QUOTIENT HYPER HOOP-ALGEBRAS

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Abstract. In this paper, by considering the notion of hyper hoop-algebras, we define the concepts of (strong) regular relations on hyper hoop-algebras and investigate some properties of them. Then we construct a quotient (hyper) hoop-algebra by a (strong)regular relation on hyper hoop-algebras. Finally, we define the notion of maximal filter and we investigate the relation between quotient simple hyper hoop-algebras and maximal filters.

Keywords: Hyper hoop-algebra, (strong) regular relation, quotient hyper hoop-algebra, maximal filter.

Mathematical Subject Classification (2010): 20N20, 03G25,06F35.

1. Introduction

Hoop-algebras are naturally ordered commutative residuated integral monoids, introduced by B. Bosbach in [5] and then were investigated by Büchi and Owens in an unpublished manuscript [8] of 1975, and they have been studied by Aglianò, Blok and Ferreirim [1], [6], [7]. The study of hoops is motivated by their occurrence both in universal algebra and algebraic logic. In recent years, hoop theory was enriched with deep structure theorems. Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops one obtains an elegant short proof of the completeness theorem for propositional basic logic introduced by Hájek [13].

Hyperstructure theory was introduced in 1934 [14], when Marty at the 8th congress of scandinavian mathematicians, gave the definition of hypergroup and illustrated some applications and showed its utility in the study of groups, algebraic functions and rational fraction. Till now, the hyperstructures have been studied from the theoretical point of view for their applications to many subject of pure and applied mathematics. Some fields of applications of the mentioned structures are lattices, graphs, coding, ordered sets, median algebra, automata, and cryptography [9]. Many researchers have worked on this area. The authors applied hyper structure theory on hyper hoop and introduced hyper hoop algebra in [4]. In this paper, we continue the study of hyper hoop-algebras and introduce (strong) regular relations on hyper hoop-algebras and study the quotient (hyper) hoop-algebras that are the results of (strong) regular relations. Then we define the notion of maximal filter and we investigate the relation between quotient simple hyper hoop-algebras and maximal filters.

2. Preliminaries

Let H be a non-empty set. A hypergroupoid is a pair $(H, \odot)$, where $\odot : H \times H \longrightarrow P(H) - \{\emptyset\}$ is a binary hyperoperation on H. If $a \odot (b \odot c) = (a \odot b) \odot c$ holds, for all $a, b, c \in H$ then $(H, \odot)$ is called a semihypergroup, and it is said to be commutative if $\odot$ is commutative. An element $1 \in H$ is called a unit, if $a \in 1 \odot a \cap a \odot 1$, for all $a \in H$ and is called a scaler unit, if $1 \odot a = a \odot 1 = \{a\}$, for all $a \in A$. If $a \odot H = H = H \odot a$, for any element $a \in H$, then the pair $(H, \odot)$ is called a hypergroup. Note that if $A, B \subseteq H$, then we consider $A \odot B$ by $A \odot B = \bigcup\limits_{a \in A, b \in B} (a \odot b)$.

Definition 2.1. [7] A *hoop-algebra* or a *hoop* is an algebra $(A, \odot, \rightarrow, 1)$ of type $(2, 2, 0)$ such that $(A, \odot, 1)$ is a commutative monoid and for all $x, y, z \in A$, $x \rightarrow x = 1$, $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ and $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y$. On hoop A we define "$x \leq y$" if and only if $x \rightarrow y = 1$. It is easy to see that $\leq$ is a partial order relation on A.

Definition 2.2. [4] A *quasi hyper hoop algebra* or briefly, a *quasi hyper hoop* is a non-empty set A endowed with two binary hyperoperations $\odot, \rightarrow$ and a constant 1 such that, for all $x, y, z \in A$ satisfying the following conditions:
(HHA1) $(A, \odot, 1)$ is a commutative semihypergroup with 1 as the unit,
(HHA2) $1 \in x \rightarrow x$,
(HHA3) $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y$,
(HHA4) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$,

A quasi hyper hoop $(A, \odot, \to, 1)$ is called a *hyper hoop* if the following hold;
(HHA5) $1 \in x \to 1$,
(HHA6) if $1 \in x \to y$ and $1 \in y \to x$ then $x = y$,
(HHA7) if $1 \in x \to y$ and $1 \in y \to z$ then $1 \in x \to z$.

In the sequel we will refer to the (quasi) hyper hoop $(A, \odot, \to, 1)$ by its universe A. On (quasi) hyper hoop A, we define $x \leq y$ if and only if $1 \in x \to y$. If A is a hyper hoop, it is easy to see that $\leq$ is a partial order relation on A. Moreover, for all $B, C \subseteq A$ we define $B \ll C$ iff there exist $b \in B$ and $c \in C$ such that $b \leq c$ and define $B \leq C$ if and only if for any $b \in B$ there exists $c \in C$ such that $b \leq c$. A (quasi) hyper hoop A is bounded if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$.(See [4])

Proposition 2.3. *In (quasi) hyper hoop $(A, \odot, \to, 1)$, if for any $x, y \in A$, $x \odot y$ and $x \to y$ are singletons, then $(A, \odot, \to, 1)$ is a hoop. Hence (quasi) hyper hoops are a generalization of hoops.*

Proposition 2.4. [4] *Let A be a hyper hoop. Then for all $x, y, z \in A$ and $B, C, D \subseteq A$, the following hold;*
(HHA8) $x \odot y \ll z \Leftrightarrow x \leq y \to z$,
(HHA9) $B \odot C \ll D \Leftrightarrow B \ll C \to D$,
(HHA10) $z \to y \leq (y \to x) \to (z \to x)$,
(HHA11) $z \to y \ll (x \to z) \to (x \to y)$,
(HHA12) $1 \odot 1 = \{1\}$.

