A NEW CHARACTERIZATION OF THE ALTERNATING GROUP A_8 BY ITS ORDER AND LARGE DEGREES OF ITS IRREDUCIBLE CHARACTER

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Abstract. It is well-known that characters of a finite group can give information about its structure. Also it is known that a finite simple group can uniquely determined by its character table. Here the authors attempt to investigate how to characterize a finite group by using less information of its character table, and successfully characterize the alternating group A_8 by its order and at most two irreducible character degrees of the character table.

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1. Introduction and preliminary results

Let G be a finite group, $Irr(G)$ be the set of all irreducible complex characters of G, and denote the set of character degrees of G by $cd(G) = \{\chi(1)|\chi \in Irr(G)\}$. In stances, where the context is clear, we will refer to character degrees as degrees. On the other hand, we use $cd^*(G)$ to denote the multi-set of degrees of irreducible characters, i.e., each element of the set $cd^*(G)$ can occur many times upon on the number of characters of the same degree. In particular, $|cd^*(G)| = |Irr(G)|$. Let $\pi(G)$ denote the set of all primes dividing the order of G. For each prime $p \in \pi(G)$, Let $Syl_p(G)$ denote the set of all Sylow p-subgroups of G and G_p denote a Sylow p-subgroup of G. $L_1(G)$ and $L_2(G)$ are the largest and the second largest irreducible character degree of character table of G, respectively. The other notations and terminologies in this paper are standard and can be found in [1], [2], for instance.

It is a well-known fact that characters of a finite group can give some information of its structure. It was proved in [3] that a finite simple group can be uniquely determined by its character table. In 2000, Huppert in [4] conjectured that each finite non-abelian simple group G can be characterized by the set $cd(G)$ of degrees of its complex irreducible characters; in [4], [5], [6], Huppert confirmed that the conjecture holds for the simple groups such as $L_2(q)$ and $S_z(q)$; moreover, he also proved that the conjecture follows for 19 out of 26 sporadic simple groups, and a few others (cf.[4], [5], [6]). In [7], [8], Daneshkhah, et al. showed that the conjecture holds for another three sporadic simple groups Co_1, Co_2 and Co_3. In [9], Xu, Chen and Yan attempted to characterize the finite simple groups by less information of its characters, and for the first time successfully characterized the simple K_3-groups by their orders and one or both of its largest and second largest irreducible character degrees. Simple K_3-groups are eight simple groups with their orders having exactly three distinct prime divisors, which are A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, and $U_4(2)$. In [9], the authors proved that the simple groups A_5, $L_2(7)$, $L_2(17)$, $L_3(3)$ and $U_4(2)$ can be uniquely determined by their orders and the largest degrees of its irreducible characters, the alternating group A_6 can be uniquely determined by its order and the second largest degree of its irreducible characters and the remaining simple groups $L_2(8)$ and $U_3(3)$ can be uniquely determined by their orders and the largest and the second largest degrees of their irreducible characters.

In this paper, we continue this investigation, and show that the alternating group A_8 can be characterized by its order and at most two irreducible character degrees of its character table.

We come to the following main theorem:

Main Theorem. *Let G be a finite group and $|G| = |A_8|$. Then $G \cong A_8$ if and only if*

(i) $L_1(G){=}L_1(A_8)$;

(ii) $L_2(G){=}L_2(A_8)$.

In order to prove the above main theorem, we need the following lemmas:

Lemma 1. (c.f [9]) *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$.*

Lemma 2. (c.f [9]) *Let G be a finite solvable group of order $q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$, where $q_1, q_2, \cdots, q_s$ are distinct primes. If $(kq_s + 1) \nmid q_i^{\alpha_i}$ for each $i \leq s-1$ and $k > 0$, then the Sylow q_s-subgroup is normal in G.*

Lemma 3. (cf.[1]) *Let S be a finite non-abelian simple group with $\pi(S) \subseteq \{2, 3, 5, 7\}$, then S is isomorphic to one of the following simple groups listed in Table 1. In particular, if $|\pi(Out(S))| \neq 1$, then $\pi(Out(S)) \subseteq \{2, 3\}$.*

Table 1. Finite nonabelian simple groups with $\pi(S) \subseteq \{2, 3, 5, 7\}$

G	$\|G\|$	$Out(G)$	G	$\|G\|$	$Out(G)$
A_5	$2^2 \cdot 3 \cdot 5$	2	$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2

Proof of Main Theorem

In this section, we give the proof of the main theorem.

