ON SOME ALGEBRAIC PROPERTIES OF SOFT SETS

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Abstract. This paper studies some algebraic and lattice properties of soft sets. A soft binary operation is introduced and a few interesting results are investigated in this context.

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Introduction

The soft set theory was introduced by Molodtsov [6] in 1999 as a general mathematical tool to deal with problems involving uncertainty or vagueness. Molodtsov recognized the importance of the role of parameters in modeling various problems involving uncertainty and the soft set models worked very well in dealing with such problems. He has shown several applications of this theory in many fields like economics, engineering, medical sciences, etc. Among many theories like Fuzzy set theory, Rough set theory etc., the soft set theory became very significant because of its wide range of applicability. The development in the fields of soft set theory and its application has been taking place in a rapid pace.

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In 2010, Babitha and Sunil [2] introduced the concepts of Soft set relations and Soft equivalence relation. In this present work, we define a soft binary operation and investigate some interesting results. We also introduce a soft relation in a different approach unlike the work of Babitha and Sunil. Some algebraic and lattice properties of soft sets are established.

In what follows, U, E and $P(U)$ stand for the universe set, the set of parameters and the collection of all subsets of U respectively.

1. Preliminaries

Definition 1.1. A pair (F, A) is called a soft set over U if $A \subset E$ and $F : A \to P(U)$. We write F_A for the pair (F, A). The soft set $F : E \to P(U)$ is denoted by F_E .

Definition 1.2. Let F_A and G_B be soft sets over a common universe set U and $A, B \subset E$. Then we say that

(a) F_A is a *soft subset* of G_B, denoted by $F_A \tilde{\subset} G_B$, if

  (i) $A \subset B$, and

  (ii) $F(e) \subset G(e) \forall e \in A$.

(b) F_A equals G_B , denoted by $F_A \cong G_B$, if $F_A \tilde{\subset} G_B$ and $G_B \tilde{\subset} F_A$.

Definition 1.3. A soft set F_A over U is called a *null soft set*, denoted by Φ, if $e \in A$, $F(e) = \phi$.

Definition 1.4. A soft set F_A over U is called an *absolute soft set*, denoted by $\widetilde{A}$, if $e \in A$, $F(e) = U$

Definition 1.5. The *union* of two soft sets F_A and G_B over a common universe U is the soft set H_C, where $C = A \cup B$, and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B, \\ G(e) & \text{if } e \in B - A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

We write $F_A \cup G_B \cong H_C$.

Definition 1.6. The intersection of two soft sets F_A and G_B over a common universe U is the soft set H_C, where $C = A \cap B$, and for all $e \in C$, $H(e) = F(e) \cap G(e)$. We write $F_A \cap G_B \cong H_C$.

Definition 1.7. For a soft set F_A over U, the relative *complement* of F_A is denoted by F_A^C and is defined by $F_A^C \cong F_A^1$, where $F^1 : A \to P(U)$ is a mapping given by $F^1(e) = U - F(e)$ for all $e \in A$.

2. Algebraic properties

This section is devoted to some algebraic notions of Soft set theory which are different from others. We introduce and define some soft operations and also we establish a few interesting results in this context.

Definition 2.1. Let $*$ be a binary operation on U. If F_E and G_E are two soft sets over U, then we define $F_E \circ G_E$ as a soft set $H : E \to P(U)$ such that

$$H(e) = \{x * y : x \in F(e) \text{ and } y \in G(e)\}.$$

We denote the collection of all soft sets over U with domain E by the symbol $S_E(U)$.

Proposition 2.2. *$\circ$ is a binary operation on $S_E(U)$.*

Proof. Let $F_E \in S_E(U)$ and $G_E \in S_E(U)$. Then $F_E \circ G_E$ is a soft set $H : E \to P(U)$, defined by

$$H(e) = \{x * y : x \in F(e) \text{ and } y \in G(e)\}.$$

Since $*$ is a binary operation on U, $x * y \in U$ for every $x, y \in U$, it results

$$H(e) = \{x * y : x \in F(e) \text{ and } y \in G(e)\} \subseteq U.$$

Hence $H_E \in S_E(U)$. This proves that $\circ$ is a binary operation on $S_E(U)$. ∎

Proposition 2.3. *If $(U, *)$ is a group then so is $((S_E(U), \circ)$.*

Proof. Suppose that $(U, *)$ is a group. Then, by Proposition 2.2, $\circ$ is a binary operation on $S_E(U)$.

Let F_E, G_E and H_E be any three soft sets in $S_E(U)$.

$$\begin{aligned}[F_E \circ (G_E \circ H_E)](e) &= \{x * (y * z) : x \in F(e),\ y \in G(e) \text{ and } z \in H(e)\}\\ &= \{x * y) * z) : x \in F(e),\ y \in G(e) \text{ and } z \in H(e)\}\\ &= [(F_E \circ G_E) \circ H_E](e)\ \ \forall\ e \in E.\end{aligned}$$

Hence

$$F_E \circ (G_E \circ H_E) = (F_E \circ G_E) \circ H_E.$$

Thus $\circ$ is associative on $S_E(U)$.

Let $\theta \in U$ be the identity element in $(U, *)$. Define a soft set $I : E \to P(U)$ by

$$I(e) = \{\theta\}\ \ \forall\ e \in E.$$

Clearly, $I_E \in S_E(U)$. Then,

$$\begin{aligned}(F_E \circ I_E)(e) &= \{x * \theta : x \in F(e)\}\\ &= F_E(e)\ \ \forall\ e \in E\\ &\Rightarrow\ F_E \circ I_E \cong F_E.\end{aligned}$$

Similarly, $I_E \circ F_E \cong F_E$. Thus I_E is the identity element in $S_E(U)$.