Definition 2.5. [4] Let A be a hyper hoop and F be a non-empty subset of A. Then,
(i) F is called an *upset* of A, if $x \in F$ and $x \leq y$ imply $y \in F$, for all $x, y \in A$,
(ii) F is called a *weak filter* of A, if F is an upset and for all $x, y \in F$, $x \odot y \cap F \neq \emptyset$,
(iii) F is called a *filter* of A, if F is an upset and for all $x, y \in F$, $x \odot y \subseteq F$.

Let F be a (weak) filter of A and $x \in F$. Since F is an upset and $x \leq 1$, we get $1 \in F$.

Notation. From now on, in this paper we let A be a hyper hoop, unless otherwise is stated.

3. (Strong) Regular relations on hyper hoops

In this section, we introduce the concepts of regular, strong regular, and fundamental relations on hyper hoops and investigate some properties of them.

Notations. Let $\mathbf{R}$ be an equivalence relation on A and $B, C \subseteq A$. Then $B\mathbf{R}C$, $B\overline{\mathbf{R}}C$ and $B\overline{\overline{\mathbf{R}}}C$ denoted as follows:

(i) $B\mathbf{R}C$ if there exist $b \in B$ and $c \in C$ such that $b\mathbf{R}c$,
(ii) $B\overline{\mathbf{R}}C$ if for all $b \in B$ there exists $c \in C$ such that $b\mathbf{R}c$ and for all $c \in C$ there exists $b \in B$ such that $b\mathbf{R}c$,
(iii) $B\overline{\overline{\mathbf{R}}}C$ if for all $b \in B$ and $c \in C$, we have $b\mathbf{R}c$.

Remark 3.1. It is clear that $B\overline{\mathbf{R}}C$ and $C\overline{\mathbf{R}}D$ imply that $B\overline{\mathbf{R}}D$, for all $B, C, D \subseteq A$.

Definition 3.2. Let $\mathbf{R}$ be an equivalence relation on A. Then $\mathbf{R}$ is called a *regular relation* on A if and only if for all $x, y, z \in A$,

(i) if $x\mathbf{R}y$, then $x \odot z\overline{\mathbf{R}}y \odot z$,

(ii) if $x\mathbf{R}y$, then $x \to z\overline{\mathbf{R}}y \to z$ and $z \to x\overline{\mathbf{R}}z \to y$,

(iii) if $x \to y\mathbf{R}\{1\}$ and $y \to x\mathbf{R}\{1\}$, then $x\mathbf{R}y$.

Example 3.3. Let $A = \{1, a, b, c\}$. Define the hyperoperations $\odot$ and $\to$ on A as follows:

$\odot$	1	a	b	c
1	$\{1\}$	$\{1, a\}$	$\{1, b\}$	$\{1, a, b, c\}$
a	$\{1, a\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$
b	$\{1, b\}$	$\{1, a, b, c\}$	$\{1, b\}$	$\{1, a, b, c\}$
c	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$

$\to$	1	a	b	c
1	$\{1, a, b, c\}$	$\{a, c\}$	$\{b, c\}$	$\{c\}$
a	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{b, c\}$
b	$\{1, a, b, c\}$	$\{a, c\}$	$\{1, a, b, c\}$	$\{a, c\}$
c	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$

Then $(A, \odot, \to, 1)$ is a hyper hoop. Let

$$\mathbf{R} = \{(1, 1), (a, a), (b, b), (c, c), (1, b), (b, 1), (a, c), (c, a)\}$$

Then $\mathbf{R}$ is a regular relation on A.

In the following example, we will show that the condition (iii) in the Definition 3.2, is independent from the other conditions.

Example 3.4. Let $A = \{1, a, b, c\}$. Define hyperoperations $\odot$ and $\to$ on A as follows:

$\odot$	1	a	b	c
1	$\{1\}$	$\{1, a, c\}$	$\{1, b\}$	$\{1, c\}$
a	$\{1, a, c\}$	$\{1, a, c\}$	$\{1, a, b, c\}$	$\{1, a, c\}$
b	$\{1, b\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, b, c\}$
c	$\{1, c\}$	$\{1, a, c\}$	$\{1, b, c\}$	$\{1, a, c\}$

$\to$	1	a	b	c
1	$\{1, a, c\}$	$\{a\}$	$\{a, b\}$	$\{a, c\}$
a	$\{1, a, c\}$	$\{1, a, c\}$	$\{1, a, b, c\}$	$\{1, a, c\}$
b	$\{1, a, b, c\}$	$\{a, b\}$	$\{1, a, b, c\}$	$\{a, b, c\}$
c	$\{1, a, c\}$	$\{a, c\}$	$\{a, b, c\}$	$\{1, a, c\}$

Then $(A, \odot, \to, 1)$ is a bounded hyper hoop.

Let

$$\mathbf{R} = \{(a,a),(b,b),(c,c),(1,1),(1,c),(c,1),(a,c),(c,a),(a,1),(1,a)\}$$

Then $\mathbf{R}$ is an equivalence relation on A and conditions (i) and (ii) hold. We have $b \to c\mathbf{R}\{1\}$ and $c \to b\mathbf{R}\{1\}$, but b, c are not related.

In the follows, by definition of homomorphism on hyper hoops, we get a class of examples of regular relations on hyper hoops.