Proof of Main Theorem. It is enough to prove the sufficiency.

By hypotheses, we have that $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$. Let $\beta, \chi \in Irr(G)$ such that $\beta(1) = L_1(G) = 70$, $\chi(1) = L_2(G) = 64$. In the following, we will prove that $G \cong A_8$.

We first assert that G is nonsolvable. If G is solvable, let T be a Hall-subgroup of G with order $|T| = 2^6 \cdot 3^2 \cdot 7$. Let $T_G = \bigcap_{g \in G} T^g \leq T$, then we have $G/T_G \lesssim S_5$. Since orders of solvable subgroups of S_5 divided by 5 are 5, 10 and 20, it follows that $|T_G|$ may be one of $2^6 \cdot 3^2 \cdot 7$, $2^5 \cdot 3^2 \cdot 7$ or $2^4 \cdot 3^2 \cdot 7$.

If $|T_G| = 2^6 \cdot 3^2 \cdot 7$, let $\theta \in Irr(T_G)$ such that $[\chi_{T_G}, \theta] \neq 0$, then $\chi(1)/\theta(1) \big| |G : T_G| = 5$. Thus $\theta(1) = 64$. But $\theta(1)^2 > |T_G|$, a contradiction.

If $|T_G| = 2^5 \cdot 3^2 \cdot 7$, since T_G is solvable, then T_G has a subgroup D such that $|T_G : D| = 7$. Considering the permutation representation of T_G on the right cosets of D with kernel D_{T_G}, one has that $T_G/D_{T_G} \lesssim S_7$. Using Magma soft, we know that orders of solvable subgroups of S_7 divided by 7 are 7, 14 and 21. Hence, $|D_{T_G}|$ can be equal to one of $2^5 \cdot 3^2$, $2^4 \cdot 3^2$ or $2^5 \cdot 3$.

If $|D_{T_G}|=2^5 \cdot 3^2$, let $\varsigma \in Irr(D_{T_G})$ such that $[\chi_{D_{T_G}}, \varsigma] \neq 0$, then $\chi(1)/\varsigma(1) \big| |G : D_{T_G}|=5$, and so $\varsigma(1) = 32$. But $\varsigma(1)^2 > |D_{T_G}|$, a contradiction.

Similarly, we can prove that $|D_{T_G}| \neq 2^4 \cdot 3^2$ or $2^5 \cdot 3$.

If $|T_G| = 2^4 \cdot 3^2 \cdot 7$, since T_G is solvable, let I be a Hall-subgroup of T_G with order $|I| = 2^4 \cdot 3^2$ and $I_{T_G} = \bigcap_{g \in T_G} I^g \leq I$, then $T_G/I_{T_G} \lesssim S_7$. Since orders of solvable subgroups of S_7 divided by 7 are 7, 14 and 21, it follows that the order of I_{T_G} is one of $2^4 \cdot 3^2$, $2^3 \cdot 3^2$ and $2^4 \cdot 3$.

If $|I_{T_G}|=2^4 \cdot 3^2$, let $\Lambda \in Irr(I_{T_G})$ such that $[\chi_{I_{T_G}}, \Lambda] \neq 0$, then $\chi(1)/\Lambda(1) \big| |G : I_{T_G}|=70$. Hence $\Lambda(1) = 16$. But $\Lambda(1)^2 > |I_{T_G}|$, a contradiction.

