ON GENERALIZED RITT ORDER OF ENTIRE FUNCTIONS REPRESENTED BY VECTOR VALUED DIRICHLET SERIES

Sanjib Kumar Datta
Department of Mathematics
University of Kalyani
P.O.-Kalyani, Dist-Nadia, PIN-741235
West Bengal
India
e-mail: sanjib_kr_datta@yahoo.co.in

Tanmay Biswas
Rajbari, Rabindrapalli, R.N. Tagore Road
P.O.-Krishnagar, Dist-Nadia, PIN-741101
West Bengal
India
e-mail: tanmaybiswas_math@rediffmail.com

Jinarul Haque Shaikh
Department of Mathematics
University of Kalyani
P.O.-Kalyani, Dist-Nadia, PIN-741235
West Bengal
India
email: jnrlhqshkh188@gmail.com

Abstract. In this paper, we introduce the idea of generalized Ritt order of entire function (respectively generalized Ritt lower order) represented by a vector valued Dirichlet series. Hence we study some growth properties of of two of entire functions represented by a vector valued Dirichlet series on the basis of their generalized Ritt orders and generalized Ritt lower orders.

Keywords and phrases: vector valued Dirichlet series (VVDS), generalized Ritt order, generalized Ritt lower order, growth.
AMS Subject Classification (2010): 30B50, 30D15, 30D99.

1. Introduction, definitions and notations

Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ (σ and t are real variables) defined by everywhere absolutely convergent vector valued Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1}$$

where a_n's belong to a Banach space $(E, \|.\|)$ and λ_n's are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1} \, (n \geq 1) \, , \lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ and satisfy the conditions

$$\limsup_{n\rightarrow\infty} \frac{\log n}{\lambda_n} = D < \infty$$

and

$$\limsup_{n\rightarrow\infty} \frac{\log \|a_n\|}{\lambda_n} = -\infty \ .$$

If σ_a and σ_c are denoted respectively the abscissa of convergence and absolute convergence of (1), then in this case clearly $\sigma_a = \sigma_c = \infty$.

The function $M(\sigma, f)$ known as maximum modulus function corresponding to an entire function $f(s)$ defined by (1), is written as follows

$$M(\sigma, f) = \underset{-\infty<t<\infty}{l.u.b.} |f(\sigma + it)| \ .$$

Then, the Ritt order [1] of $f(s)$, denoted by ρ_f is given by

$$\begin{aligned} \rho_f &= \inf \{\mu > 0 : \log M(\sigma, f) < \exp(\sigma\mu) \ \text{ for all } \sigma > R(\mu)\} \\ &= \limsup_{\sigma\rightarrow\infty} \frac{\log^{[2]} M(\sigma, f)}{\sigma} \end{aligned}$$

where

$$\begin{aligned} \log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \ \text{ for } k = 1, 2, 3, \cdots ; \\ \log^{[0]} x &= x \ . \end{aligned}$$

Similarly, one can define the Ritt lower order of $f(s)$, denoted by λ_f in the following manner:

$$\lambda_f = \liminf_{\sigma\rightarrow\infty} \frac{\log^{[2]} M(\sigma, f)}{\sigma} \ .$$

During the past decades, several authors (see for example [1],[2],[3],[4],[6]) made close investications on the properties of entire Dirichlet series related to Ritt order. Further, B.L. Srivastava [5] defined different growth parameters such as order and lower order of entire functions represented by vector valued Dirichlet series. He also obtained the results for coefficient characterization of order. Now in the line of Sato [7], it therefore seems reasonable to define suitably the generalized Ritt order denoted as $\rho_f^{[l]}$ of entire function represented by a vector valued Dirichlet series which is as follows:

$$\rho_f^{[l]} = \limsup_{\sigma\rightarrow\infty} \frac{\log^{[l]} M(\sigma, f)}{\sigma}$$

where l is any positive integer.

