A STUDY ON FUZZY INTERIOR HYPERIDEALS IN ORDERED SEMIHYPERGROUPS

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Abstract. In this paper, we first introduce the concept of fuzzy interior hyperideals of an ordered semihypergroup S by the ordered fuzzy points of S, and investigate its related properties. In particular, we give the characterization of fuzzy interior hyperideal generated by a fuzzy subset in an ordered semihypergroup. Furthermore, the idea of normal fuzzy interior hyperideals in ordered semihypergroups is given and several related characterization theorems are provided. Finally, some new characterizations of semisimple ordered semihypergroups by the properties of fuzzy interior hyperideals are given.

Keywords: ordered semihypergroup, ordered fuzzy point, interior hyperideal, fuzzy interior hyperideal, normal fuzzy interior hyperideal.

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1. Introduction

As we know, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups play a prominent role in mathematics with a wider range of applications in many disciplines such as sequential machines, formal languages, computer arithmetics, error-correcting codes and many others. Ordered semigroups have been investigated by several researchers, for example, Alimov [1], Clifford [4] and Conrad [5], also see [38]. The theory of fuzzy sets, which was initially introduced by Zadeh [39], has been applied to many mathematical branches. In particular, Kehayopulu and Tsingelis [22] applied the concept of fuzzy sets to the theory of ordered semigroups. Then they defined "fuzzy" analogue of several notations, which appeared to be useful in the theory of ordered semigroups. The theory of fuzzy sets on ordered semigroups has been recently developed. For more details, the reader is referred to [23], [24], [34], [36]. In [32], Sardar et al. studied the notions of interior ideals and characteristic interior ideals of a Γ-semigroups. In [33], they studied the notion of generalized fuzzy interior ideals in Γ-semigroups. Also, see [13].

The theory of algebraic hyperstructures which is a generalization of the concept of algebraic structures was first introduced by Marty [27]. Later on, hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics. One of the main reason which attracts researches towards hyperstructures is its unique property that in hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an element. Thus hyperstructures are natural extension of classical algebraic structures. Recently, many researchers have worked on algebraic hyperstructures and generalized various classical algebraic structures, for instance, see [6], [7], [11], [17], [28], [37]. Especially, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Nowadays many authors have studied different aspects of semihypergroups, for example, Davvaz [8], Davvaz and Poursalavati [14], Hila et al. [21] and Leoreanu [25], also see [10], [29]. It is worth pointing out that Heidari and Davvaz [19] applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of ordered semigroups, also see [3], [12].

The study of fuzzy hyperstructures is an interesting research topic of fuzzy set theory. We noticed that the relationships between the fuzzy sets and algebraic hyperstructures have been already considered by Corsini, Davvaz, Leoreanu, Dudek, Jun, Zhan, Hila and others, for instance, the reader can refer to [2], [9], [15], [20], [26], [40]. In [16], Ersoy et al. investigated the properties of fuzzy interior hyperideals of Γ-semihypergroups. It is now natural to investigate similar type of the existing fuzzy subsystems of ordered semihypergroups. In [30], [31], Pibaljommee and Davvaz studied fuzzy hyperideals of ordered semihypergroups. Also, see [35]. As a further study of ordered semihypergroups theory, we attempt in the present paper to study fuzzy interior hyperideals of ordered semihypergroups in detail.

The rest of this paper is organized as follows. After an introduction, in Section 2 we recall some basic definitions and results of ordered semihypergroups which will be used throughout this paper. In Section 3, we define and study the interior hyperideals of an ordered semihypergroup. Furthermore, we introduce the concept of fuzzy interior hyperideals of ordered semihypergroups by the ordered fuzzy points, and investigate its related properties. In particular, we characterize the fuzzy interior hyperideal generated by a fuzzy subset in an ordered semihypergroup. The idea of normal fuzzy interior hyperideals in ordered semihypergroups is given and several related characterization theorems are provided in Section 4. In Section 5, we give some new characterizations of semisimple ordered semihypergroups in terms of fuzzy interior hyperideals.

2. Preliminaries and some notations

Recall first the basic terms and definitions from the hyperstructure theory.

As we know, a *hypergroupoid* $(S, \circ)$ is a nonempty set S together with a hyperoperation, that is a map $\circ : S \times S \to P^*(S)$, where $P^*(S)$ denotes the set of all nonempty subsets of S (see [6]). The image of the pair (x, y) is denoted by $x \circ y$. If $x \in S$ and A, B are nonempty subsets of S, then $A \circ B$ is defined by $A \circ B = \bigcup\limits_{a\in A, b\in B} a \circ b$. The notations $A \circ x$ and $x \circ A$ are used for $A \circ \{x\}$ and $\{x\} \circ A$, respectively.

We say that a hypergroupoid $(S, \circ)$ is a *semihypergroup* if the hyperoperation "$\circ$" is associative, that is, $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$ (see [7]).

Now, we recall the notion of ordered semihypergroups from [19].

Definition 2.1. An algebraic hyperstructure $(S, \circ, \leq)$ is called an *ordered semihypergroup* (also called *po-semihypergroup* in [19]) if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that: for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \preceq a \circ y$ and $x \circ a \preceq y \circ a$. Here, if $A, B \in P^*(S)$, then we say that $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. In particular, if $A = \{a\}$, then we write $a \preceq B$ instead of $\{a\} \preceq B$.

Throughout this paper, unless otherwise mentioned, S will denote an ordered semihypergroup.

Let S be an ordered semihypergroup. For $\emptyset \neq H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

By a *subsemihypergroup* of an ordered semihypergroup S we mean a nonempty subset A of S such that $A \circ A \subseteq A$. A nonempty subset A of an ordered semihypergroup S is called a *left* (resp. *right*) *hyperideal* of S if (1) $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$) and (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A is both a left and a right hyperideal of S, then it is called a *(two-sided) hyperideal* of S (see [19]).

Lemma 2.2. *Let S be an ordered semihypergroup. Then the following statements hold:*

(1) $A \subseteq (A]$, $\forall A \subseteq S$.

(2) *If* $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$.

(3) $(A] \circ (B] \subseteq (A \circ B]$ *and* $((A] \circ (B]] = (A \circ B],\ \forall A, B \subseteq S$.

(4) $((A]] = (A],\ \forall A \subseteq S$.

(5) *For any nonempty subset* A, B, C *of* S *such that* $A \preceq B$ *and* $B \preceq C$, *we have* $A \preceq C$.

(6) *For any two nonempty subsets* A, B *of* S *such that* $A \preceq B$, *we have* $C \circ A \preceq C \circ B$ *and* $A \circ C \preceq B \circ C$ *for any nonempty subset* C *of* S.

Proof. Straightforward. ∎

For the sake of simplicity, throughout this paper, we denote $A^n = A \circ A \circ \cdots \circ A$ (n-copies).

An ordered semihypergroup $(S, \circ, \leq)$ is called *regular* if for each $a \in S$, there exists $x \in S$ such that $a \preceq a \circ x \circ a$.

Equivalent Definitions: (1) $A \subseteq (A \circ S \circ A], \forall A \subseteq S$. (2) $a \in (a \circ S \circ a], \forall a \in S$.

An ordered semihypergroup $(S, \circ, \leq)$ is called *semisimple* if $(I^2] = I$ holds for every hyperideal I of S (see [35]).

We next state some fuzzy logic concepts.

Let S be an ordered semihypergroup. By a *fuzzy subset* of S, we mean a function from S into the real closed interval [0,1], that is, $f : S \to [0, 1]$. For an ordered semihypergroup S, the fuzzy subset 1 of S is defined as follows:

$$1 :\ S \to [0, 1],\ x \mapsto 1(x) := 1,\ \forall x \in S.$$

We denote by $F(S)$ the set of all fuzzy subsets of S. Let $f, g \in F(S)$. Then the inclusion relation $f \subseteq g$ is defined by $f(x) \leq g(x)$ for all $x \in S$, and $f \cap g$ is defined by

$$(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x)$$

for all $x \in S$.