By the same arguments as before, we can prove that $|I_{T_G}| \neq 2^4 \cdot 3$.

If $|I_{T_G}| = 2^3 \cdot 3^2$, let $\mu \in Irr(I_{T_G})$ such that $[\chi_{I_{T_G}}, \mu] \neq 0$, then we have $\chi(1)/\mu(1) \big| |G : I_{T_G}| = 2^3 \cdot 5 \cdot 7$, and so $\mu(1) = 8$. In this case, we see that there exists an irreducible character of degree 8 in I_{T_G}. By using Magma soft (cf. [10]), it is easy to check that the set $cd^*(I_{T_G})$ satisfying $8 \in Irr(I_{T_G})$ can only be one of

$$cd^*(I_{T_G}) = \{1, 1, 1, 1, 1, 1, 1, 1, 8\} \tag{1}$$

$$cd^*(I_{T_G}) = \{1, 1, 1, 1, 2, 8\} \tag{2}$$

Since $|T_G/I_{T_G}| = 2 \cdot 7$, $U/I_{T_G} \in Syl_7(T_G/I_{T_G})$, by Sylow Theorem, one has that $U/I_{T_G} \trianglelefteq T_G/I_{T_G}$. Hence $U \trianglelefteq T_G$ and $|U| = 2^3 \cdot 3^2 \cdot 7$. Let $\vartheta \in Irr(U)$ such that $[\beta_U, \vartheta] \neq 0$. Then $\beta(1)/\vartheta(1) \big| |G : U| = 2^3 \cdot 5$. Furthermore, we get that $\vartheta(1) = 7$ or 14. Let $\pi \in Irr(I_{T_G})$ such that $[\vartheta_{I_{T_G}}, \pi] \neq 0$. Set $e = [\vartheta_{I_{T_G}}, \pi]$, $t = |U : I_U(\pi)|$, one has that $\vartheta(1) = et\pi(1)$. By the structure of $cd^*(I_{T_G})$ in (1) and (2), we deduce that $e = 1$ and $t = 7$ since otherwise $[\vartheta_{I_{T_G}}, \vartheta_{I_{T_G}}] = e^2 t > |U : I_{T_G}|$, a contradiction. By the structure of I_{T_G}, it is easy to see that there exists no such irreducible character with an orbit of length 7 in I_{T_G}, a contradiction.

Therefore, G is nonsolvable. By Lemma 1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \big| |Out(K/H)|$. Since $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$, then by Table 1 we get that K/H can only be isomorphic to one of the groups A_5, A_6, $L_2(7)$, $L_2(8)$, A_7, $L_3(4)$ or A_8. And we write the proof up to what K/H is case by case.

Case 1. $K/H \ncong A_5$.

If $K/H \cong A_5$, then by Lemma 1 one has that $|G:K| = 1$ or 2.

If $|G:K| = 1$, then $|H| = 2^4 \cdot 3 \cdot 7$. If H is solvable, then H has a subgroup T with index 3. Considering the permutation of H on T with kernel T_H, we have $H/T_H \lesssim S_3$. Hence $|T_H| = 2^4 \cdot 7$ or $2^3 \cdot 7$.

If $|T_H|{=}2^4 \cdot 7$, let $\Pi{\in}Irr(T_H)$ such that $[\chi_{T_H}, \Pi]{\neq}0$, then $\chi(1)/\Pi(1)\big||G:T_H|{=}2^2 \cdot 3^2 \cdot 5$. Thus $\Pi(1) = 16$. But $\Pi(1)^2 > |T_H|$, a contradiction. By the similar reason we have $|T_H| \neq 2^3 \cdot 7$.

We may assume that H is nonsolvable. By Lemma 1, H has a normal series: $1 \trianglelefteq N \trianglelefteq M \trianglelefteq H$ such that M/N is a direct product of isomorphic non-abelian simple groups and $|H/M|\big||Out(M/N)|$. Since $|H| = 2^4 \cdot 3 \cdot 7$, we obtain that $M/N \cong L_2(7)$. Hence, we have $|H/M|\big||Out(L_2(7))| = 2$, and so $|H:M| = 1$ or 2.