Definition 3.5. Let A_1 and A_2 be two hyper hoops. A mapping $f : A_1 \to A_2$ is said to be a *homomorphism* if $f(1) = 1$, $f(x \to y) = f(x) \to f(y)$, and $f(x \odot y) = f(x) \odot f(y)$, for any $x, y \in A$. If f is both one to one and onto, we say that f is an isomorphism.

Example 3.6. Let $f : A_1 \to A_2$ be a homomorphism of hyper hoops. We define relation ρ on A_1, for all $a, b \in A_1$, as following:

$$a\rho b \Leftrightarrow f(a) = f(b)$$

We claim that ρ is a regular relation on A_1. It is clear that ρ is an equivalence relation on A_1. So, we show that ρ is regular. Let $x\rho y$, for $x, y \in A_1$. Then $f(x) = f(y)$ and so $f(x) \to f(z) = f(y) \to f(z)$, for all $z \in A_1$. Then $f(x \to z)$ $= f(y \to z)$. Hence for all $u \in x \to z$ there exists $v \in y \to z$ such that $u\mathbf{R}v$ and for all $v \in y \to z$ there exists $u \in x \to z$ such that $v\rho u$ and so $x \to z\overline{\rho}y \to z$. By the same way, we can show that $x\rho y$ implies $x \odot z\overline{\rho}y \odot z$, for all $z \in A_1$. Now, if $x \to y\rho\{1\}$ and $y \to x\rho\{1\}$, then there exist $u \in (x \to y)$ and $v \in (y \to x)$ such that $u\rho 1$ and $v\rho 1$ and so $f(u) = f(v) = f(1) = 1$. Then $1 \in f(x \to y) = f(x) \to f(y)$ and $1 \in f(y \to x) = f(y) \to f(x)$. Since A_2 is a hyper hoop, we get $f(x) = f(y)$ and so $x\rho y$. Hence ρ is a regular relation on A_1.

Lemma 3.7. *Let F be a non-empty subset of A. Then F is a filter of A if and only if $1 \in F$ and $F \ll x \to y$ and $x \in F$ implies $y \in F$, for any $x, y \in A$.*

Proof. ($\Rightarrow$) Let F be a filter, $F \ll x \to y$ and $x \in F$, for $x, y \in A$. Hence there exist $u \in F$ and $v \in x \to y$ such that $u \leq v$. Since $u \in F$ and F is an upset, we get $v \in F$ and since F is a filter, we get $x \odot v \subseteq F$. By $v \in x \to y$ we have $v \leq x \to y$. Then by (HHA10), $v \odot x \ll y$ and so there exists $t \in v \odot x \subseteq F$ such that $t \leq y$. Since F is an upset, we get $y \in F$.
($\Leftarrow$) Let $x \leq y$ and $x \in F$, for $x, y \in A$. Then $1 \in x \to y$ and since $1 \in F$, we get $F \ll x \to y$. Then, by hypothesis $y \in F$ and so F is an upset. Now, let $x, y \in F$ and $u \in x \odot y$. Then $x \odot y \ll u$ and so by (HHA10), $y \leq x \to u$. Since $y \in F$, we get $F \ll x \to u$ and so by hypothesis, $u \in F$. Hence $x \odot y \subseteq F$ and so F is a filter of A. ∎

Proposition 3.8. *If $\boldsymbol{R}$ is a regular relation on A, then the class $[1]$ is a filter of A.*

Proof. Let $F = [1]$. It is clear that $1 \in F$. Let $F \ll x \to y$ and $x \in F$, for $x, y \in A$. Then there exist $u \in F$ and $v \in x \to y$ such that $u \leq v$. Then $1 \in u \to v$ and so $u \to v\mathbf{R}\{1\}$. Since $u \in F$, we have $u\mathbf{R}1$ and by Definition 3.2, $v \to u\overline{\mathbf{R}}v \to 1$. By (HHA5), $1 \in v \to 1$, and so $v \to 1\mathbf{R}\{1\}$. Then by Remark 3.1, $v \to u\mathbf{R}\{1\}$ and then by Definition 3.2(iii), $u\mathbf{R}v$. Since $u\mathbf{R}1$ and $u\mathbf{R}v$, we get $v\mathbf{R}1$ and since $v \in x \to y$, we get $x \to y\mathbf{R}\{1\}$. By $x \in F$, we have $x\mathbf{R}1$ and so by Definition 3.2(ii), $y \to x\overline{\mathbf{R}}y \to 1$. By (HHA5), $1 \in y \to 1$ and so $y \to x\mathbf{R}\{1\}$. By Definition 3.2(iii), since $x \to y\mathbf{R}\{1\}$ and $y \to x\mathbf{R}\{1\}$, we get $x\mathbf{R}y$. Hence $y\mathbf{R}1$ and so $y \in F$. Therefore by Lemma 3.7, F is a filter of A. ∎

Definition 3.9. Let $\mathbf{R}$ be an equivalence relation on A. Then $\mathbf{R}$ is called a *strong regular relation* on A if and only if, for all $x, y, z \in A$,

(i) if $x\mathbf{R}y$, then $x \odot z\overline{\overline{\mathbf{R}}}y \odot z$,

(ii) if $x\mathbf{R}y$, then $x \to z\overline{\overline{\mathbf{R}}}y \to z$ and $z \to x\overline{\overline{\mathbf{R}}}z \to y$.