Let $(S, \circ, \leq)$ be an ordered semihypergroup. For $x \in S$, we define $H_x := \{(y, z) \in S \times S \mid x \preceq y \circ z\}$. For any $f, g \in F(S)$, the product $f * g$ of f and g is defined by

$$(\forall x \in S)\ (f * g)(x) = \begin{cases} \bigvee_{(y,z) \in H_x} [f(y) \wedge g(z)], & \text{if } H_x \neq \emptyset\ , \\ 0, & \text{if } H_x = \emptyset. \end{cases}$$

As we know, the multiplication "$*$" on $F(S)$ is associative and $(F(S), *, \subseteq)$ forms an ordered semigroup (see [35]).

Let S be an ordered semihypergroup. A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) *hyperideal* of S if

(1) $x \leq y$ implies $f(x) \geq f(y)$ for all $x, y \in S$, and

(2) $\bigwedge_{z \in x \circ y} f(z) \geq f(y)$ (resp. $\bigwedge_{z \in x \circ y} f(z) \geq f(x)$) for all $x, y \in S$. Equivalently, $1 * f \subseteq f$ (resp. $f * 1 \subseteq f$).

A *fuzzy hyperideal* of S is a fuzzy subset of S which is both a fuzzy left and a fuzzy right hyperideal of S (see [35]).

Definition 2.3. Let S be an ordered semihypergroup and $f \in F(S)$. The set $f_t := \{x \in S \mid f(x) \geq t\}$, where $t \in [0, 1]$ is called a *level subset* of f.

Definition 2.4. ([35]) Let S be an ordered semihypergroup, $a \in S$ and $\lambda \in [0,1]$. An *ordered fuzzy point* a_λ of S is defined by the rule that

$$(\forall x \in S)\ a_\lambda(x) = \begin{cases} \lambda, & \text{if } x \in (a], \\ 0, & \text{if } x \notin (a]. \end{cases}$$

It is evident that every ordered fuzzy point of S is a fuzzy subset of S. For any fuzzy subset f of S, we also denote $a_\lambda \subseteq f$ by $a_\lambda \in f$ in the sequel.

Definition 2.5. ([35]) Let f be a fuzzy subset of an ordered semihypergroup S. We define $(f]$ by the rule that

$$(f](x) = \bigvee_{y \geq x} f(y)$$

for all $x \in S$. A fuzzy subset f of S is called *strongly convex* if $f = (f]$.

Lemma 2.6. ([35]) *Let f be a fuzzy subset of an ordered semihypergroup S. Then f is a strongly convex fuzzy subset of S if and only if $x \leq y$ implies $f(x) \geq f(y)$, for all $x, y \in S$.*

Lemma 2.7. ([35]) *Let a_λ, b_μ $(\lambda > 0, \mu > 0)$ be ordered fuzzy points of S, and $f, g \in F(S)$. Then the following statements are true:*

(1) $a_\lambda * b_\mu = \bigcup_{c \in (a \circ b]} c_{\lambda \wedge \mu}$.

(2) $(a_\lambda * b_\mu) * c_\delta = a_\lambda * (b_\mu * c_\delta) = \bigcup_{d \in (a \circ b \circ c]} d_{\lambda \wedge \mu \wedge \delta}$ *for any ordered fuzzy point a_λ, b_μ and c_δ of S.*

(3) *If $f \subseteq g$ and $h \in F(S)$, then $f * h \subseteq g * h$, $h * f \subseteq h * g$.*

(4) *If f is a strongly convex fuzzy subset of S, then $a_\lambda \in f$ if and only if $f(a) \geq \lambda$.*

Let A be a nonempty subset of an ordered semihypergroup S. Then the *characteristic function* f_A of A is a fuzzy subset of S defined by

$$f_A : S \to [0,1] \mid x \mapsto f_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Lemma 2.8. ([35]) *Let S be an ordered semihypergroup. A nonempty subset A of S is a hyperideal of S if and only if the characteristic function f_A of A is a fuzzy hyperideal of S.*

The reader is referred to [7], [35], [38] for notation and terminology not defined in this paper.

3. Fuzzy interior hyperideals of ordered semihypergroups

In what follows, let Z^+ denote the set of all positive integers. In this section, we introduce the concepts of interior hyperideals and fuzzy interior hyperideals of an ordered semihypergroup, and investigate their related properties. In particular, we give the characterization of fuzzy interior hyperideal generated by a fuzzy subset in an ordered semihypergroup.

Definition 3.1. Let S be an ordered semihypergroup. A subsemihypergroup A of S is called an *interior hyperideal* of S if

(1) $S \circ A \circ S \subseteq A$, and
(2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. Equivalently, $(A] \subseteq A$.

Remark 3.2. The concept of interior hyperideals defined in Definition 3.1 is a generalization of the concept of interior hyperideals of semihypergroups (without order) to ordered semihypergroup theory, see [18]. Moreover, it is clear that every hyperideal of an ordered semihypergroup S is an interior hyperideal of S. However, the converse is not true, in general, as shown in the following example.

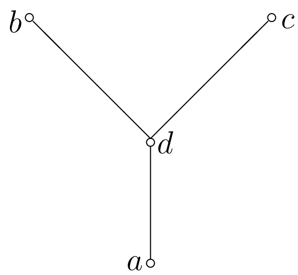
Example 3.3. We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
d	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a\}$

$$\leq := \{(a, a), (a, b), (a, c), (a, d), (b, b), (c, c), (d, b), (d, c), (d, d)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a, d), (d, b), (d, c)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup. We can easily verify that the sets $\{a\}$, $\{a, d\}$, $\{a, b, d\}$ and S are all interior hyperideals of S. But $\{a, d\}$ is not a hyperideal of S. In fact, since $c \in S, d \in \{a, d\}$, but $c \circ d = \{a, b\} \nsubseteq \{a, d\}$, i.e., $\{a, d\}$ is not a left hyperideal of S.

In the following theorem, we give a condition for an interior hyperideal of an ordered semihypergroup S to be a hyperideal of S.

Theorem 3.4. *Let S be a regular ordered semihypergroup and A an interior hyperideal of S. Then A is a hyperideal of S.*

Proof. Suppose that A is an interior hyperideal of a regular ordered semihypergroup S. Let $x \in A, y \in S$. Then, since S is regular, there exists $x_1 \in S$ such that $x \preceq x \circ x_1 \circ x$, and by Lemma 2.2(6), we have

$$x \circ y \preceq x \circ x_1 \circ x \circ y = (x \circ x_1) \circ x \circ y \subseteq S \circ A \circ S \subseteq A \text{ and}$$
$$y \circ x \preceq y \circ x \circ x_1 \circ x = y \circ x \circ (x_1 \circ x) \subseteq S \circ A \circ S \subseteq A.$$

Then it can be easily shown that $x \circ y \subseteq A, y \circ x \subseteq A$. Thus A is a hyperideal of S. ∎

Theorem 3.5. *Let S be an ordered semihypergroup and $\{A_i \mid i \in I\}$ a family of interior hyperideals of S. Then $\bigcap_{i\in I} A_i$ is an interior hyperideal of S if $\bigcap_{i\in I} A_i \neq \emptyset$.*

Proof. The proof is straightforward verification, and hence we omit the details.

Let now S be an ordered semihypergroup and $\emptyset \neq A \subseteq S$. We denote

$$\Omega = \{I \mid I \text{ is an interior hyperideal of } S \text{ containing } A\}.$$

Clearly, Ω is not empty since $S \in \Omega$. Let $In(A) = \bigcap_{I\in\Omega} I$. It is clear that $In(A) \neq \emptyset$ because $A \subseteq In(A)$. By Theorem 3.5, $In(A)$ is an interior hyperideal of S. Moreover, $In(A)$ is the smallest interior hyperideal of S containing A. The interior hyperideal $In(A)$ is called the *interior hyperideal of S generated by A*. For $A = \{a\}$, let $In(a)$ denote the interior hyperideal of S generated by $\{a\}$.