Let $F_E \in S_E(U)$. Define a soft set $F^{-1} : E \to P(U)$ by

$$F^{-1}(e) = \{x^{-1} : x \in F(e)\}\ \ \forall\ e \in E.$$

Then $F_E^{-1} \in S_E(U)$ and $F_E \circ F_E^{-1}\ F_E^{-1} \circ F_E \cong I_E$. Hence $(S_E(U), \circ)$ is a group. ∎

Proposition 2.4. *$*$ is Commutative on $U \Leftrightarrow \circ$ is Commutative on $S_E(U)$.*

Proof. Suppose $*$ is commutative on U. Let $F_E, G_E \in (S_E(U)$. Then

$$\begin{aligned}(F_E \circ G_E)(e) &= \{x * y : x \in F(e) \ \text{ and } \ y \in G(e)\} \\ &= \{y * x : y \in G(e) \ \text{ and } \ x \in F(e)\} \\ &= (G_E \circ F_E)(e) \ \ \forall\, e \in E.\end{aligned}$$

Hence

$$F_E \circ G_E \cong G_E \circ F_E.$$

Thus $\circ$ is commutative on $S_E(U)$.

Conversely, suppose that $\circ$ is commutative on $S_E(U)$. Let $x, y \in U$. Define $F : E \to P(U)$ and $G : E \to P(U)$ by

$$F(e) = \{x\} \text{ and } G(e) = \{y\} \ \forall\, e \in E.$$

Since $\circ$ is commutative on $S_E(U)$, we have

$$\begin{aligned}F_E \circ G_E \cong G_E \circ F_E &\Rightarrow \ (F_E \circ G_E)(e) = (G_E \circ F_E)(e) \ \ \forall\, e \in E \\ &\Rightarrow \ \{x * y\} = \{y * x\} \\ &\Rightarrow \ x * y = y * x.\end{aligned}$$

Hence $*$ is commutative on U. ∎

Definition 2.5. Let $(U, +, \cdot)$ be a ring. Let $F_E, G_E \in S_E(U)$. Then we define

$$\begin{aligned}(F_E \oplus G_E)(e) &= \{x + y : x \in F(e) \ \text{ and } \ y \in G(e)\} \\ (F_E \odot G_E)(e) &= \{x \cdot y : x \in F(e) \ \text{ and } \ y \in G(e)\}.\end{aligned}$$

Proposition 2.6. *If $(U, +, \cdot)$ is a commutative ring then $(S_E(U), \oplus, \odot)$ is a soft commutative ring.*

3. Lattice properties

Proposition 3.1. *The collection $S_E(U)$ is a partially ordered set with respect to the relation $\leq$ defined by*

$$F_E \leq G_E \Leftrightarrow F(x) \subseteq G(x) \textit{ for all } x \in E.$$

Proof. Since $F(x) \subseteq F(x)$ for all $x \in E$, we have $F_E \leq F_E$. Hence $\leq$ is reflexive on $S_E(U)$.

Let F_E and G_E be two soft sets over U such that $F_E \leq G_E$ and $G_E \leq F_E$.

$\Rightarrow$ $F(x) \subseteq G(x)$ and $G(x) \subseteq F(x)$ for all $x \in E$.
$\Rightarrow$ $F(x) = G(x)$ for all $x \in E$.
$\Rightarrow$ $F_E \cong G_E$.

Hence $\leq$ is anti-symmetric on $S_E(U)$.

Let F_E, G_E and H_E be three soft sets over U such that $F_E \leq G_E$ and $G_E \leq H_E$.

$\Rightarrow$ $F(x) \subseteq G(x)$ and $G(x) \subseteq H(x)$ for all $x \in E$.
$\Rightarrow$ $F(x) \subseteq H(x)$ for all $x \in E$.
$\Rightarrow$ $F_E \leq H_E$.
$\Rightarrow$ $\leq$ is transitive on $S_E(U)$.

Thus $\leq$ is a partially ordered relation on $S_E(U)$. ∎

Proposition 3.2. $(S_E(U), \leq)$ *is a lattice.*

Proof. If $F_E \in S_E(U)$ and $G_E \in S_E(U)$ then the lub and glb of F_E and G_E are defined by the soft sets $F_E \vee G_E$ and $F_E \wedge G_E$ respectively as follows.

$(F_E \vee G_E)(e) = F(e) \cup (G(e)$ for all $e \in E$

$(F_E \wedge G_E)(e) = F(e) \cap (G(e)$ for all $e \in E$

Hence $F_E \vee G_E$ and $F_E \wedge G_E$ are lub and glb of F_E and G_E respectively in $S_E(U)$. ∎

Definition 3.3. Let $\{(F_\alpha)_E : \alpha \in \Delta\}$ be any collection of soft sets in $S_E(U)$. Then we define $\bigcup\limits_{\alpha \in \Delta} (F_\alpha)_E$ to be a soft set $H : E \to P(U)$ such that

$$H(x) = \bigcup_{\alpha \in \Lambda} (F_\alpha(x))$$

for all $x \in E$.

Definition 3.4. Let $\{(F_\alpha)_E : \alpha \in \Delta\}$ be any collection of soft sets in $S_E(U)$. Then we define $\bigcap\limits_{\alpha \in \Delta} (F_\alpha)_E$ to be a soft set $H : E \to P(U)$ such that

$$H(x) = \bigcap_{\alpha \in \Lambda} (F_\alpha(x))$$

for all $x \in E$.

Proposition 3.5. $(S_E(U), \leq)$ *is a complete lattice.*

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