Example 3.10. Let $A = \{1, a, b, c\}$. Define the hyperoperations $\odot, \to$ on A as follows:

$\odot$	1	a	b	c
1	$\{1\}$	$\{a,b\}$	$\{b\}$	$\{1,c\}$
a	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	$\{b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{1,c\}$	$\{a,b\}$	$\{a,b\}$	$\{1,c\}$

$\to$	1	a	b	c
1	$\{1,c\}$	$\{a\}$	$\{a,b\}$	$\{c\}$
a	$\{1,c\}$	$\{1,c\}$	$\{1,c\}$	$\{1,c\}$
b	$\{1,c\}$	$\{c\}$	$\{1,c\}$	$\{1,c\}$
c	$\{1,c\}$	$\{a,b\}$	$\{a,b\}$	$\{1,c\}$

Then $(A, \odot, \to, 1)$ is a hyper hoop. Let

$$\mathbf{R} = \{(a,a), (b,b), (c,c), (1,1), (1,c), (c,1), (a,b), (b,a)\}.$$

Then $\mathbf{R}$ is a strong regular relation on A.

Proposition 3.11. *If $\mathbf{R}$ is a strong regular relation on A, then $\mathbf{R}$ is a regular relation on A.*

Proof. Let $\mathbf{R}$ be a strong regular relation on A. Clearly the properties (i) and (ii), of Definition 3.2 are satisfied. If $x \to y\mathbf{R}\{1\}$ and $y \to x\mathbf{R}\{1\}$ then by the condition (ii) of Definition 3.9 we have, $x \odot (x \to y)\overline{\overline{\mathbf{R}}}x \odot 1$ and $y \odot (y \to x)\overline{\overline{\mathbf{R}}}y \odot 1$. By (HHA1)and (HHA3), we get $x\mathbf{R}y$ and so the condition (iii) of Definition 3.2 holds. Hence $\mathbf{R}$ is a regular relation on A. ∎

Example 3.12. In Example 3.3, $\mathbf{R}$ is a regular relation but is not a strong regular relation. Hence every regular relation may not be a strong regular relation.

In the following, by definition of fundamental relation on hyper hoops, we get an example of strong regular relations.

Let $(H, \{*_i\}_{i \in I})$ be an algebraic hyperstructure, where $\{*_i\}_{i \in I}$ are hyperoperations. Consider U be the set of all finite combinations of elements of A with respect to $*_i$, where $*_i \in \{*_i\}_{i \in I}$. We define the relation β on A by $a\beta b$ if and only if there exists $u \in U$ such that $\{a, b\} \subseteq u$. Let β^* be the transitive closure of β, that is $a\beta^* b$ if and only if there exist $z_1, ..., z_{m+1} \in A$ and $u_i \in U$ such that $z_1 = a$, $z_{m+1} = b$ and $\{z_i, z_{i+1}\} \subseteq u_i$, for $i = 1, ..., m$. Then the relation β^* is an equivalence relation on A and is called the fundamental relation on A (See [10]).

Now, in the following, the well-known idea of β^* relation on hyperstructures that is mentioned above is applied to hyper hoops.

Let $U(A)$ denote the set of all finite combinations of elements A with $\odot$ and $\to$. Then, for all $a, b \in A$, we have $a\beta b$ if and only if $\{a, b\} \subseteq u$, where $u \in U(A)$, and $a\beta^* b$ if and only if there exist $z_1, ..., z_{m+1} \in A$ with $z_1 = a, z_{m+1} = b$ such that $\{z_i, z_{i+1}\} \subseteq u_i \subseteq U(A)$, for $i = 1, ..., m$. (In fact β^* is the transitive closure of the relation β and so is an equivalence relation on A).

Example 3.13. β^* is a strong regular relation on A. Since, if $a\beta^* b$, for $a, b \in A$, then there exist $x_1, ..., x_{n+1} \in A$ with $x_1 = a, x_{n+1} = b$ and $u_i \in U(A)$ such that $\{x_i, x_{i+1}\} \subseteq u_i$ for $1 \le i \le n$. Let $z_i \in x_i \to c$, for all $1 \le i \le n+1, c \in A$. Then we have,

$$\{z_i, z_{i+1}\} \subseteq (x_i \to c) \cup (x_{i+1} \to c) \subseteq u_i \to c \subseteq U(A), \text{ for all } 1 \le i \le n.$$

Hence $z_1 \beta^* z_{n+1}$, where $z_1 \in a \to c$ and $z_{n+1} \in b \to c$ and so $a \to c\,\overline{\overline{\beta^*}}\,b \to c$. Similarly, we can show that $c \to a\,\overline{\overline{\beta^*}}\,c \to b$. Now, by the same way we can prove that $a\beta^* b$ implies $a \odot c\,\overline{\overline{\beta^*}}\,b \odot c$, for all $c \in A$. Hence β^* is a strong regular relation on A.

4. Quotient hyper hoops

In this section we construct the quotient hyper hoops by using the concept of regular relations and the quotient hoops by using the concept of strong regular relations and investigate some properties of them.

Lemma 4.1. *Let $\boldsymbol{R}$ be a regular relation on A. If $x \to y\boldsymbol{R}\{1\}$ and $y \to z\boldsymbol{R}\{1\}$, then $x \to z\boldsymbol{R}\{1\}$, for any $x, y, z \in A$.*

Proof. Let $x \to y\mathbf{R}\{1\}$ and $y \to z\mathbf{R}\{1\}$. Then it is easy to see that $(x \to y) \odot (y \to z)\overline{\mathbf{R}}\{1\} \odot \{1\} = \{1\}$ and so for all $u \in (x \to y) \odot (y \to z)$, we have $u\mathbf{R}1$. By (HHA9) and (HHA10), $(x \to y) \odot (y \to z) \ll x \to z$ and so there exists $v \in (x \to y) \odot (y \to z)$ and $r \in x \to z$, such that $v \le r$. By considering $u = v$, we get $u \le r$. Since $u \le r$, we get $1 \in u \to r$ and so $u \to r\mathbf{R}\{1\}$. Since $u\mathbf{R}1$, we have $r \to u\overline{\mathbf{R}}r \to 1$ and since by (HHA5), $1 \in r \to 1$, we get $r \to u\mathbf{R}\{1\}$. Hence $r\mathbf{R}u$. Now, since $u\mathbf{R}1$, we have $r\mathbf{R}1$ and so $x \to z\mathbf{R}\{1\}$. ∎

