Fredholm* operator. From [13, Proposition 2.1] if $T_{[n]}$ is a semi-Fredholm operator then $T_{[m]}$ is also a semi-Fredholm operator for each $m \geq n$, and $ind(T_{[m]}) = ind(T_{[n]})$. Then the *index* of a semi-B-Fredholm operator is defined as the index of the semi-Fredholm operator $T_{[n]}$ (see [12, 13]). $T \in \mathcal{L}(X)$ is said to be a *B-Weyl* operator if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a B-Weyl operator}\}.$$

We shall denote by $SBF_+^-(X)$ (or $SBF_-^+(X)$) the class of all T upper semi-B-Fredholm operators such that $\text{ind}(T) \leq 0$ (respectively, T lower semi-B-Fredholm operators such that $\text{ind}(T) \geq 0$). The spectrum associated with $SBF_+^-(X)$ is called the *B-Weyl essential approximate point spectrum* and is denoted by

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_+^-(X)\},$$

while the spectrum associated with $SBF_-^+(X)$ is denoted by

$$\sigma_{SBF_-^+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_-^+(X)\}.$$

The *ascent* $a(T)$ and the *descent* $d(T)$ of T are given by

$$a(T) = \inf\{n : N(T^n) = N(T^{n+1})\} \text{ and } d(T) = \inf\{n : R(T^n) = R(T^{n+1})\},$$

with $\inf \emptyset = \infty$. It is well-known that if $a(T)$ and $d(T)$ are both finite then they are equal, see [17, Proposition 38.3]. An operator $T \in \mathcal{L}(X)$ is said to be Browder if it is Weyl of finite ascent and descent. Let $\sigma_b(T)$ be the Browder spectrum defined by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a Browder operator}\}.$$

Following Coburn [15], Weyl's theorem holds for T if $\sigma(T) \setminus \sigma_w(T) = E^0(T)$ where $E^0(T) := \{\lambda \in iso\sigma(T) : 0 < \alpha(T-\lambda) < \infty\}$. Here and elsewhere in this paper, for $K \subset \mathbb{C}$, $isoK$ is the set of isolated points of K.

We say that a-Weyl's theorem holds for T if $\sigma_{ap}(T)\setminus\sigma_{SF_-^+}(T)=E_a^0(T)$ where $E_a^0(T):=\{\lambda\in iso\sigma_{ap}(T)\,:\,0<\alpha(T-\lambda)<\infty\}$.

T is said to have the property (w) if $\sigma_a(T)\setminus\sigma_{SF_-^+}(T)=E^0(T)$. Originally introduced by Rakočević in [19], property (w) has been intensively studied in the recent past, see [2], [3], [4], [5], [6]. Following Berkani and Koliha [10], [12], generalized Weyl's theorem holds of T if $\sigma(T)\setminus\sigma_{BW}(T)=E(T)$ where $E(T):=\{\lambda\in iso\sigma(T)\,:\,\alpha(T-\lambda>0)\}$; and generalized a-Weyl's theorem holds for T if $\sigma_{ap}(T)\setminus\sigma_{SBF_+^-}(T)=E_a(T)$ where $E_a(T):=\{\lambda\in iso\sigma_{ap}(T)\,:\,\alpha(T-\lambda)>0\}$.

Operators satisfying property (gw), an extension of property (w) have been introduced and studied in [9], [11], [7]. T is said to have the property (gw), if $\sigma_{ap}(T)\setminus\sigma_{SBF_+^-}(T)=E(T)$. Note that the property (gw) implies generalized Weyl's theorem but (gw) is not intermediate between Weyl's theorems and generalized a-Weyl's theorem, see [9].

The problem of transferring Weyl's theorem, a-Weyl's theorem and Property (w) from operators A and B to their tensor product $A\otimes B$ was considered in [16], [20], [18]. The main objective of this paper is to study the transfer of property (gw) from a bounded linear operator A acting on a Banach space X and a bounded linear operator B acting on a Banach space Y to their tensor product $A\otimes B$. In Section 2, after having recalled some preliminary definitions and facts, we give necessary and sufficient condition for transferring property (gw) from isoloid operators A and B to their tensor product $A\otimes B$ in term of the B-Weyl essential approximate point spectrum equality. Section 3 is devoted to characterize the transference of generalized a-Weyl's theorem from a-isoloid operators A and B to their tensor product $A\otimes B$.