Theorem 3.6. *Let S be an ordered semihypergroup. Then*

(1) *For every $a \in S$, $In(a) = (a \cup a^2 \cup (S \circ a \circ S)]$.*

(2) *For every $\emptyset \neq A \subseteq S$, $In(A) = (A \cup A^2 \cup (S \circ A \circ S)]$.*

Proof. (1) Let $a \in S$. Clearly, $(a\cup a^2\cup(S\circ a\circ S)]$ is a subsemihypergroup of S. Let $x, y \in S$ and $b \in (a\cup a^2\cup(S\circ a\circ S)]$. Then there exists $w \in a\cup a^2\cup(S\circ a\circ S)$ such that $b \leq w$. Thus, by Lemma 2.2(6), we have $x\circ b\circ y \preceq x\circ w\circ y \subseteq a\cup a^2\cup(S\circ a\circ S)$. It implies that $x\circ b\circ y \subseteq (a\cup a^2\cup(S\circ a\circ S)]$. Consequently, $S\circ(a\cup a^2\cup(S\circ a\circ S)]\circ S \subseteq (a \cup a^2 \cup (S \circ a \circ S)]$. Hence $(a \cup a^2 \cup (S \circ a \circ S)]$ is an interior hyperideal of S containing a. Furthermore, we claim that $In(a) = (a\cup a^2\cup(S\circ a\circ S)]$. To prove our claim, let I be an interior hyperideal of S containing a. Let $c \in (a\cup a^2\cup(S\circ a\circ S)]$. Then, there exists $u \in a\cup a^2\cup(S\circ a\circ S)$ such that $c \leq u$. Since $a \in I$ and I is an interior hyperideal of S, we have $a^2 \subseteq I, S\circ a\circ S \subseteq S\circ I\circ S \subseteq I$. Thus, $c \leq u \in I$ and then $c \in I$. Hence $(a \cup a^2 \cup (S \circ a \circ S)]$ is the smallest interior hyperideal of S containing a. Therefore, we obtain the request result.

(2) It can be proved similarly as (1). ∎

Example 3.7. Let S be the ordered semihypergroup of Example 3.3 and $A = \{a, b\} \subseteq S$. Then we apply the result of the above theorem, and obtain that $In(A) = \{a, b, d\}$.

Now, we define and study the fuzzy interior hyperideals of ordered semihypergroups.

Definition 3.8. Let S be an ordered semihypergroup. A fuzzy subset f of S is called a *fuzzy subsemihypergroup* of S if $x_t \in f$ and $y_r \in f$ imply $x_t * y_r \in f$ for all $t, r \in (0, 1]$ and $x, y \in S$.

Theorem 3.9. *Let S be an ordered semihypergroup and f a strongly convex fuzzy subset of S. Then f is a fuzzy subsemihypergroup of S if and only if*

$$\bigwedge_{z\in x\circ y} f(z) \geq f(x) \wedge f(y) \textit{ for all } x, y \in S.$$

Proof. Let f be a fuzzy subsemihypergroup of S. Assume that $\bigwedge_{z\in x\circ y} f(z) < f(x)\wedge f(y)$ for some $x, y \in S$. Choose $t \in (0, 1]$ such that $\bigwedge_{z\in x\circ y} f(z) < t \leq f(x) \wedge f(y)$. Then, since f is strongly convex, $x_t \in f$ and $y_t \in f$. Thus, by Lemma 2.7(1), $\bigcup_{z\in(x\circ y]} z_t = x_t * y_t \in f$. Then it can be easily shown that $\bigwedge_{z\in x\circ y} f(z) \geq t$, which is a contradiction. Hence $\bigwedge_{z\in x\circ y} f(z) \geq f(x) \wedge f(y)$ for all $x, y \in S$. Conversely, suppose that $\bigwedge_{z\in x\circ y} f(z) \geq f(x) \wedge f(y)$ for all $x, y \in S$. Let $x_t \in f$ and $y_r \in f$ for all $t, r \in (0, 1]$. Then $f(x) \geq t$ and $f(y) \geq r$. By hypothesis, we have $\bigwedge_{z\in x\circ y} f(z) \geq f(x) \wedge f(y) \geq t \wedge r$, which means that $f(z) \geq t \wedge r$ for any $z \in x \circ y$. Let w be any element of $(x \circ y]$. Then there exists $u \in x \circ y$ such that $w \leq u$. Since f is a strongly convex fuzzy subset of S, by Lemma 2.6, $f(w) \geq f(u) \geq t \wedge r$, and we have $w_{t\wedge r} \in f$. Thus, by Lemma 2.7(1), $x_t * y_r = \bigcup_{w\in(x\circ y]} w_{t\wedge r} \in f$. Therefore, f is a fuzzy subsemihypergroup of S. ∎

Definition 3.10. Let S be an ordered semihypergroup. A fuzzy subsemihypergroup f of S is called a *fuzzy interior hyperideal* of S if for all $x, y, z \in S$ and $t \in (0, 1]$, the following conditions hold:

(1) $x \leq y \Rightarrow f(x) \geq f(y)$.

(2) $y_t \in f$ and $x, z \in S \Rightarrow x_t * y_t * z_t \in f$.

Theorem 3.11. *Let S be an ordered semihypergroup and f a fuzzy subsemihypergroup of S. Then f is a fuzzy interior hyperideal of S if and only if f satisfies the following assertions:*

(1) $x \leq y \Rightarrow f(x) \geq f(y)$ *for all* $x, y \in S$.

(2) $\bigwedge_{w\in x\circ y\circ z} f(w) \geq f(y)$.

Proof. The proof is similar to that of Theorem 3.9 with suitable modification by using Lemma 2.7(2). ∎

Lemma 3.12. *Let S be an ordered semihypergroup and f a strongly convex fuzzy subset of S. Then f is a fuzzy subsemihypergroup of S if and only if $f * f \subseteq f$.*

Proof. Suppose that f is a fuzzy subsemihypergroup of S. To prove that $f*f \subseteq f$, it is enough to show that $(f * f)(x) \leq f(x)$ for any $x \in S$. Indeed, let $x \in S$. If $H_x = \emptyset$, then $(f*f)(x) = 0$. Since $f(x) \geq 0$ for all $x \in S$, we have $(f*f)(x) \leq f(x)$. Let $H_x \neq \emptyset$. Then there exist $y, z \in S$ such that $x \preceq y\circ z$, and there exists $w \in y\circ z$ such that $x \leq w$. Since f is a strongly convex fuzzy subsemihypergroup of S, by Lemma 2.6 and Theorem 3.9, we have $f(y) \wedge f(z) \leq \bigwedge_{w\in y\circ z} f(w) \leq \bigwedge_{\substack{w\in y\circ z\\ x\leq w}} f(w) \leq \bigwedge_{\substack{w\in y\circ z\\ x\leq w}} f(x) = f(x)$. Hence we have $f(y) \wedge f(z) \leq f(x)$ for any $x \preceq y \circ z$. Thus

$(f * f)(x) = \bigvee_{x \preceq y \circ z} [f(y) \wedge f(z)] \leq f(x)$. Thus $(f * f)(x) \leq f(x)$ for any $x \in S$.