Theorem 4.2. *Let $\boldsymbol{R}$ be a regular relation on A and $\frac{A}{\boldsymbol{R}}$ be the set of all equivalence classes respect to $\boldsymbol{R}$, that is $\frac{A}{\boldsymbol{R}} = \{[x] \mid x \in A\}$. Then $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop, which is called the quotient hyper hoop of A respect to $\boldsymbol{R}$, where for all $[x], [y] \in \frac{A}{\boldsymbol{R}}$,*

$$[x] \otimes [y] = \{[t] \mid t \in x \odot y\} \quad \textit{and} \quad [x] \hookrightarrow [y] = \{[z] \mid z \in x \to y\}$$

Proof. First, we show that $\otimes$ and $\hookrightarrow$ are well-defined on $\frac{A}{\mathbf{R}}$. Let $[x_1]=[x_2]$ and $[y_1]=[y_2]$. Then $x_1\mathbf{R}x_2$ and $y_1\mathbf{R}y_2$. Since $\mathbf{R}$ is a regular relation on A, it follows that $x_1\odot y_1\overline{\mathbf{R}}x_2\odot y_1$ and $x_2\odot y_1\overline{\mathbf{R}}x_2\odot y_2$ and so $x_1\odot y_1\overline{\mathbf{R}}x_2\odot y_2$. Let $[t]\in[x_1]\otimes[y_1]$. Then $[t]=[s]$ for some $s\in x_1\odot y_1$. From $x_1\odot y_1\overline{\mathbf{R}}x_2\odot y_2$, we get that $s\mathbf{R}u$ for some $u\in x_2\odot y_2$. Thus $[s]=[u]$ and then $[t]=[u]$. Hence $[x_1]\otimes[y_1]\subseteq[x_2]\otimes[y_2]$. Similarly, we have $[x_2]\otimes[y_2]\subseteq[x_1]\otimes[y_1]$. Therefore, $\otimes$ is well-defined. By the same way we can prove that $\hookrightarrow$ is well-defined on $\frac{A}{\mathbf{R}}$. Now, we show that $(\frac{A}{\mathbf{R}},\otimes,\hookrightarrow,[1])$ is a hyper hoop.

(HHA1): First we check the associativity of $\otimes$. Let $[x],[y]$ and $[z]$ be arbitrary elements in $\frac{A}{\mathbf{R}}$ and $[u]\in([x]\otimes[y])\otimes[z]$. Then there exists $[v]\in[x]\otimes[y]$ such that$[u]\in[v]\otimes[z]$. In the other words, there exist $v_1\in x\odot y$ and $u_1\in v\odot z$, such that $v\mathbf{R}v_1$ and $u\mathbf{R}u_1$. Since $\mathbf{R}$ is regular and A is a hyper hoop, there exists $u_3\in v_1\odot z\subseteq(x\odot y)\odot z=x\odot(y\odot z)$ such that u_1Ru_3. Then there exists $u_4\in y\odot z$ such that $u_3\in x\odot u_4$. Hence $[u]=[u_1]=[u_3]\in[x]\otimes[u_4]\subseteq[x]\otimes([y]\otimes[z])$ and so $([x]\otimes[y])\otimes[z]\subseteq[x]\otimes([y]\otimes[z])$. Similarly, we can prove that $[x]\otimes([y]\otimes[z])\subseteq([x]\otimes[y])\otimes[z]$. Then $\otimes$ is associative. Now, since $\odot$ is commutative, then $\otimes$ is commutative. Since by (HHA1), $x\in 1\odot x$, for all $x\in A$, then $[x]\in[x]\otimes[1]$, for all $x\in A$. Hence $[1]$ is a unit of $(\frac{A}{\mathbf{R}},\otimes)$. Therefore $(\frac{A}{\mathbf{R}},\otimes)$ is a semihypergroup with $[1]$ as unit.

(HHA2): Since for all $x\in A$, $1\in x\to x$, we get $[1]\in[x]\hookrightarrow[x]$.

(HHA3): Let $[x],[y],[z]$ be arbitrary elements in $\frac{A}{\mathbf{R}}$ and $[u]\in([x]\otimes[y])\hookrightarrow[z]$. This means that there exists $[v]\in[x]\otimes[y]$ such that$[u]\in[v]\hookrightarrow[z]$. Then, there exist $v_1\in x\odot y$ and $u_1\in v\to z$, such that $v\mathbf{R}v_1$ and $u\mathbf{R}u_1$. Since $\mathbf{R}$ is regular and A is a hyper hoop, it follows that there exists $u_3\in v_1\to z\subseteq(x\odot y)\to z=x\to(y\to z)$ such that $u_1\mathbf{R}u_3$. Then there exists $u_4\in y\to z$ such that $u_3\in x\to u_4$. Hence $[u]=[u_1]=[u_3]\in[x]\hookrightarrow[u_4]\subseteq[x]\hookrightarrow([y]\hookrightarrow[z])$. Then $([x]\otimes[y])\hookrightarrow[z]\subseteq[x]\hookrightarrow([y]\hookrightarrow[z])$. Similarly, we can prove that $[x]\hookrightarrow([y]\hookrightarrow[z])\subseteq([x]\otimes[y])\hookrightarrow[z]$.