Likewise, one can define the generalized Ritt lower order of $f(s)$, denoted by $\lambda_f^{[l]}$ for any integer $l \geq 1$ in the following manner:

$$\lambda_f^{[l]} = \liminf_{\sigma\rightarrow\infty} \frac{\log^{[l]} M(\sigma, f)}{\sigma} \ .$$

In this paper we study some growth properties of of two entire functions represented by vector valued Dirichlet series on the basis of generalized Ritt order and generalized Ritt lower order.

2. Theorems

In this section, we present the main results of the paper.

Theorem 1 *If f, g be any two entire vector valued Dirichlet series such that $0 < \lambda_f^{[m]} \leq \rho_f^{[m]} < \infty$ and $0 < \lambda_g^{[n]} \leq \rho_g^{[n]} < \infty$ where m and n are any two positive integers, then*

$$\frac{\lambda_f^{[m]}}{\rho_g^{[n]}} \leq \liminf_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \leq \frac{\lambda_f^{[m]}}{\lambda_g^{[n]}} \leq \limsup_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \leq \frac{\rho_f^{[m]}}{\lambda_g^{[n]}} .$$

Proof. From the definitions of $\rho_g^{[n]}$ and $\lambda_f^{[m]}$, we have for arbitrary positive ε and for all sufficiently large values of σ that

$$\log^{[m]} M(\sigma, f) \geqslant \left(\lambda_f^{[m]} - \varepsilon\right)\sigma \tag{1}$$

and

$$\log^{[n]} M(\sigma, g) \leq \left(\rho_g^{[n]} + \varepsilon\right)\sigma . \tag{2}$$

Now from (1) and (2) it follows for all sufficiently large values of σ that

$$\frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \geqslant \frac{\left(\lambda_f^{[m]} - \varepsilon\right)\sigma}{\left(\rho_g^{[n]} + \varepsilon\right)\sigma} .$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \geqslant \frac{\lambda_f^{[m]}}{\rho_g^{[n]}} . \tag{3}$$

Again for a sequence of values of σ tending to infinity,

$$\log^{[m]} M(\sigma, f) \leq \left(\lambda_f^{[m]} + \varepsilon\right)\sigma \tag{4}$$

and for all sufficiently large values of σ,

$$\log^{[n]} M(\sigma, g) \geqslant \left(\lambda_g^{[n]} - \varepsilon\right)\sigma . \tag{5}$$

Combining (4) and (5) we get for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \leq \frac{\left(\lambda_f^{[m]} + \varepsilon\right)\sigma}{\left(\lambda_g^{[n]} - \varepsilon\right)\sigma} .$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{\sigma\to\infty}\frac{\log^{[m]}M(\sigma,f)}{\log^{[n]}M(\sigma,g)}\leq\frac{\lambda_f^{[m]}}{\lambda_g^{[n]}}\ . \tag{6}$$

Also for a sequence of values of σ tending to infinity that

$$\log^{[n]}M(\sigma,g)\leq\left(\lambda_g^{[n]}+\varepsilon\right)\sigma\ . \tag{7}$$

Now from (1) and (7) we obtain for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]}M(\sigma,f)}{\log^{[n]}M(\sigma,g)}\geq\frac{\left(\lambda_f^{[m]}-\varepsilon\right)\sigma}{\left(\lambda_g^{[n]}+\varepsilon\right)\sigma}\ .$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{\sigma\to\infty}\frac{\log^{[m]}M(\sigma,f)}{\log^{[n]}M(\sigma,g)}\geq\frac{\lambda_f^{[m]}}{\lambda_g^{[n]}}\ . \tag{8}$$

Also for all sufficiently large values of σ,

$$\log^{[m]}M(\sigma,f)\leq\left(\rho_f^{[m]}+\varepsilon\right)\sigma\ . \tag{9}$$

Now, it follows from (5) and (9), for all sufficiently large values of σ that

$$\frac{\log^{[m]}M(\sigma,f)}{\log^{[n]}M(\sigma,g)}\leq\frac{\left(\rho_f^{[m]}+\varepsilon\right)\sigma}{\left(\lambda_g^{[n]}-\varepsilon\right)\sigma}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty}\frac{\log^{[m]}M(\sigma,f)}{\log^{[n]}M(\sigma,g)}\leq\frac{\rho_f^{[m]}}{\lambda_g^{[n]}}\ . \tag{10}$$