Conversely, let $x, y \in S$. Then we can prove that $f(z) \geq f(y) \wedge f(z)$ for any $z \in x \circ y$. In fact, since $z \in x \circ y, z \leq z$, we have $z \preceq x \circ y$. Thus, by hypothesis, we have $f(z) \geq (f * f)(z) = \bigvee_{z \preceq u \circ v} [f(u) \wedge f(v)] \geq f(x) \wedge f(y)$ for every $x, y \in S$ such that $z \in x \circ y$, from which we can deduce that $\bigwedge_{z \in x \circ y} f(z) \geq f(x) \wedge f(y)$. By Theorem 3.9, f is a fuzzy subsemihypergroup of S. ∎

Theorem 3.13. *Let S be an ordered semihypergroup and f a strongly convex fuzzy subset of S. Then f is a fuzzy interior hyperideal of S if and only if $f * f \subseteq f$ and $1 * f * 1 \subseteq f$.*

Proof. Let f be a fuzzy interior hyperideal of S. Then $(1 * f * 1)(a) \leq f(a)$ for all $a \in S$. Indeed, if $(1 * f * 1)(a) = 0$, clearly, $(1 * f * 1)(a) \leq f(a)$. Let $(1 * f * 1)(a) \neq 0$. Then we can also prove that $(1 * f * 1)(a) \leq f(a)$. In fact, let $(x, y) \in H_a$ and $(p, q) \in H_x$, i.e., $a \preceq x \circ y, x \preceq p \circ q$. Then, by Lemma 2.2, $a \preceq p \circ q \circ y$, and there exists $u \in p \circ q \circ y$ such that $a \leq u$. Since f is a fuzzy interior hyperideal of S, by Theorem 3.11 we have $f(a) \geq \bigwedge_{\substack{u \in p \circ q \circ y \\ a \leq u}} f(u) \geq \bigwedge_{u \in p \circ q \circ y} f(u) \geq f(q)$. Thus $(1 * f * 1)(a) = \bigvee_{(x,y) \in H_a} [(1 * f)(x) \wedge 1(y)] = \bigvee_{(x,y) \in H_a} (1 * f)(x) = \bigvee_{(x,y) \in H_a} (\bigvee_{(p,q) \in H_x} [1(p) \wedge f(q)]) = \bigvee_{\substack{a \preceq x \circ y \\ x \preceq p \circ q}} f(q) \leq f(a)$, which means that $1 * f * 1 \subseteq f$.

Conversely, for any $x, y, z \in S$, let $w \in x \circ y \circ z$. Then, there exists $u \in x \circ y \subseteq (x \circ y]$ such that $w \in u \circ z \subseteq (u \circ z]$, and we have $(x, y) \in H_u, (u, z) \in H_w$. Since $1 * f * 1 \subseteq f$, we have $f(w) \geq (1 * f * 1)(w) = \bigvee_{(p,q) \in H_w} [(1 * f)(p) \wedge 1(q)] \geq (1 * f)(u) \wedge 1(z) = (1 * f)(u) = \bigvee_{(s,t) \in H_u} [1(s) \wedge f(t)] \geq 1(x) \wedge f(y) = f(y)$. It thus follows that $\bigwedge_{w \in x \circ y \circ z} f(w) \geq f(y)$.

The rest of the proof is a consequence of Theorem 3.11 and Lemma 3.12. ∎

Theorem 3.14. *Let S be an ordered semihypergroup. A nonempty subset A of S is an interior hyperideal of S if and only if the characteristic function f_A of A is a fuzzy interior hyperideal of S.*

Proof. Suppose that A is an interior hyperideal of S. Let $x, y \in S, x \leq y$. Then $f_A(x) \geq f_A(y)$. In fact, if $y \in A$, then $f_A(y) = 1$. Since $S \ni x \leq y \in A$, by hypothesis we have $x \in A$, then $f_A(x) = 1$. Thus $f_A(x) \geq f_A(y)$. If $y \notin A$, then $f_A(y) = 0$. Since $x \in S$, we have $f_A(x) \geq 0$. Thus $f_A(x) \geq 0 = f_A(y)$, and by Lemma 2.6, f_A is a strongly convex fuzzy subset of S. Let now $x, y \in S$. Then we can show that $\bigwedge_{z \in x \circ y} f_A(z) \geq f_A(x) \wedge f_A(y)$. Indeed, if $x \circ y \not\subseteq A$, then there exists $z \in x \circ y$ such that $z \notin A$, and we have $\bigwedge_{z \in x \circ y} f_A(z) = 0$. Besides that, $x \circ y \not\subseteq A$ implies that $x \notin A$ or $y \notin A$. Then $f_A(x) = 0$ or $f_A(y) = 0$, and

$\bigwedge_{z\in x\circ y} f_A(z) = f_A(x) \wedge f_A(y)$. Let $x \circ y \subseteq A$. Then $f_A(z) = 1$ for any $z \in x \circ y$. It implies that $\bigwedge_{z\in x\circ y} f_A(z) = 1$. Since f_A is a fuzzy subset of S, we have $f_A(x) \leq 1$ for any $x \in A$. Thus, in this case, $\bigwedge_{z\in x\circ y} f_A(z) \geq f_A(x) \wedge f_A(y)$. Similar to the above proof, we can prove that $\bigwedge_{w\in x\circ y\circ z} f_A(w) \geq f_A(y)$ for all $x, y, z \in S$. Therefore, f_A is a fuzzy interior hyperideal of S by Theorem 3.11.

Conversely, let $\emptyset \neq A \subseteq S$ such that f_A is a fuzzy interior hyperideal of S. We claim that $A \circ A \subseteq A$. To prove our claim, let $x, y \in A$. By hypothesis, $\bigwedge_{z\in x\circ y} f_A(z) \geq f_A(x) \wedge f_A(y) = 1$, which implies that $f_A(z) \geq 1$ for any $z \in x \circ y$. On the other hand, $f_A(z) \leq 1$ for all $z \in S$. Thus for any $z \in x \circ y$, $f_A(z) = 1$, i.e., $z \in A$. It thus follows that $A \circ A \subseteq A$. In a similar way, we can show that $S \circ A \circ S \subseteq A$. Furthermore, let $x \in A, S \ni y \leq x$. Then $y \in A$. Indeed, it is enough to prove that $f_A(y) = 1$. From $x \in A$, we have $f_A(x) = 1$. Since f_A is a fuzzy interior hyperideal of S and $y \leq x$, we have $f_A(y) \geq f_A(x) = 1$. Since $y \in S$, we have $f_A(y) \leq 1$. Thus A is an interior hyperideal of S. ■

Now, we shall characterize fuzzy interior hyperideals of ordered semihypergroups by the level subsets.

Theorem 3.15. *Let S be an ordered semihypergroup and f a fuzzy subset of S. Then f is a fuzzy interior hyperideal of S if and only if the level subset f_t ($t \in (0, 1]$) of f is an interior hyperideal of S for $f_t \neq \emptyset$.*

Proof. Assume that f is a fuzzy interior hyperideal of S. Let $t \in (0, 1]$ be such that $f_t \neq \emptyset$. Let now $x, y \in f_t$. Then $f(x) \geq t, f(y) \geq t$. By hypothesis, we have $\bigwedge_{z\in x\circ y} f(z) \geq f(x) \wedge f(y) \geq t \wedge t = t$. Thus for any $z \in x \circ y$, we have $f(z) \geq t$, i.e., $z \in f_t$. It follows that $x \circ y \subseteq f_t$. Hence f_t is a subsemihypergroup of S. Let $y \in f_t, x, z \in S$. Then $f(y) \geq t$. It follows from Theorem 3.11 that $\bigwedge_{w\in x\circ y\circ z} f(w) \geq f(y) \geq t$. Hence $f(w) \geq t$ for any $w \in x \circ y \circ z$. Thus it can be easily shown that $x \circ y \circ z \subseteq f_t$. Furthermore, let $x \in f_t, S \ni y \leq x$. Then $y \in f_t$. Indeed, since $x \in f_t, f(x) \geq t$ and f is a fuzzy interior hyperideal of S, we have $f(y) \geq f(x) \geq t$, so $y \in f_t$. Consequently, f_t is an interior hyperideal of S for $f_t \neq \emptyset$.