(HHA4): Let $[x]$ and $[y]$ be arbitrary elements in $\frac{A}{\mathbf{R}}$ and $[u]\in([x]\hookrightarrow[y])\otimes[x]$. This means that there exists $[v]\in[x]\hookrightarrow[y]$ such that $[u]\in[v]\otimes[x]$. Then, there exist $v_1\in x\to y$ and $u_1\in v\odot x$, such that $v\mathbf{R}v_1$ and $u\mathbf{R}u_1$. Since $\mathbf{R}$ is regular and A is a hyper hoop, it follows that there exists $u_3\in v_1\odot x\subseteq(x\to y)\odot x=(y\to x)\odot y$ such that $u_1\mathbf{R}u_3$. Then there exists $u_4\in y\to x$ such that $u_3\in u_4\odot y$. Hence $[u]=[u_1]=[u_3]\in[u_4]\otimes[y]\subseteq([y]\hookrightarrow[x])\otimes[y]$. Then $([x]\hookrightarrow[y])\otimes[x]\subseteq([y]\hookrightarrow[x])\otimes[y]$. Similarly, we can prove the converse and conclude that $([x]\hookrightarrow[y])\otimes[x]=([y]\hookrightarrow[x])\otimes[y]$

(HHA5): Since for all $x\in A$, $1\in x\to 1$, we get $[1]\in[x]\hookrightarrow[1]$.

(HHA6): Let $[x]$ and $[y]$ be arbitrary elements in $\frac{A}{\mathbf{R}}$, $[1]\in[x]\hookrightarrow[y]$ and $[1]\in[y]\hookrightarrow[x]$. Then there exist $u\in x\to y$ and $v\in y\to x$ such that $[u]=[v]=[1]$. Since $\mathbf{R}$ is a regular relation on A, we get $x\mathbf{R}y$ and so $[x]=[y]$.

(HHA7): Let $[x],[y],[z]$ be arbitrary elements in $\frac{A}{\mathbf{R}}$, $[1]\in[x]\hookrightarrow[y]$ and $[1]\in[y]\hookrightarrow[z]$. Then there exist $u\in x\to y$ and $v\in y\to z$ such that $[u]=[v]=[1]$. Since $\mathbf{R}$ is a regular relation on A, by Lemma 4.1, $x\to z\mathbf{R}\{1\}$ and so there exists $t\in x\to z$ such that $[t]=[1]$. So $[1]\in[x]\hookrightarrow[z]$.

Therefore, $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop. ■

Example 4.3. Let $(A, \otimes, \hookrightarrow, 1)$ be hyper hoop as in Example 3.3. Then $[1] = [b] = \{1, b\}$ and $[a] = [c] = \{a, c\}$. Hence $\frac{A}{\mathbf{R}} = \{[1], [a]\}$ and hyperoperations $\otimes, \hookrightarrow$ are as following tables,

$\otimes$	[1]	[a]
[1]	$\{[1]\}$	$\{[a], [1]\}$
$[a]$	$\{[a], [1]\}$	$\{[1],$[a]$\}$

$\hookrightarrow$	[1]	[a]
[1]	$\{[a], [1]\}$	$\{[a]\}$
$[a]$	$\{[1], [a]\}$	$\{[a], [1]\}$

It is clear that $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop.

Theorem 4.4. *Let $\boldsymbol{R}$ be an equivalence relation on A. If $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop, then $\boldsymbol{R}$ is a regular relation.*

Proof. Let $\mathbf{R}$ be an equivalence relation on A, $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ be a hyper hoop, and $a\mathbf{R}b$, for $a, b \in A$. Then for any $x \in A$ and $u \in a \odot x$, we have $[u] \in [a] \otimes [x] = [b] \otimes [x] = \{[v] | v \in b \odot x\}$ and so, there exists $v \in b \odot x$ such that $u\mathbf{R}v$. Thus $(a \odot x)\bar{\mathbf{R}}(b \odot x)$. Similarly, if $a\mathbf{R}b$, then $(a \to x)\bar{\mathbf{R}}(b \to x)$. Now, let $x \to y\mathbf{R}\{1\}$ and $y \to x\mathbf{R}\{1\}$, for $x, y \in A$. Then $[1] \in [x] \hookrightarrow [y]$ and $[1] \in [y] \hookrightarrow [x]$. Since $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop by (HHA6), we get $[x] = [y]$ and so $x\mathbf{R}y$. Hence $\mathbf{R}$ is a regular relation. ■

Corollary 4.5. *Let $\boldsymbol{R}$ be an equivalence relation on A. Then $\boldsymbol{R}$ is a regular relation on A if and only if $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop.*

Proof. By Theorems 4.2 and 4.4, the proof is clear. ■

Theorem 4.6. *Let $\boldsymbol{R}$ be a strong regular relation on A. Then $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ is a hoop which is called the quotient hoop of A respect to $\boldsymbol{R}$.*

Proof. Similar to the proof of Theorem 4.2, we can prove that the hyperoperations $\otimes$ and $\hookrightarrow$ are well-defined on $\frac{A}{\mathbf{R}}$ and axioms (HHA1), (HHA2)and (HHA3) hold on $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$. Hence $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ is a quasi hyper hoop. Now, for any $x, y \in A$, since $x\mathbf{R}x$ and $\mathbf{R}$ is a strong regular relation on A, we get $(x \odot y)\overline{\overline{\mathbf{R}}}(x \odot y)$ and $(x \to y)\overline{\overline{\mathbf{R}}}(x \to y)$. Hence for any $z, z' \in x \odot y$ and $w, w' \in x \to y$, $z\mathbf{R}z'$ and $w\mathbf{R}w'$. Then $[x] \otimes [y] = \{[z] | z \in x \odot y\} = \{[z]\}$ and $[x] \hookrightarrow [y] = \{[w] | w \in x \to y\} = \{[w]\}$, which means that $[x] \otimes [y]$ and $[x] \hookrightarrow [y]$ are singletons, for any $[x], [y] \in \frac{A}{\mathbf{R}}$. Therefore, By Proposition 2.3, $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ is a hoop. ■

