COMPARATIVE GROWTH ANALYSIS OF FUNCTIONS ANALYTIC IN THE UNIT DISC DEPENDING UPON THEIR RELATIVE L^*-ORDERS AND RELATIVE L^*-LOWER ORDERS

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Abstract. In the paper the ideas of relative Nevanlinna L^*-order and relative Nevanlinna L^*-lower order of an analytic function with respect to an entire function in the unit disc $U = \{z : |z| < 1\}$ are introduced. Hence, we study some comparative growth properties of composition of two analytic functions in the unit disc U on the basis of relative Nevanlinna L^*-order and relative Nevanlinna L^*-lower order.

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1. Introduction, definitions and notations

A function f, analytic in the unit disc $U = \{z : |z| < 1\}$, is said to be of finite Nevanlinna order [2] if there exist a number μ such that Nevanlinna characteristic function

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta$$

satisfies $T(r,f) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$. The greatest lower bound of all such numbers μ is called the Nevanlinna order of f. Thus the Nevanlinna order ρ_f of f is given by

$$\rho_f = \limsup_{r\to 1} \frac{\log T(r,f)}{-\log(1-r)} .$$

Similarly, the Nevanlinna lower order λ_f of f is given by

$$\lambda_f = \liminf_{r\to 1} \frac{\log T(r,f)}{-\log(1-r)} .$$

Datta et. al. [1] introduced the notion of Nevanlinna L-order for an analytic function f in the unit disc $U = \{z : |z| < 1\}$ where $L = L\left(\frac{1}{1-r}\right)$ is a positive continuous function in the unit disc U increasing slowly i.e., $L\left(\frac{a}{1-r}\right) \sim L\left(\frac{1}{1-r}\right)$ as $r \to 1$, for every positive constant 'a', in the following manner:

Definition 1 If f be analytic in U, then the Nevanlinna L-order ρ_f^L and the Nevanlinna L-lower order λ_f^L of f are defined as

$$\rho_f^L = \frac{\log T(r,f)}{\log\left(\frac{L\left(\frac{1}{1-r}\right)}{(1-r)}\right)} \quad \text{and} \quad \lambda_f = \liminf_{r\to 1} \frac{\log T(r,f)}{\log\left(\frac{L\left(\frac{1}{1-r}\right)}{(1-r)}\right)} .$$

Now we introduce the concepts of relative Nevanlinna L^*-order and relative Nevanlinna L^*-lower order of an analytic function f with respect to another analytic function g in the unit disc U which are as follows:

Definition 2 If f be analytic in U and g be entire, then the relative Nevanlinna L^*-order of f with respect to g, denoted by $\rho_g^{L^*}(f)$ is defined by

$$\rho_f^L = \inf\left\{\mu > 0 : T_f(r) < T_g\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\mu} \text{ for all } 0 < r_0(\mu) < r < 1\right\} .$$

Similarly, the relative Nevanlinna L^*-order of f with respect to g, denoted by $\lambda_g^{L^*}(f)$ is given by

$$\lambda_g^{L^*}(f) = \liminf_{r\to 1} \frac{\log T_g^{-1} T_f(r)}{\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)} .$$

When $g(z) = \exp z$, the definition coincides with the definition of the Nevanlinna L^*-order and the Nevanlinna L^*-lower order.

In this paper, we study some growth properties of composition of two analytic functions in the unit disc $U = \{z : |z| < 1\}$ on the basis of relative Nevanlinna L^*-order (relative Nevanlinna L^*-lower order). We do not explain the standard definitions and notations in the theory of entire functions as those are available in [3].

2. Theorems

In this section, we present the main results of the paper.

Theorem 1 *If f, g be any two analytic functions in U and h be an entire function such that* $0 < \lambda_h^{L^*}(f \circ g) \leq \rho_h^{L^*}(f \circ g) < \infty$ *and* $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ *then*

$$\frac{\lambda_h^{L^*}(f \circ g)}{\rho_h^{L^*}(f)} \leq \liminf_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \leq \frac{\lambda_h^{L^*}(f \circ g)}{\lambda_h^{L^*}(f)}$$
$$\leq \limsup_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \leq \frac{\rho_h^{L^*}(f \circ g)}{\lambda_h^{L^*}(f)} .$$

Proof. From the definition of $\rho_h^{L^*}(f)$ and $\lambda_h^{L^*}(f \circ g)$, we have for arbitrary positive ε and for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$\log T_h^{-1} T_{f \circ g}(r) \geqslant \left(\lambda_h^{L^*}(f \circ g) - \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right) \tag{1}$$

and

$$\log T_h^{-1} T_f(r) \leq \left(\rho_h^{L^*}(f) + \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right) . \tag{2}$$

Now from (1) and (2) it follows for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \geqslant \frac{\left(\lambda_h^{L^*}(f \circ g) - \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right)}{\left(\rho_h^{L^*}(f) + \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right)} .$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \geqslant \frac{\lambda_h^{L^*}(f \circ g)}{\rho_h^{L^*}(f)} . \tag{3}$$