2. Property (gw) and tensor product

Let $T\in\mathcal{L}(X)$, then T is said to be *isoloid* , if $iso\sigma(T)=E(T)$. In [20, Theorem 2.3], it was stated that if A and B are isoloid operators that possess property (gw), then $A\otimes B$ satisfies property (gw) if and only if

$$\sigma_{SBF_+^-}(A\otimes B)=\sigma_{SBF_+^-}(A)\sigma_{ap}(B)\cup\sigma_{ap}(A)\sigma_{SBF_+^-}(B).$$

We point out that Theorem 2.3 in [20] is accurate as shown by the following example.

Example 2.1 Let A be a nonzero nilpotent operator and let B be a quasi-nilpotent which is not nilpotent. Then it is easy to see that

$$\sigma_{ap}(A)=\{0\},\ \sigma_{SBF_+^-}(A)=\emptyset \text{ and } \sigma_{ap}(B)=\sigma_{SBF_+^-}(B)=\{0\}.$$

Hence A and B satisfy property (gw). Since $A\otimes B$ is nilpotent then 0 is a pole and then $\sigma_{SBF_+^-}(A\otimes B)=\emptyset$. Hence $A\otimes B$ satifies property (gw). However

$$\sigma_{SBF_+^-}(A)\sigma(B)\cup\sigma(A)\sigma_{SBF_+^-}(B)=\{0\}\neq\sigma_{SBF_+^-}(A\otimes B).$$

Here $0\in iso\sigma(A\otimes B)$.

However, under the assumption $0 \notin iso\sigma(A \otimes B)$, we have the following result.

Theorem 2.2 *Let X and Y two Banach spaces and consider $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ such that A and B are isoloid and $0 \notin iso\sigma(A \otimes B)$. If property (gw) holds for A and B, then the following statements are equivalent.*

(i) *$A \otimes B$ satisfies property (gw).*

(ii) $\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B)$.

Proof. (ii) $\Rightarrow$ (i): Since A and B obey property (gw), then

$$\sigma_{ap}(A) \setminus_{SBF_+^-} (A) = E(A) \text{ and } \sigma_{ap}(B) \setminus_{SBF_+^-} (B) = E(B).$$

Assume that

$$\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B).$$

Let $\lambda \in E(A \otimes B)$. Then there exists $\mu \in iso\sigma(A)$ and $\nu \in iso\sigma(B)$ such that $\lambda = \mu\nu$. Since A and B are isoloid, then $\mu \in E(A)$ and $\nu \in E(A)$. Hence $\mu \notin \sigma_{SBF_+^-}(A)$ and $\nu \notin \sigma_{SBF_+^-}(B)$. Then $\lambda \notin \sigma_{SBF_+^-}(A \otimes B)$. Thus

$$E(A \otimes B) \subseteq \sigma_{ap}(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B).$$

Conversely, suppose that $\lambda \notin \sigma_{ap}(A \otimes B) \setminus \sigma_{SF_+^-}(A \otimes B)$, then there exists $\mu \in \sigma_{ap}(A) \setminus \sigma_{SBF_+^-}(A)$ and $\nu \in \sigma_{ap}(B) \setminus \sigma_{SBF_+^-}(B)$ such that $\lambda = \mu\nu$. Since

$$A \otimes B = (A - \mu) \otimes B + \mu I \otimes (B - \nu),$$

then we can see that $\lambda \in E(A \otimes B)$. Therefore $A \otimes B$ satisfies property (gw).

(i) $\Rightarrow$ ii): Assume that $A \otimes B$ satisfies property (gw). Let

$$\lambda \in E(A \otimes B) = \sigma_{ap}(A \otimes B) \setminus_{SBF_+^-} (A \otimes B).$$

Since $0 \notin iso\sigma(A \otimes B)$, then $\lambda \neq 0$. Hence $\lambda \in iso\sigma(A \otimes B) = iso\sigma(A)iso\sigma(B)$. That is $\lambda = \mu\nu$ with $\mu \in iso\sigma(A)$ and $\nu \in iso\sigma(B)$. Since A and B are isoloid, then $\mu \in E(A) = \sigma_{ap}(A) \setminus_{SBF_+^-} (A)$ and $\nu \in E(B) = \sigma_{ap}(B) \setminus_{SBF_+^-} (B)$, and hence $\lambda = \mu\nu \notin \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B)$. Thus

$$\sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B) \subseteq \sigma_{SBF_+^-}(A \otimes B).$$