Conversely, suppose that f_t ($\neq \emptyset$) is an interior hyperideal of S. If $\bigwedge_{z\in x\circ y} f(z) < f(x) \wedge f(y)$ for some $x, y \in S$, then there exists $t \in (0, 1]$ such that $\bigwedge_{z\in x\circ y} f(z) < t \leq f(x) \wedge f(y)$, which implies that $x, y \in f_t$ and $x \circ y \not\subseteq f_t$. It contradicts the fact that f_t is an interior hyperideal of S. Consequently, $\bigwedge_{z\in x\circ y} f(z) \geq f(x) \wedge f(y)$ for all $x, y \in S$. In a similar way, we can show that $\bigwedge_{w\in x\circ y\circ z} f(w) \geq f(y)$ for all $x, y, z \in S$. Moreover, let $x, y \in S$ and $x \leq y$. Then $f(x) \geq f(y)$. In fact, let

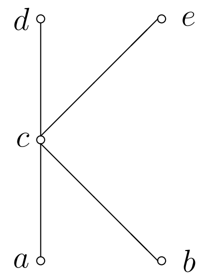
$t = f(y)$. Then $y \in f_t$. Since f_t is an interior hyperideal of S, we have $x \in f_t$. Then $f(x) \geq t = f(y)$. Therefore, f is a fuzzy interior hyperideal of S. ■

Example 3.16. We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d	e
a	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{c\}$	$\{e\}$
d	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{d\}$	$\{e\}$
e	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{c\}$	$\{e\}$

$\leq := \{(a,a),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),(c,c),(c,d),(c,e),(d,d),(e,e)\}$.
We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a,c),(b,c),(c,d),(c,e)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup, and the sets $\{a, b\}$, $\{a, b, c, e\}$ and S are all interior hyperideals of S. Now let f be a fuzzy subset of S such that $f(a) = f(b) = 0.8, f(c) = f(d) = f(e) = 0.6$. Then

$$f_t = \begin{cases} S, & \text{if } t \in (0, 0.6], \\ \{a,b\}, & \text{if } t \in (0.6, 0.8], \\ \emptyset, & \text{if } t \in (0.8, 1]. \end{cases}$$

Thus all nonempty level subsets f_t ($t \in (0, 1]$) of f are interior hyperideals of S and by Theorem 3.15, f is a fuzzy interior hyperideal of S.

Theorem 3.17. *Let S be an ordered semihypergroup. If f is a fuzzy hyperideal of S, then f is a fuzzy interior hyperideal of S.*

Proof. Assume that f is a fuzzy hyperideal of S. Then, by Lemma 2.7(3), we have $f * f \subseteq f * 1 \subseteq f$ and $1 * f * 1 = 1 * (f * 1) \subseteq 1 * f \subseteq f$. Thus, by Theorem 3.13, f is a fuzzy interior hyperideal of S. ■

The converse of Theorem 3.17 is not true in general. We can illustrate it by the following example:

Example 3.18. Suppose that S is the ordered semihypergroup of Example 3.3. Let f be a fuzzy subset of S such that $f(a) = 0.8, f(b) = f(c) = 0.4, f(d) = 0.6$. Then

$$f_t = \begin{cases} S, & \text{if } t \in (0, 0.4], \\ \{a, d\}, & \text{if } t \in (0.4, 0.6], \\ \{a\}, & \text{if } t \in (0.6, 0.8], \\ \emptyset, & \text{if } t \in (0.8, 1]. \end{cases}$$

Similar to Example 3.16, we can verify that f is a fuzzy interior hyperideal of S. But we claim that f is not a fuzzy hyperideal of S. Indeed, since $c \circ d = \{a, b\}$, we have

$$\bigwedge_{z \in c \circ d} f(z) = f(a) \wedge f(b) = 0.4 < 0.6 = f(d),$$

i.e., f is not a fuzzy left hyperideal of S.

The following theorem can be obtained under the assumption of an additional condition.

Theorem 3.19. *Let S be a regular ordered semihypergroup. If f is a fuzzy interior hyperideal of S, then f is a fuzzy hyperideal of S.*

Proof. Assume that f is a fuzzy interior hyperideal of a regular ordered semihypergroup S. Let $x, y \in S$. Then, since S is regular, there exists $x_1 \in S$ such that $x \preceq x \circ x_1 \circ x$. Then, by Lemma 2.2(6), we have $x \circ y \preceq (x \circ x_1 \circ x) \circ y = (x \circ x_1) \circ x \circ y$. Thus, for any $z \in x \circ y$, there exists $w \in (x \circ x_1) \circ x \circ y$ such that $z \leq w$. By $w \in (x \circ x_1) \circ x \circ y$, we have $w \in u \circ x \circ y$ for some $u \in x \circ x_1$. Since f is a fuzzy interior hyperideal of S, by Theorem 3.11 we have

$$f(z) \geq f(w) \geq \bigwedge_{w \in u \circ x \circ y} f(w) \geq f(x).$$

Thus it can be obtained that $\bigwedge_{z \in x \circ y} f(z) \geq f(x)$. Hence f is a fuzzy right hyperideal of S. In the same way we can show that f is also a fuzzy left hyperideal of S. Therefore, f is indeed a fuzzy hyperideal of S. ∎

By Theorems 3.17 and 3.19, we immediately obtain the following corollary:

Corollary 3.20. *In regular ordered semihypergroups the concepts of fuzzy hyperideals and fuzzy interior hyperideals coincide.*

Theorem 3.21. *Let $\{f_i \mid i \in I\}$ be a family of fuzzy interior hyperideals of an ordered semihypergroup S. Then $f := \bigcap_{i \in I} f_i$ is a fuzzy interior hyperideal of S, where $(\bigcap_{i \in I} f_i)(x) = \bigwedge_{i \in I}(f_i(x))$.*

Proof. Let $x, y \in S$. Then, since each f_i $(i \in I)$ is a fuzzy interior hyperideal of S, $\bigwedge_{z \in x \circ y} f_i(z) \geq f_i(x) \wedge f_i(y)$. Thus, for any $z \in x \circ y$, $f_i(z) \geq f_i(x) \wedge f_i(y)$, and we have $f(z) = (\bigcap_{i \in I} f_i)(z) = \bigwedge_{i \in I}(f_i(z)) \geq \bigwedge_{i \in I}(f_i(x) \wedge f_i(y)) = (\bigwedge_{i \in I}(f_i(x))) \wedge (\bigwedge_{i \in I}(f_i(y))) = (\bigcap_{i \in I} f_i)(x) \wedge (\bigcap_{i \in I} f_i)(y) = f(x) \wedge f(y)$, which implies that $\bigwedge_{z \in x \circ y} f(z) \geq f(x) \wedge f(y)$. Similar to the above proof, it can be easily shown that $\bigwedge_{w \in x \circ y \circ z} f(w) \geq f(y)$ for all

$x, y, z \in S$. Furthermore, if $x \leq y$, then $f(x) \geq f(y)$. Indeed, since every f_i $(i \in I)$ is a fuzzy interior hyperideal of S, it can be obtained that $f_i(x) \geq f_i(y)$ for all $i \in I$. Thus $f(x) = (\bigcap_{i\in I} f_i)(x) = \bigwedge_{i\in I}(f_i(x)) \geq \bigwedge_{i\in I}(f_i(y)) = (\bigcap_{i\in I} f_i)(y) = f(y)$. By Theorems 3.9 and 3.11, f is a fuzzy interior hyperideal of S.

Definition 3.22. Let S be an ordered semihypergroup and $f \in F(S)$. The smallest fuzzy interior hyperideal of S containing f is called the *fuzzy interior hyperideal of S generated by f*, denoted by $< f >$.

In order to characterize the fuzzy interior hyperideal generated by a fuzzy subset in an ordered semihypergroup, we need the following lemma.