Again for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity,

$$\log T_h^{-1} T_{f \circ g}(r) \leq \left(\lambda_h^{L^*}(f \circ g) + \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right) \tag{4}$$

and for all sufficiently large values of $\left(\frac{1}{1-r}\right)$,

$$\log T_h^{-1} T_f(r) \geqslant \left(\lambda_h^{L^*}(f) - \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right) . \tag{5}$$

Combining (4) and (5), we get for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \leq \frac{\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}{\left(\lambda_h^{L^*}(f)-\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$(6)\qquad \liminf_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \leq \frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)}.$$

Also for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$(7)\qquad \log T_h^{-1}T_f(r) \leq \left(\lambda_h^{L^*}(f)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right).$$

Now from (1) and (7), we obtain for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \geq \frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}{\left(\lambda_h^{L^*}(f)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$(8)\qquad \limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \geq \frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)}.$$

Also for all sufficiently large values of $\left(\frac{1}{1-r}\right)$,

$$(9)\qquad \log T_h^{-1}T_{f\circ g}(r) \leq \left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right).$$

Now it follows from (5) and (9) for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \leq \frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}{\left(\lambda_h^{L^*}(f)-\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$(10)\qquad \limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \leq \frac{\rho_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)}.$$

Thus the theorem follows from (3), (6), (8) and (10). ■

The following theorem can be proved in the line of Theorem 1 and so its proof is omitted.

Theorem 2 *If f, g be any two analytic functions in U and h be entire function with $0 < \lambda_h^{L^*}(f \circ g) \leq \rho_h^{L^*}(f \circ g) < \infty$ and $0 < \lambda_h^{L^*}(g) \leq \rho_h^{L^*}(g) < \infty$ then*

$$\frac{\lambda_h^{L^*}(f \circ g)}{\rho_h^{L^*}(g)} \leq \liminf_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_g(r)} \leq \frac{\lambda_h^{L^*}(f \circ g)}{\lambda_h^{L^*}(g)}$$
$$\leq \limsup_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_g(r)} \leq \frac{\rho_h^{L^*}(f \circ g)}{\lambda_h^{L^*}(g)} .$$

Theorem 3 *If f, g be any two analytic functions in U and h be entire function such that $0 < \rho_h^{L^*}(f \circ g) < \infty$ and $0 < \rho_h^{L^*}(f) < \infty$ then*

$$\liminf_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \leq \frac{\rho_h^{L^*}(f \circ g)}{\rho_h^{L^*}(f)} \leq \limsup_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} .$$

Proof. From the definition of $\rho_h^{L^*}(f)$, we get for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\log T_h^{-1} T_f(r) \geqslant \left(\rho_h^{L^*}(f) - \varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right) . \tag{11}$$

Now from (9) and (11), it follows for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \leq \frac{\left(\rho_h^{L^*}(f \circ g) + \varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}{\left(\rho_h^{L^*}(f) - \varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)} .$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \leq \frac{\rho_h^{L^*}(f \circ g)}{\rho_h^{L^*}(f)} . \tag{12}$$

Again for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity,

$$\log T_h^{-1} T_{f \circ g}(r) \geqslant \left(\rho_h^{L^*}(f \circ g) - \varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right) . \tag{13}$$

So combining (2) and (13), we get for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \geqslant \frac{\left(\rho_h^{L^*}(f \circ g) - \varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}{\left(\rho_h^{L^*}(f) + \varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \geqslant \frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)}\ . \tag{14}$$

Thus the theorem follows from (12) and (14). ∎

The following theorem can be carried out in the line of Theorem 3 and therefore we omit its proof.

Theorem 4 *If f, g be any two analytic functions in U and h be an entire function with $0<\rho_h^{L^*}(f\circ g)<\infty$ and $0<\rho_h^{L^*}(g)<\infty$ then*

$$\liminf_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)} \leq \frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(g)} \leq \limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)}\ .$$

The following theorem is a natural consequence of Theorem 1 and Theorem 3.

Theorem 5 *If f, g be any two analytic functions in U and h be an entire function such that $0<\lambda_h^{L^*}(f\circ g)\leq\rho_h^{L^*}(f\circ g)<\infty$ and $0<\lambda_h^{L^*}(f)\leq\rho_h^{L^*}(f)<\infty$ then*

$$\begin{aligned}\liminf_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} &\leq \min\left\{\frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)},\frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)}\right\}\\ &\leq \max\left\{\frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)},\frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)}\right\} \leq \limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)}\ .\end{aligned}$$

The proof is omitted.

Analogously, one may state the following theorem without its proof:

Theorem 6 *If f, g be any two analytic functions in U and h be an entire function with $0<\lambda_h^{L^*}(f\circ g)\leq\rho_h^{L^*}(f\circ g)<\infty$ and $0<\lambda_h^{L^*}(g)\leq\rho_h^{L^*}(g)<\infty$ then*

$$\begin{aligned}\liminf_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)} &\leq \min\left\{\frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(g)},\frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(g)}\right\}\\ &\leq \max\left\{\frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(g)},\frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(g)}\right\} \leq \limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)}\ .\end{aligned}$$

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