Conversely, let $\lambda \in \sigma_{ap}(A \otimes B) \setminus (\sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B))$, then for $\lambda = \mu\nu$ we have that $\mu \in \sigma_{ap}(A)$ and $\nu \in \sigma_{ap}(B)$, hence $\mu \in E(A)$ and $\nu \in E(B)$. Thus $\lambda = \mu\nu \in E(A \otimes B) = \sigma_{ap}(A \otimes B) \setminus_{SBF_+^-} (A \otimes B)$. Finally,

$$\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B).$$

■

Remark 2.3 Let $T \in \mathcal{L}(X)$ without eigenvalue, then $\sigma_{SBF_+^-}(T) = \sigma_{ap}(T)$. Indeed, Let $\lambda \in \sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T)$. Without loss of generality we may assume that $\lambda = 0$. Then there exists some positive integer n such that $T_{[n]}$ is upper semi-Fredhom. Hence $R(T_{[n]}) = R(T^{n+1})$ is closed. Since T has no eigenvalue, then $N(T^{n+1}) = \{0\}$. Thus $0 \notin \sigma_{ap}(T^{n+1})$ which is a contradiction. Therefore, $\sigma_{SBF_+^-}(T) = \sigma_{ap}(T)$. In particular T satisfies generalized a-Weyl's theorem.

Also, the assumption "A and B are isoloid " is crucial in Theorem 2.2, as shown by the following example.

Example 2.4 Let I_1 and I_2 be the identities acting on $\mathbb{C}$ and l_2, respectively. Let S_1 and S_2 defined on l_2 by

$$S_1(x_1, x_2, ...) = \left(0, \frac{1}{3}x_1, \frac{1}{3}x_2, ...\right) \text{ and } S_2(x_1, x_2, ...) = \left(0, \frac{1}{2}x_1, \frac{1}{3}x_2, ...\right).$$

Let $A = I_1 \oplus S_1$ and $B = S_2 - I_2$. Clearly,

$$\sigma(A) = \left\{\lambda \in \mathbb{C} : |\lambda| \le \frac{1}{3}\right\} \cup \{1\} \text{ and } \sigma(B) = \{-1\}.$$

We have $\sigma(B) = \sigma_{ap}(B) = \sigma_{BW}(B) = \{-1\}$. Since B and B^* have the SVEP, then by [1, Corollary 3.53] and [1, Theorem 3.66]

$$\sigma_{SBF_+^-}(B) = \sigma_{BW}(B) = \sigma(B) = \{-1\}.$$

Now, $\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\} \cup \{1\}$. We claim that $\sigma_{SBF_+^-}(A) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\}$. Indeed, we have that $\sigma_{SBF_+^-}(A) \subseteq \sigma_{ap}(A) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\} \cup \{1\}$ and $1 \notin \sigma_{SBF_+^-}(A)$. If there exists $\lambda \in \mathbb{C}$, $|\lambda| = \frac{1}{3}$ such that $A - \lambda(I_1 \oplus I_2)$ is upper semi-B-Fredholm of negative index, then $S_1 - \lambda I_2$ is upper semi-B-Fredholm of negative index. Since S_1 has no eigenvalue, then it follows from Remark 2.3 that $\lambda \notin \sigma_{ap}(S_1) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\}$. This is a contradiction. Since A and B have the SVEP, then from [1, Corollary 3.72], we conclude that

$$\sigma_{SBF_+^-}(A^2) = \sigma_{SBF_+^-}(A)^2 = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{9}\} \text{ and } \sigma_{SBF_+^-}(B^2) = \sigma_{SBF_+^-}(B)^2 = \{1\}.$$

Now,

$$\sigma_{ap}(A) \setminus \sigma_{SBF_+^-}(A) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\} \cup \{1\}) \setminus \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\} = \{1\} = E(A)$$

and $\sigma_{ap}(B) \setminus \sigma_{SBF_+^-}(B) = \emptyset = E(B)$. Also,

$$\sigma_{ap}(A^2) \setminus \sigma_{SBF_+^-}(A^2) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{9}\} \cup \{1\}) \setminus \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{9}\} = \{1\} = E(A^2)$$

and $\sigma_{ap}(B^2) \setminus \sigma_{SBF_+^-}(B^2) = \emptyset = E(B^2)$. Thus the property (gw) holds for A, B, A^2 and B^2.