If $|H:M| = 1$, then $|N| = 2$. By [2, Clifford Theorem 6.2], N has an irreducible character of degree 2, which is impossible. If $|H:M| = 2$, then $|N| = 1$. In this case, we have $M \cong L_2(7) \cdot 2$. Let $\nu \in Irr(M)$ such that $[\chi_M, \nu] \neq 0$, then we have $\chi(1)/\nu(1)\big||G:M| = 2^2 \cdot 3 \cdot 5$, and hence $\nu(1) = 16$. By [1], we see that the largest irreducible character degree of $L_2(7) \cdot 2$ is 8. Therefore, M has no such irreducible character of degree 16, a contradiction.

If $|G:K| = 2$, then $|H| = 2^3 \cdot 3 \cdot 7$. If H is solvable, then H has a Hall-subgroup P with index 3. Considering the permutation of H on the right cosets of P with kernel P_H, one has that $H/P_H \lesssim S_3$. Therefore, $|P_H| = 2^3 \cdot 7$ or $2^2 \cdot 7$.

If $|P_H| = 2^3 \cdot 7$, let $\varpi \in Irr(P_H)$ such that $[\chi_{P_H}, \varpi] \neq 0$, $\chi(1)/\varpi(1)\big||G:P_H| = 2^3 \cdot 3^2 \cdot 5$, so $\varpi(1) = 8$. But $\varpi(1)^2 > |P_H|$, a contradiction. If $|P_H| = 2^2 \cdot 7$, let $P_7 \in Syl_7(P_H)$, by Sylow Theorem, we have $P_7 \trianglelefteq P_H$. On the other hand, since $P_H \lhd \lhd G$, one has that $P_7 \trianglelefteq G$. Thus $\beta(1)\big||G:P_7|$, a contradiction.

If H is nonsolvable, by Table 1, we can get that $H \cong L_2(7)$. Therefore, $K \cong A_5 \times L_2(7)$. Let $\xi \in Irr(K)$ such that $[\beta_K, \xi] \neq 0$, then we have $\beta(1)/\xi(1)\big||G:K| = 2$, and thus $35\big|\xi(1)$. Let $e = [\beta_K, \xi]$, $t = |G:I_G(\xi)|$, one has that $\beta(1) = et\xi(1)$. By the structure of K, we have $\xi(1) = 35$, $e = 2$ and $t = 1$. It is easy to check that $[\beta_K, \beta_K] = e^2 t = 4 > |G:K| = 2$, a contradiction to [2, Lemma 2.29].

Case 2. $K/H \ncong A_6$.

If not, since $|Out(A_6)| = 4$, then by Lemma 1 we have $|G:K| = 1$, 2 or 4.

If $|G:K| = 1$, then $|H| = 2^3 \cdot 7$. Let $\mu \in Irr(H)$ such that $[\chi_H, \mu] \neq 0$, then we get that $\chi(1)/\mu(1)\big||G:H| = 2^3 \cdot 3^2 \cdot 5$, and hence $\mu(1) = 8$. But $\mu(1)^2 > |H|$, a contradiction.

If $|G:K|=2$, then $|H|=2^2\cdot 7$. Let $\kappa\in Irr(H)$ such that $[\chi_H,\kappa]\neq 0$, one has that $\chi(1)/\kappa(1)\big||G:H|=2^4\cdot 3^2\cdot 5$, and so $\kappa(1)=7$. But $\kappa(1)^2>|H|$, a contradiction. Similarly, we can prove that $|G:K|\neq 4$.

Case 3. $K/H\ncong L_2(7)$.

Otherwise, we assume that $K/H\cong L_2(7)$. In this case, we have $|G:K|=1$ or 2.