Lemma 3.23. *Let S be an ordered semihypergroup and $f \in F(S)$. Then $f(x) = \sup\{k \mid x \in f_k\}$ for any $x \in S$, where f_k $(k \in (0, 1])$ is the level subset of f.*

Proof. Let $\alpha = \sup\{k \mid x \in f_k\}$. Then, for any $\varepsilon > 0$, we have $\sup\{k \mid x \in f_k\} > \alpha - \varepsilon$, and thus there exists $t \in \{k \mid x \in f_k\}$ such that $t > \alpha - \varepsilon$. Since $x \in f_t$, we have $f(x) \geq t$. Thus $f(x) > \alpha - \varepsilon$. By the arbitrariness of ε, it follows that $f(x) \geq \alpha$. On the other hand, let $t = f(x)$. Then $x \in f_t$, and $t \in \{k \mid x \in f_k\}$. It thus implies that $f(x) = t \leq \sup\{k \mid x \in f_k\} = \alpha$. Therefore, we obtain the requested result. ∎

Theorem 3.24. *Let S be an ordered semihypergroup and $f \in F(S)$. Then the fuzzy set f^* of S defined by*

$$(\forall x \in S)\ f^*(x) = \sup\{k \mid x \in In(f_k)\}$$

is the fuzzy interior hyperideal $< f >$ generated by f in S, where $In(f_k)$ $(k \in (0, 1])$ is the interior hyperideal of S generated by f_k.

Proof. In order to prove that f^* is the fuzzy interior hyperideal $< f >$ of S generated by f, we now consider the following three steps:

(1) $f \subseteq f^*$. In fact, for any $x \in S$, let $t \in \{k \mid x \in f_k\}$. Then $x \in f_t$, and we have $x \in In(f_t)$. It thus follows that $t \in \{k \mid x \in In(f_k)\}$. Hence we have shown that $\{k \mid x \in f_k\} \subseteq \{k \mid x \in In(f_k)\}$. By Lemma 3.23, we have

$$f(x) = \sup\{k \mid x \in f_k\} \leq \sup\{k \mid x \in In(f_k)\} = f^*(x),$$

from which we can deduce that $f \subseteq f^*$.

(2) f^* is a fuzzy interior hyperideal of S. Indeed, let $t \in (0, 1]$ be such that $f^* \neq \emptyset$. Let $\alpha_n = t - \frac{1}{n}$ for any $n \in Z^+$. We claim that $f^*_t = \bigcap_{n\in Z^+} In(f_{\alpha_n})$. To prove our claim, let $x \in f^*_t$. Then $f^*(x) \geq t$, that is, $\sup\{k \mid x \in In(f_k)\} \geq t > t - \frac{1}{n} = \alpha_n$ for any $n \in Z^+$. It implies that $k_n > \alpha_n$ for some $k_n \in \sup\{k \mid x \in In(f_k)\}$. Thus $f_{k_n} \subseteq f_{\alpha_n}$, and we have $x \in In(f_{k_n}) \subseteq In(f_{\alpha_n})$, which means that $x \in \bigcap_{n\in Z^+} In(f_{\alpha_n})$. Consequently, $f^*_t \subseteq \bigcap_{n\in Z^+} In(f_{\alpha_n})$. To show the inverse inclusion, let $x \in \bigcap_{n\in Z^+} In(f_{\alpha_n})$. Then $\alpha_n \in \{k \mid x \in In(f_k)\}$ for any $n \in Z^+$. Thus we have

$$t - \frac{1}{n} = \alpha_n \leq \sup\{k \mid x \in In(f_k)\} = f^*(x),\ n \in Z^+.$$

Since n is an arbitrary positive integer, we have $t \leq f^*(x)$. Thus $x \in f^*_t$. Therefore, $f^*_t = \bigcap_{n \in Z^+} In(f_{\alpha_n})$, and by Theorem 3.5, f^*_t is an interior hyperideal of S. Thus f^* is a fuzzy interior hyperideal of S by Theorem 3.15.

(3) $f^* = < f >$. In fact, suppose that g is a fuzzy interior hyperideal of S and $f \subseteq g$. Let $x \in S$. If $f^*(x) = 0$, then, obviously, $f^*(x) \leq g(x)$. Let $f^*(x) = t \neq 0$. Then, by (2), we have $x \in f^*_t = \bigcap_{n \in Z^+} In(f_{\alpha_n})$. Since $f \subseteq g$, we have $f_{\alpha_n} \subseteq g_{\alpha_n}$, and $x \in In(f_{\alpha_n}) \subseteq In(g_{\alpha_n}) = (g_{\alpha_n} \cup g^2_{\alpha_n} \cup (S \circ g_{\alpha_n} \circ S)]$, $n \in Z^+$. Then $x \leq w$ for some $w \in g_{\alpha_n} \cup g^2_{\alpha_n} \cup (S \circ g_{\alpha_n} \circ S)$. We consider the following three cases:

Case 1. If $w \in g_{\alpha_n}$, then $g(w) \geq \alpha_n$.

Case 2. If $w \in g^2_{\alpha_n}$, then $w \in w_1 \circ w_2$ for some $w_1, w_2 \in g_{\alpha_n}$. Hence $g(w_1) \geq \alpha_n$, $g(w_2) \geq \alpha_n$. Since g is a fuzzy interior hyperideal of S, and it is a fuzzy subsemihypergroup of S, by Theorem 3.9, $g(w) \geq \bigwedge_{w \in w_1 \circ w_2} g(w) \geq g(w_1) \wedge g(w_2) \geq \alpha_n \wedge \alpha_n = \alpha_n$.

Case 3. Let $w \in S \circ g_{\alpha_n} \circ S$. Then there exist $y, z \in S, u \in g_{\alpha_n}$ such that $w \in y \circ u \circ z$, and we have $g(u) \geq \alpha_n$. Since g is a fuzzy interior hyperideal of S, by Theorem 3.11, $g(w) \geq \bigwedge_{w \in y \circ u \circ z} g(w) \geq g(u) \geq \alpha_n$.

Thus, in any case, we have $g(w) \geq \alpha_n$, $n \in Z^+$. Since g is a fuzzy interior hyperideal of S and $x \leq w$, we have $g(x) \geq g(w) \geq \alpha_n = t - \frac{1}{n}$, $n \in Z^+$. By the arbitrariness of n in Z^+, $g(x) \geq t = f^*(x)$. Hence $f^* \subseteq g$, and it is shown that f^* is the fuzzy interior hyperideal $< f >$ of S generated by f. This completes the proof. ■

4. Normal fuzzy interior hyperideals of ordered semihypergroups

In the current section we define a normal fuzzy interior hyperideal of an ordered semihypergroup, and investigate some related properties.

An element 0 of an ordered semihypergroup S is called *zero element* if $0 \leq x$ and $0 \circ x = x \circ 0 = \{0\}$ for all $x \in S$. In this section, unless stated otherwise S means an ordered semihypergroup containing the zero element 0.

Example 4.1. Suppose that S is the ordered semihypergroup of Example 3.3. Then S is an ordered semihypergroup containing the zero element, and a is the zero element of S.

For a given fuzzy interior hyperideal f of S, we note that $f(0)$ is the largest value of f, that is, $f(0) \geq f(x)$ for all $x \in S$. In general, $f(0) \neq 1$. Particularly, we have the following concept.

Definition 4.2. A fuzzy interior hyperideal f of S is said to be *normal*, if $f(0) = 1$.

Theorem 4.3. *Let f be a fuzzy subset of S. We define the subset S_f of S as follows:*

$$S_f := \{x \in S \mid f(x) = f(0)\}.$$

Then the following statements are true:

(1) *If f is a fuzzy interior hyperideal of S, then S_f is an interior hyperideal of S.*

(2) *If A is an interior hyperideal of S, then the characteristic function f_A of A is a normal fuzzy interior hyperideal of S and $S_{f_A} = A$.*

Proof. (1) Let $x, y \in S_f$. Then $f(x) = f(y) = 0$. Since f is a fuzzy subsemihypergroup of S, by Theorem 3.9, $\bigwedge_{z \in x \circ y} f(z) \geq f(x) \wedge f(y) = f(0)$, and for any $z \in x \circ y$, we have $f(z) \geq f(0)$. Noticing that $f(z) \leq f(0)$ for all $z \in S$, we have $f(z) = f(0)$, that is, $z \in S_f$. It thus follows that $x \circ y \subseteq S_f$. Let now $x, z \in S, y \in S_f$. Then, similar to the above proof, we can deduce that $x \circ y \circ z \subseteq S_f$. Furthermore, let $y \in S_f, S \ni x \leq y$. Then $x \in S_f$. Indeed, it is enough to prove that $f(x) \geq f(0)$. By $y \in S_f$, we have $f(y) = 0$. Since f is a fuzzy interior hyperideal of S, we have $f(x) \geq f(y) = 0$. Hence S_f is an interior hyperideal of S.