Thus the theorem follows from (3), (6), (8) and (10). ∎

Theorem 2 *If f, g be any two entire vector valued Dirichlet series such that $0<\rho_f^{[m]}<\infty$ and $0<\rho_g^{[n]}<\infty$ where m and n are any two positive integers, then*

$$\liminf_{\sigma\to\infty}\frac{\log^{[m]}M(\sigma,f)}{\log^{[n]}M(\sigma,g)}\leq\frac{\rho_{f\circ g}^{[m]}}{\rho_g^{[n]}}\leq\limsup_{\sigma\to\infty}\frac{\log^{[m]}M(\sigma,f)}{\log^{[n]}M(\sigma,g)}\ .$$

Proof. From the definition of $\rho_g^{[n]}$, we get for a sequence of values of σ tending to infinity that

$$\log^{[n]}M(\sigma,g)\geqslant\left(\rho_g^{[n]}-\varepsilon\right)\sigma\ . \tag{11}$$

Now from (9) and (11), it follows for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \leq \frac{\left(\rho_f^{[m]} + \varepsilon\right)\sigma}{\left(\rho_g^{[n]} - \varepsilon\right)\sigma} .$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \leq \frac{\rho_f^{[m]}}{\rho_g^{[n]}} . \tag{12}$$

Again for a sequence of values of σ tending to infinity,

$$\log^{[m]} M(\sigma, f) \geqslant \left(\rho_f^{[m]} - \varepsilon\right)\sigma . \tag{13}$$

So combining (2) and (13), we get for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \geqslant \frac{\left(\rho_f^{[m]} - \varepsilon\right)\sigma}{\left(\rho_g^{[n]} + \varepsilon\right)\sigma} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \geqslant \frac{\rho_f^{[m]}}{\rho_g^{[n]}} . \tag{14}$$

Thus the theorem follows from (12) and (14). ∎

The following theorem is a natural consequence of Theorem 1 and Theorem 2.

Theorem 3 *If f, g be any two entire vector valued Dirichlet series such that $0 < \lambda_f^{[m]} \leq \rho_f^{[m]} < \infty$ and $0 < \lambda_g^{[n]} \leq \rho_g^{[n]} < \infty$ where m and n are any two positive integers, then*

$$\begin{aligned} \liminf_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} &\leq \min\left\{\frac{\lambda_f^{[m]}}{\lambda_g^{[n]}}, \frac{\rho_f^{[m]}}{\rho_g^{[n]}}\right\} \\ &\leq \max\left\{\frac{\lambda_f^{[m]}}{\lambda_g^{[n]}}, \frac{\rho_f^{[m]}}{\rho_g^{[n]}}\right\} \\ &\leq \leq \limsup_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} . \end{aligned}$$

The proof is omitted.

References

[1] Rahaman, Q.I., *The Ritt order of the derivative of an entire function*, Annales Polonici Mathematici, 17 (1965), 137-140.

[2] Rajagopal, C.T., Reddy, A.R., *A note on entire functions represented by Dirichlet series*, Annales Polonici Mathematici, 17 (1965), 199-208.

[3] Ritt, J.F., *On certain points in the theory of Dirichlet series*, Amer. J. Math., 50 (1928), 73-86.

[4] Srivastava, G.S., Sharma, A., *On generalized order and generalized type of vector valued Dirichlet series of slow growth*, Int. J. Math. Archive, 2 (12) (2011), 2652-2659.

[5] Srivastava, B.L., *A study of spaces of certain classes of vector valued Dirichlet series*, Thesis, I.I.T., Kanpur, 1983.

[6] Srivastav, R.P., Ghosh, R.K., *On entire functions represented by Dirichlet series*, Annales Polonici Mathematici, 13 (1963), 93-100.

[7] Sato, D., *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc., 69 (1963), 411-414.

Accepted: 03.08.2015