If $|G:K|=1$, then $|H|=2^3\cdot 3\cdot 5$. If H is solvable, by Lemma 2, the Sylow 5-subgroup H_5 is normal in H. Since $H\trianglelefteq G$, we have $H_5\trianglelefteq G$. Therefore, $\beta(1)\big||G:H_5|=2^6\cdot 3^2\cdot 7$, a contradiction. If H is nonsolvable, then by Lemma 1 one has that H has a normal series: $1\trianglelefteq N\trianglelefteq M\trianglelefteq H$ such that M/N is a direct product of isomorphic non-abelian simple groups and $|H/M|\big||Out(M/N)|$. Since $|H|=2^3\cdot 3\cdot 5$, then by Table 1 we have $M/N\cong A_5$. Thus $|H/M|\big||Out(A_5)|=2$. And so $|H:M|=1$ or 2.

If $|H:M|=1$, then $|N|=2$. By [2, Clifford Theorem 6.2], N has an irreducible character of degree 2, a contradiction. If $|H:M|=2$, then $|N|=1$. In the case that $M\cong A_5\cdot 2$. Let $\nu\in Irr(M)$ such that $[\chi_M,\nu]\neq 0$, then we have $\chi(1)/\nu(1)\big||G:M|=2^3\cdot 3\cdot 7$, and hence $\nu(1)=8$. By [1], we see that the largest irreducible character degree of $A_5\cdot 2$ is 6. Therefore, M has no such irreducible character of degree 8, a contradiction.

If $|G:K|=2$, then $|H|=2^2\cdot 3\cdot 5$. If H is solvable, then by Lemma 2 we have that $H_5\trianglelefteq H$, where $H_5\in Syl_5(H)$. Since $H\trianglelefteq G$, we have $H_5\trianglelefteq G$. Hence $\beta(1)\big||G:H_5|=2^6\cdot 3^2\cdot 7$, a contradiction. If H is nonsolvable, then by Lemma 1 and Table 1 one has that $H\cong A_5$. Therefore $K\cong A_5\times L_2(7)$. Let $\xi\in Irr(K)$ such that $[\beta_K,\xi]\neq 0$, then we have $\beta(1)/\xi(1)\big||G:K|=2$, and thus $35\big|\xi(1)$. Let $e=[\beta_K,\xi]$, $t=|G:I_G(\xi)|$, one has that $\beta(1)=et\xi(1)$. By the structure of K, we obtain that $\xi(1)=35$, and so $e=2$, $t=1$. But $[\beta_K,\beta_K]=e^2t=4>|G:K|=2$, a contradiction to [2, Lemma 2.29].

Case 4. $K/H\ncong L_2(8)$.

Assume the contrary, suppose that $K/H\cong L_2(8)$. By Lemma 1, one has that $|G:K|=1$ or 3. Since $3^2\parallel |L_2(8)|$, we have $|G:K|=1$. In the case that $|H|=2^3\cdot 5$. By Sylow Theorem, the 5-Sylow subgroup H_5 is normal in H. As $H\trianglelefteq G$, we have $H_5\trianglelefteq G$. Therefore, $\beta(1)\big||G:H_5|=2^6\cdot 3^2\cdot 7$, a contradiction.

Case 5. $K/H\ncong A_7$.

If $K/H\cong A_7$, then by Lemma 1 we have $|G:K|=1$ or 2, and so $|H|=4$ or 8. By [2, Clifford Theorem 6.2], H has an irreducible character of degree 4 or 8, a contradiction to the structure of H.

Case 6. $K/H \ncong L_3(4)$.

We may suppose that $K/H \cong L_3(4)$. By comparing the orders of G and $L_3(4)$, we have $H = 1$, and so $G \cong L_3(4)$. But we know that the largest irreducible character degree of $L_3(4)$ equals to 64, a contradiction to $L_1(G) = 70$.

Case 7. $K/H \cong A_8$.

In this case, we have $H = 1$, and hence $G \cong A_8$.

Thus, we complete the proof of Main Theorem.

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