Example 4.7. Let A be hyper hoop as in Example 3.10. Then $[a] = [b] = \{a, b\}$ and $[c] = [1] = \{1, c\}$. Hence $\frac{A}{\mathbf{R}} = \{[1], [a]\}$ and

$\otimes$	[1]	[a]
[1]	$\{[1]\}$	$\{[a]\}$
$[a]$	$\{[a]\}$	$\{[a]\}$

$\hookrightarrow$	[1]	[a]
[1]	$\{[1]\}$	$\{[a]\}$
$[a]$	$\{[1]\}$	$\{[1]\}$

It is clear that $(A, \otimes, \hookrightarrow, [1])$ is a hoop.

Theorem 4.8. *Let $\boldsymbol{R}$ be an equivalence relation on A. If $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ is a hoop, then $\boldsymbol{R}$ is a strong regular relation on A.*

Proof. Let $\mathbf{R}$ be an equivalence relation on A, $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ be a hoop , $a\mathbf{R}b$, for $a, b \in A$, and $x \in A$. Then for any $u \in a \odot x$ and $v \in b \odot x$, we have $[u] = [a] \otimes [x] = [b] \otimes [x] = [v]$, which means that $u\mathbf{R}v$. Hence, $(a \odot x)\overline{\overline{\mathbf{R}}}(b \odot x)$. Similarly, if $a\mathbf{R}b$ then for any $x \in A$, $(a \to x)\overline{\overline{R}}(b \to x)$ and $(x \to a)\overline{\overline{R}}(x \to b)$. Hence $\mathbf{R}$ is a strong regular relation on A. ∎

Corollary 4.9. *Let $\boldsymbol{R}$ be an equivalence relation on A. Then $\boldsymbol{R}$ is a strong regular relation if and only if $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ is a hoop.*

Proof. By Theorems 4.6 and 4.8, the proof is clear. ∎

Lemma 4.10. *Let $f : A_1 \to A_2$ be a homomorphism of hyper hoops. Then,*

(i) *if $x \leq y$, for all $x, y \in A_1$ then $f(x) \leq f(y)$,*

(ii) *if F is a (weak) filter of A_2, then $f^{-1}(F)$ is a (weak) filter of A_1,*

(iii) *$ker(f) = \{x \in A_1 | f(x) = 1\}$ is a filter of A_1,*

(iv) *f is one to one if and only if $ker(f) = \{1\}$,*

(v) *if f is onto and F is a filter of A_1 which contains ker(f), then f(F) is a filter of A_2.*

Proof. (i). The proof is easy.

(ii) Let $x \leq y$ and $x \in f^{-1}(F)$. Then there exists $u \in F$ such that $f(x) = u$. Since $x \leq y$, by (i), $u = f(x) \leq f(y)$. Since F is a filter and $u \in F$, we get $f(y) \in F$ and so $y \in f^{-1}(F)$. Then $f^{-1}(F)$ is an upset. Let $x, y \in f^{-1}(F)$. Then $f(x) \in F$ and $f(y) \in F$. Hence $f(x \odot y) = f(x) \odot f(y) \subseteq F$ and so $x \odot y \subseteq f^{-1}(F)$. Then $f^{-1}(F)$ is a filter. If F is a weak filter, then $f(x \odot y) = f(x) \odot f(y) \cap F \neq \emptyset$. Hence $x \odot y \cap f^{-1}(F) \neq \emptyset$ and then $f^{-1}(F)$ is a weak filter, too.

(iii) Let $x \leq y$ and $x \in ker(f)$. Then by (i), $f(x) \leq f(y)$. Since $f(x) = 1$, we get $f(y) = 1$ and so $y \in ker(f)$. Hence $ker(f)$ is an upset. Let $x, y \in ker(F)$. Then by (HHA12), $f(x \odot y) = f(x) \odot f(y) = 1 \odot 1 = 1$. Hence $x \odot y \subseteq F$ and so F is a filter of A_1.

(iv) The proof is clear.

(v) Since $1 \in F$, we get $1 = f(1) \in f(F)$. Let $x, y \in A_2$ such that $f(F) \ll x \to y$ and $x \in f(F)$. Since $x \in f(F)$ and f is onto, there are $a \in F$ and $b \in A_1$ such that $y = f(b)$ and $x = f(a)$. Hence $f(a \to b) = f(a) \to f(b) = x \to y \gg f(F)$. Then there are $u \in a \to b$ and $v \in F$ such that $f(v) \leq f(u)$ and so $1 \in f(v) \to f(u) = f(v \to u)$. Then there exists $t \in v \to u$, such that $f(t) = 1$ and so $t \in ker(f) \subseteq F$. Hence $F \ll v \to u$. Since F is a filter of A_1 and $v \in F$, by Theorem 3.7, we get $u \in F$. Then $F \ll a \to b$ and so by Theorem 3.7, $b \in F$. Hence $y = f(b) \in f(F)$ and so, by Theorem 3.7 $f(F)$ is a filter of A_2. ∎

Let $\mathbf{R}$ be a regular relation on A. We define the relation $\ll$ on $\frac{A}{\mathbf{R}}$ by $[x] \ll [y]$ if and only if $[1] \in [x] \hookrightarrow [y]$.