(2) Assume that A is an interior hyperideal of S. By Theorem 3.14, f_A is a fuzzy interior hyperideal of S. To prove that f_A is normal, let $x \in A$. Then, since A is an interior hyperideal of S, $0 \in 0 \circ x \circ 0 \subseteq S \circ A \circ S \subseteq A$, and we have $f_A(0) = 1$. Thus f_A is a normal fuzzy interior hyperideal of S. Moreover, it can be easily shown that $S_{f_A} = A$. ∎

Theorem 4.4. *Let f, g be fuzzy interior hyperideals of S. Then the following statements are true:*

(1) $S_f \cap S_g \subseteq S_{f \cap g}$.

(2) *If $f \subseteq g$ and $f(0) = g(0)$, then $S_f \subseteq S_g$.*

Proof. The proof is obvious, and hence we omit the details. ∎

By Theorem 4.4, we immediately obtain the following corollary:

Corollary 4.5. *Let f, g be normal fuzzy interior hyperideals of S. Then the following statements are true:*

(1) $S_f \cap S_g = S_{f \cap g}$.

(2) *If $f \subseteq g$, then $S_f \subseteq S_g$.*

The hypothesis that f and g are normal cannot be removed in the above corollary. Otherwise, Corollary 4.5 does not hold in general. We can illustrate it by the following example.

Example 4.6. Let S be an ordered semihypergroup containing the zero element 0 and $|S| \geq 2$. Define two fuzzy subsets f, g of S as follows:

$$f(x) := 0 \quad \text{and} \quad g(x) := \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0, \end{cases}$$

for any $x \in S$. We can easily verify that f and g are two fuzzy interior hyperideals of S, and $S_f = S, S_g = \{0\}$. Clearly, $f \subseteq g$. But $S_f \cap S_g = \{0\} \subset S = S_f = S_{f \cap g}$ and $S_f \supset S_g$.

Theorem 4.7. *Let f be a fuzzy interior hyperideal of S and let f^+ be a fuzzy subset of S defined by $(\forall x \in S)\ f^+(x) = f(x) + 1 - f(0)$. Then f^+ is a normal fuzzy interior hyperideal of S containing f.*

Proof. Suppose that f is a fuzzy interior hyperideal of S. Let $x, y \in S$. Then $\bigwedge_{z\in x\circ y} f^+(z) = \bigwedge_{z\in x\circ y} (f(z) + 1 - f(0)) = (\bigwedge_{z\in x\circ y} f(z)) + 1 - f(0) \geq (f(x) \wedge f(y)) + 1 - f(0)) = (f(x) + 1 - f(0)) \wedge (f(y) + 1 - f(0)) = f^+(x) \wedge f^+(y)$. In a similar way we can show that $\bigwedge_{w\in x\circ y\circ z} f^+(w) \geq f^+(y)$ for all $x, y, z \in S$. Furthermore, let $x, y \in S$ be such that $x \leq y$. Then $f^+(x) \geq f^+(y)$. In fact, since f is a fuzzy interior hyperideal of S, we have $f(x) \geq f(y)$. Thus

$$f^+(x) = f(x) + 1 - f(0) \geq f(y) + 1 - f(0) = f^+(y).$$

Consequently, f^+ is a fuzzy interior hyperideal of S. Clearly, $f^+(0) = 1$ and $f \subseteq f^+$. The proof is completed.

Corollary 4.8. *Let f be a fuzzy interior hyperideal of S satisfying $f^+(x) = 0$ for some $x \in S$. Then $f(x) = 0$ and f is normal.*

Theorem 4.9. *Let f be a fuzzy interior hyperideal of S. Then f is normal if and only if $f^+ = f$.*

Proof. The sufficiency is obvious. Assume that f is a normal fuzzy interior hyperideal of S. Let $x \in S$. Then $f^+(x) = f(x) + 1 - f(0) = f(x)$, and thus $f^+ = f$. ∎

Using Theorems 4.7 and 4.9, we obtain the following corollaries.

Corollary 4.10. *If f is a fuzzy interior hyperideal of S, then $(f^+)^+ = f^+$.*

Corollary 4.11. *If f is a normal fuzzy interior hyperideal of S, then $(f^+)^+ = f$.*

Theorem 4.12. *Let f be a fuzzy interior hyperideal of S. If there exists a fuzzy interior hyperideal g of S satisfying $g^+ \subseteq f$, then f is normal.*

Proof. Assume there exists a fuzzy interior hyperideal g of S such that $g^+ \subseteq f$. Then, by Theorem 4.7, $1 = g^+(0) \leq f(0)$, whence $f(0) = 1$. This completes the proof. ∎

Corollary 4.13. *Let f be a fuzzy interior hyperideal of S. If f is not normal, then f doesn't contain any normal fuzzy interior hyperideal of S.*

Proof. It is a direct consequence of Theorem 4.12. ∎

Theorem 4.14. *Let f be a fuzzy interior hyperideal of S and let $\varphi : [0, f(0)] \to [0, 1]$ be an increasing function. Define a fuzzy subset f_φ of S as follows:*

$$(\forall x \in S)\ f_\varphi(x) = \varphi(f(x)).$$

Then f_φ is a fuzzy interior hyperideal of S. In particular, if $\varphi(f(0)) = 1$, then f_φ is normal, and if $\varphi(t) \geq t$ for all $t \in [0, f(0)]$, then $f \subseteq f_\varphi$.

Proof. Suppose that f is a fuzzy interior hyperideal of S. Let $x, y \in S$. Then, since φ is an increasing function, we have

$$\begin{aligned}\bigwedge_{z\in x\circ y} f_\varphi(z) &= \bigwedge_{z\in x\circ y} (\varphi(f(z))) = \varphi(\bigwedge_{z\in x\circ y} f(z)) \geq \varphi(f(x) \wedge f(y)) \\ &= \varphi(f(x)) \wedge \varphi(f(y)) = f_\varphi(x) \wedge f_\varphi(y).\end{aligned}$$

Similar to the above proof, we can prove that $\bigwedge_{w \in x \circ y \circ z} f_\varphi(w) \geq f_\varphi(y)$ for all $x, y, z \in S$. Let now $x, y \in S$ be such that $x \leq y$. Then, by hypothesis, we have

$$f_\varphi(x) = \varphi(f(x)) \geq \varphi(f(y)) = f_\varphi(y).$$

Hence f_φ is a fuzzy interior hyperideal of S. If $\varphi(f(0)) = 1$, then, clearly, f_φ is normal. Assume that $\varphi(t) \geq t$ for all $t \in [0, f(0)]$. Then $f_\varphi(x) = \varphi(f(x)) \geq f(x)$ for any $x \in S$, which implies that $f \subseteq f_\varphi$. ■

Theorem 4.15. *Let f be a nonconstant normal fuzzy interior hyperideal of S, which is maximal in the poset of normal fuzzy interior hyperideals under set inclusion. Then f takes only the values 0 and 1, that is, $Im(f) = \{0, 1\}$.*

Proof. Note that $f(0) = 1$. Let $a \in S$ be such that $f(a) \neq 1$. It is sufficient to show that $f(a) = 0$. Assume that $0 < f(a) < 1$. Now define a fuzzy subset g of S as follows:

$$(\forall x \in S)\ g(x) = \frac{1}{2}(f(x) + f(a)).$$

We claim that g is a fuzzy interior hyperideal of S. To prove our claim, let $x, y \in s$. Then, since f is a fuzzy interior hyperideal of S, we have

$$\begin{aligned}
\bigwedge_{z \in x \circ y} g(z) &= \bigwedge_{z \in x \circ y} (\frac{1}{2}(f(z) + f(a))) = \frac{1}{2}(\bigwedge_{z \in x \circ y} f(z) + f(a)) \\
&\geq \frac{1}{2}(f(x) \wedge f(y) + f(a)) = [\frac{1}{2}(f(x) + f(a))] \wedge [\frac{1}{2}(f(y) + f(a))] \\
&= g(x) \wedge g(y).
\end{aligned}$$