In the other hand, since $A\otimes B$ has no eigenvalue then it follows from Remark 2.3 that

$$\sigma_{SBF_+^-}(A\otimes B)=\{\lambda\in\mathbb{C}:|\lambda|=\sigma_{ap}(A\otimes B)=\{\lambda\in\mathbb{C}:|\lambda|=\frac{1}{3}\}\cup\{-1\}$$

and

$$\sigma_{SBF_+^-}(A^2\otimes B^2)=\{\lambda\in\mathbb{C}:|\lambda|=\sigma_{ap}(A^2\otimes B^2)=\{\lambda\in\mathbb{C}:|\lambda|=\frac{1}{9}\}\cup\{1\}.$$

Hence

$$\sigma_{SBF_+^-}(A\otimes B)=\sigma_{SBF_+^-}(A)\sigma_{ap}(B)\cup\sigma_{ap}(A)\sigma_{SBF_+^-}(B),$$

and

$$\sigma_{SBF_+^-}(A^2\otimes B^2)=\sigma_{SBF_+^-}(A^2)\sigma_{ap}(B^2)\cup\sigma_{ap}(A^2)\sigma_{SBF_+^-}(B^2).$$

Also

$$\sigma_{ap}(A\otimes B)\setminus\sigma_{SBF_+^-}(A\otimes B)=\emptyset=E(A\otimes B)$$

and

$$\sigma_{ap}(A^2\otimes B^2)\setminus\sigma_{SBF_+^-}(A^2\otimes B^2)=\emptyset\neq 1=E(A^2\otimes B^2).$$

Thus the property (gw) holds for $A\otimes B$ but not for $A^2\otimes B^2$. Note here that B and B^2 are not isoloid.

The following example show that there exists two operators $A,B\in\mathcal{L}(X)$ such that $A\otimes B$ satisfies the property (gw) but A and B do not satisfy the property (gw).

Example 2.5 Let $B=U+U^*$ where U is the unilateral shift on l_2. Since B is self-adjoint, then

$$\sigma(B)=\sigma_{ap}(B)=\{\lambda\in\mathbb{C}:|\lambda|\leq 1\}$$

and from [1],

$$\sigma_{BW}(B)=\sigma_{SBF_+^-}(B)=\{\lambda\in\mathbb{C}:|\lambda|=1\}.$$

Hence

$$\sigma_{ap}(B)\setminus\sigma_{SBF_+^-}(B)=\{\lambda\in\mathbb{C}:|\lambda|\leq 1\}\setminus\{\lambda\in\mathbb{C}:|\lambda|=1\}.$$

Since $E(B)=\emptyset$, then property (gw) fails for B. In the other hand, if I is the identity acting on l_2, then $I\otimes B$ is self-adjoint, hence

$$\sigma(I\otimes B)=\sigma_{ap}(I\otimes B)=\{\lambda\in\mathbb{C}:|\lambda|\leq 1\}.$$

$\sigma_{BW}(I\otimes B)=\{\lambda\in\mathbb{C}:|\lambda|=1\}$, and $\sigma_{SBF_+^-}(I\otimes B)=\{\lambda\in\mathbb{C}:|\lambda|\leq 1\}$. Hence,

$$\sigma_{ap}(I\otimes B)\setminus\sigma_{SBF_+^-}(I\otimes B)=\emptyset=E(I\otimes B).$$

Thus $I\otimes B$ satisfies property (gw).

3. Generalized a-Weyl's theorem and tensor product

Recall that given $T \in \mathcal{L}(X)$, T is said to be *a-isoloid*, if $iso\sigma_{ap}(T) = E_a(T)$.

Theorem 3.1 *Let X and Y two Banach spaces. Suppose that $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ are a-isoloid and satisfy generalized a-Weyl's theorem. If $0 \notin iso\sigma_{ap}(A \otimes B)$ and $\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B)$, then $A \otimes B$ satisfies generalized a-Weyl's theorem.*

Proof. Let $\lambda \in \sigma_{ap}(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B)$. Assume for the sake of contradiction that $\lambda \in acc\, \sigma_{ap}(A \otimes B)$. Since $\lambda \in acc\, (\sigma_{ap}(A)\sigma_{ap}(B))$, it follows that

$$\lambda \in acc\, \sigma_{ap}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A) acc\, \sigma_{ap}(B).$$