In the same way we can show that $\bigwedge_{w \in x \circ y \circ z} g(z) \geq g(y)$ for all $x, y, z \in S$. Moreover, let $x, y \in S$ be such that $x \leq y$. Since f is a fuzzy interior hyperideal of S, by Theorem 3.11, $f(x) \geq f(y)$, and we have

$$g(x) = \frac{1}{2}(f(x) + f(a)) \geq \frac{1}{2}(f(y) + f(a)) = g(y).$$

Thus, g is indeed a fuzzy interior hyperideal of S. By Theorem 4.7, g^+ is a normal fuzzy interior hyperideal of S. In addition, for any $x \in S$, we have

$$g^+(x) = g(x) + 1 - g(0) = \frac{1}{2}(f(x) + f(a)) + 1 - \frac{1}{2}(f(0) + f(a)) = \frac{1}{2}(f(x) + 1),$$

and thus we have

$$g^+(0) = 1 > g^+(a) = \frac{1}{2}(f(a) + 1) > f(a),$$

which means that g^+ is nonconstant. From $g^+(a) > f(a)$ it follows that f is not maximal. This completes the proof.

5. Characterizations of semisimple ordered semihypergroups

In this section, we give some new characterizations of semisimple ordered semihypergroups by fuzzy hyperideals and fuzzy interior hyperideals.

Lemma 5.1 ([35]). *Let S be an ordered semihypergroup. Then S is semisimple if and only if $a \in (S \circ a \circ S \circ a \circ S]$ for all $a \in S$.*

Lemma 5.2. *Let S be a semisimple ordered semihypergroup. If f is a fuzzy interior hyperideal of S, then f is a fuzzy hyperideal of S.*

Proof. The proof is similar to that of Theorem 3.19 with suitable modification by using Lemma 5.1. ∎

In order to characterize the semisimple ordered semihypergroups, we need the following lemma.

Lemma 5.3. *Let A, B be any two nonempty subsets of an ordered semihypergroup S. Then $f_A * f_B = f_{(A\circ B]}$, where f_A, f_B and $f_{(A\circ B]}$ are the characteristic function of A, B and $(A \circ B]$, respectively.*

Proof. Let $x \in S$. If $x \in (A \circ B]$, then $f_{(A\circ B]}(x) = 1$, and $x \preceq a \circ b$ for some $a \in A$ and $b \in B$. Thus $(a, b) \in H_x$, and

$$(f_A * f_B)(x) = \bigvee_{(y,z)\in H_x} [f_A(y) \wedge f_B(z)] \geq f_A(a) \wedge f_B(b) = 1 \wedge 1 = 1.$$

On the other hand, since $f_A(y) \leq 1$ and $f_B(z) \leq 1$ for all $y, z \in S$, we have $(f_A * f_B)(x) \leq 1$. Therefore, $(f_A * f_B)(x) = 1 = f_{(A\circ B]}(x)$. If $x \notin (A \circ B]$, then $f_{(A\circ B]}(x) = 0$. We now prove that $(f_A * f_B)(x) = 0$. Indeed, if $H_x = \emptyset$, then $(f_A * f_B)(x) = 0$, and $(f_A * f_B)(x) = f_{(A\circ B]}(x)$. If $H_x \neq \emptyset$, then $x \leq y \circ z$ for all $(y, z) \in H_x$. If $y \in A$ and $z \in B$, then $y \circ z \subseteq A \circ B$, so $x \in (A \circ B]$, which is impossible. Thus $y \notin A$ or $z \notin B$. If $y \notin A$, then $f_A(y) = 0$. Since $f_B(z) \geq 0$, we have $f_A(y) \wedge f_B(z) = 0$. If $z \notin B$, then, as in the previous case, we have $f_A(y) \wedge f_B(z) = 0$. Thus we have

$$(f_A * f_B)(x) = \bigvee_{(y,z)\in H_x} [f_A(y) \wedge f_B(z)] = 0.$$

This completes the proof. ∎

Now, we give characterizations of an ordered semihypergroup which is semisimple in terms of fuzzy hyperideals and fuzzy interior hyperideals.

Theorem 5.4. *Let S be an ordered semihypergroup. Then the following statements are equivalent:*

(1) *S is semisimple.*

(2) *$f * g = f \cap g$ for all fuzzy interior hyperideals f and g of S.*

(3) *$f * g = f \cap g$ for every fuzzy interior hyperideal f and every fuzzy hyperideal g of S.*

(4) *$f * g = f \cap g$ for every fuzzy hyperideal f and every fuzzy interior hyperideal g of S.*

(5) *$f * g = f \cap g$ for all fuzzy hyperideals f and g of S.*

(6) *$f * f = f$ for every fuzzy hyperideal f of S.*

(7) *$f * f = f$ for every fuzzy interior hyperideal f of S.*

Proof. (1) $\Rightarrow$ (2). Let f and g be fuzzy interior hyperideals of S and $a \in S$. Then, since S is semisimple, by Lemma 5.1 there exist $x, y, z \in S$ such that $a \preceq x \circ a \circ y \circ a \circ z$. Thus, by Lemma 2.2, we have

$$\begin{aligned} a &\preceq x \circ a \circ y \circ a \circ z \preceq x \circ (x \circ a \circ y \circ a \circ z) \circ y \circ a \circ z \\ &= (x^2 \circ a \circ y) \circ ((a \circ z \circ y) \circ a \circ z). \end{aligned}$$

Then there exist $w_1 \in x^2 \circ a \circ y, w_2 \in (a \circ z \circ y) \circ a \circ z$ such that $a \preceq w_1 \circ w_2$, i.e., $(w_1, w_2) \in H_a$. By $w_1 \in x^2 \circ a \circ y$, there exists $u \in x^2$ such that $w_1 \in u \circ a \circ y$. Since f is a fuzzy interior hyperideal of S, by Theorem 3.11 we have

$$f(w_1) \geq \bigwedge_{w \in u \circ a \circ y} f(w) \geq f(a).$$

By $w_2 \in (a \circ z \circ y) \circ a \circ z$, in a similar way we can get $g(w_2) \geq g(a)$. Thus

$$(f * g)(a) = \bigvee_{(s,t) \in H_a} [f(s) \wedge g(t)] \geq f(w_1) \wedge g(w_2) \geq f(a) \wedge g(a) = (f \cap g)(a),$$

which implies that $f \cap g \subseteq f * g$. On the other hand, since S is a semisimple ordered semihypergroup, by Lemmas 2.7(3) and 5.2 we have

$$f * g \subseteq f * 1 \subseteq f \quad \text{and} \quad f * g \subseteq 1 * g \subseteq g,$$

and hence $f * g \subseteq f \cap g$. Thus $f * g = f \cap g$.

Using Theorem 3.17, every fuzzy hyperideal of S is a fuzzy interior hyperideal of S, and so (2) $\Rightarrow$ (3) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) $\Rightarrow$ (5) and (7) $\Rightarrow$ (6) are clear.

(5) $\Rightarrow$ (6). Take $f = g$ in (5), we get $f * f = f \cap f = f$. This proves the (6) holds.

(2) $\Rightarrow$ (7). The proof is similar to that of (5) $\Rightarrow$ (6) with slight modification.

(6) $\Rightarrow$ (1). Let I be a hyperideal of S. By Lemma 2.8, the characteristic function f_I of I is a fuzzy hyperideal of S. Then, by hypothesis and Lemma 5.3, we have

$$f_{(I^2]} = f_I * f_I = f_I,$$

from which we deduce that $(I^2] = I$. Therefore, S is semisimple. ∎

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