Since generalized a-Weyl's theorem holds for both A and B then

$$\lambda \in \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B).$$

By hypothesis, we get that $\lambda \in \sigma_{SBF_+^-}(A \otimes B)$. This is a contradiction. Hence $\lambda \in iso\, \sigma(A \otimes B)$. Since $A \otimes B - \lambda$ is upper semi B-Fredholm, then $\lambda \in E^a(A \otimes B)$. For the reverse inclusion, let $\lambda \in E^a(A \otimes B)$. By hypothesis, $\lambda \neq 0$. With the same argument as in the proof of Theorem 2.2 we get $\lambda = \mu\nu$ with $\mu \in iso\, \sigma_{ap}(A)$ and $\nu \in iso\, \sigma_{ap}(B)$. Since A and B are a-isoloid, then $\mu \in E^a(A)$ and $\nu \in E^a(B)$. Hence from the fact that generalized a-Weyl's theorem holds for A and B, we deduce that

$$\mu \in \sigma_{ap}(A) \setminus \sigma_{SBF_+^-}(A) \text{ and } \nu \in \sigma_{ap}(B) \setminus \sigma_{SBF_+^-}(B).$$

Thus

$$\lambda \notin \sigma_{ap}(A)\sigma_{SBF_+^-}(B) \cup \sigma_{SBF_+^-}(A)\sigma_{ap}(B).$$

By hypothesis, we conclude that $\lambda \notin \sigma_{SBF_+^-}(A \otimes B)$, and hence

$$\lambda \in \sigma_{ap}(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B).$$ ■

4. Concluding remarks

A bounded linear operator T is said to satisfy a-Browder's theorem if $\sigma_{ap}(T) \setminus \sigma_{SF_+^-}(T) = \pi_0^a(T)$; where $\pi_0^a(T)$ is the set of all left pole of finite rank defined by $\pi_0^a(T) = \{\lambda \in \mathbb{C} : a(T - \lambda) < \infty; R(T^{a(T-\lambda)+1})$ is closed and $\alpha(T - \lambda) < \infty\}$. It is well known that a-weyl's theorem implies a-Browder's theorem and the reverse is not true.

T is said to satisfy the generalized a-Browder's theorem if $\sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T) = \pi^a(T)$; where $\pi^a(T)$ is the set of all left pole defined by $\pi^a(T) = \{\lambda \in \mathbb{C} : a(T - \lambda) < \infty$ and $R(T^{a(T-\lambda)+1})$ is closed$\}$. Generalized a-Weyl's theorem implies generalized a-Browder's theorem and the reverse is not true. Recently we proved that a-Browder's theorem is equivalent to generalized a-Browder's theorem [8].

In [16, Theorem 1] it was shown that if $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ are isoloid operators which satisfy property (w), then equality

$$\sigma_{SF_+^-}(A \otimes B) = \sigma_{SF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SF_+^-}(B)$$

is equivalent to $A \otimes B$ satisfies a-Browder's theorem.

In [14, Remark 4.6] Boasso, Duggal and Jeon asked:

If A and B satisfy generalized a-Browder's theorem, does the equality

$$\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B)$$

equivalent to $A \otimes B$ satisfies generalized a-Browder's theorem?

In [20, Theorem 2.1] it is apparent that the author answered this question positively. However, it is not true as shown by the following example.

Example 4.1 Let A be a nonzero nilpotent operator and let B be a quasi-nilpotent which is not nilpotent. Then A and B satisfies generalizerd a-Browder's theorem. Hence

$$\sigma(A) = \{0\},\ \sigma_{SBF_+^-}(A) = \emptyset \text{ and } \sigma(B) = \sigma_{SBF_+^-}(B) = \{0\}.$$

Then

$$\sigma_{SBF_+^-}(A)\sigma(B) \cup \sigma(A)\sigma_{SBF_+^-}(B) = \{0\}.$$

However, since $A \otimes B$ is nilpotent then 0 is a pole and then $\sigma_{SBF_+^-}(A \otimes B) = \emptyset$. Here $A \otimes B$ satisfies generalized a-Browder's theorems.

Remarks 4.2

(1) In [20, Theorem 2.2], it was stated that if A and B are isoloid and satisfy generalized a-Weyl's theorem, then $\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B)$ implies that $A \otimes B$ satisfies generalized a-Weyl's theorem. In the proof, the author used Theorem 2.1 which is false as mentioned above.

(2) In the proof of [20, Lemma 2.1], the author used the following equivalence "$A \otimes B - \frac{1}{n}I$ is injective if and only if A and B are injective". We would like to point out that this equivalence is not true. For instance, if A and B are nonzero nilpotent operators then A and B are not injective however $A \otimes B - \frac{1}{n}I$ is injective.

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Accepted: 14.11.2015