Theorem 4.11. *(First isomorphism theorem) Let A_1 and A_2 be two hyper hoops and $\boldsymbol{R}$ be a regular relation on A_1. If $f : A_1 \to A_2$ is a homomorphism of hyper hoops and $ker(f) = [1]$, then $\frac{A_1}{\boldsymbol{R}} \cong Im(f)$.*

Proof. Since R is a regular relation on A_1, $\frac{A_1}{\mathbf{R}}$ is well-defined. Now, let $\overline{f} : \frac{A1}{\mathbf{R}} \to A_2$ is defined by $\overline{f}[x] = f(x)$, for all $x \in A_1$. If $[x] = [y]$, for $x, y \in A_1$ then $x\mathbf{R}y$ and so $x \to y\overline{\mathbf{R}}y \to y$. Since by (HHA2), $1 \in y \to y$, we get $x \to y\mathbf{R}\{1\}$ and so there exists $a \in x \to y$ such that $a \in [1] = ker(f)$. Then $1 = f(a) \in f(x \to y) = f(x) \to f(y)$ and so $f(x) \leq f(y)$. By the same way, we can show that $f(y) \leq f(x)$. Hence by (HHA6), $f(x) = f(y)$ and so $\overline{f}$ is well-defined. Now, we show that $\overline{f}$ is a homomorphism. Let $[x], [y] \in \frac{A_1}{\mathbf{R}}$. Then,

$$\begin{aligned}
\overline{f}([x] \hookrightarrow [y]) &= \{\overline{f}[z] \mid z \in x \to y\} = \{f(z) \mid z \in x \to y\} = f(x \to y) \\
&= f(x) \to f(y) = \overline{f}([x]) \to \overline{f}([y]) \\
\overline{f}([x] \otimes [y]) &= \{\overline{f}[z] \mid z \in x \odot y\} = \{f(z) \mid z \in x \odot y\} = f(x \odot y) \\
&= f(x) \odot f(y) = \overline{f}([x]) \odot \overline{f}([y])
\end{aligned}$$

Moreover, $\overline{f}([1]) = f(1) = 1$. Then $\overline{f}$ is a homomorphism. Now, we show that $\overline{f}$ is one to one. Suppose that $[x] \in ker(\overline{f})$. Then $f(x) = \overline{f([x])} = 1$ and so $x \in kerf = [1]$ which implies that $x\mathbf{R}1$. i.e. $[x] = [1]$. Hence $ker\overline{f} = \{[1]\}$ and so by Theorem 4.10 (iv), $\overline{f}$ is one to one. Clearly $\frac{A1}{\mathbf{R}} \to Im(f)$ is onto. Therefore, $\frac{A_1}{\mathbf{R}} \cong Im(f)$. ∎

Example 4.12. If $f : A_1 \to A_2$ is a homomorphism of hyper hoops and ρ is regular relation as Example 3.6, then $\frac{A_1}{\rho} \cong Im(f)$.

Notation. In the follows, the set of all filters of A is denoted by $\mathcal{F}(A)$ and the set of all filters of A containing S is denoted by $\mathcal{F}(A, S)$, where S is a non-empty subset of A.

Theorem 4.13. *Let $\boldsymbol{R}$ be a regular relation on A and $F = [1]$. Then there exists a one to one corresponding between $\mathcal{F}(A, F)$ and $\mathcal{F}(\frac{A}{R})$.*

Proof. Consider the canonical homomorphism $\pi : A \to \frac{A}{\mathbf{R}}$. If J is a filter of A containing F, then by Proposition 4.10(ii) and (v), the function $g : \mathcal{F}(A, F) \to \mathcal{F}(\frac{A}{R})$ by $J \to \pi(J)$ is onto. Now, we show that g is one to one. Let $G, H \in \mathcal{F}(A, F)$ and $g(G) = g(H)$. Then for $x \in H$, $[x] \in \pi(H) = \pi(G)$. Hence there exists $y \in G$ such that $[x] = [y]$ and so $x\mathbf{R}y$. Since $\mathbf{R}$ is a regular relation on A, we have $y \to x\overline{\mathbf{R}}y \to y$ and since $1 \in y \to y$ we get $y \to x\mathbf{R}\{1\}$. Then there exists $t \in y \to x$ such that $t\mathbf{R}1$, i.e. $t \in F$. Now, since $F \ll y \to x$ and $F \subseteq G$, we have $G \ll y \to x$ and so by Lemma 3.7 $x \in G$. Hence $H \subseteq G$. Similarly, we can show that $G \subseteq H$. Therefore $G = H$ and so g is one to one. ∎

Definition 4.14. Let F be a proper (weak) filter of A. Then F is called a *(weak) maximal filter* of A, if $F \subseteq J \subseteq A$ for some (weak)filter of A, then $F = J$ or $J = A$.

Proposition 4.15. *Every proper filter of A is contained in a maximal filter of A.*

Proof. Let F be a proper filter of A and S be the collection of all proper filters J of A such that $F \subseteq J$. Then $F \in S$ and $S \neq \emptyset$. S is a partially ordered set by $\subseteq$. Let $C = \{J_\alpha | \alpha \in \Gamma\}$ be a chain in S. We can show that $J = \bigcup_{\alpha\in\Gamma} J_\alpha$ is a filter of A. It is clear that $F \subseteq J$. If $J = A$ then $0 \in J$. Thus there exists $\beta \in \Gamma$ such that $0 \in J_\beta$. Since J_β is an upset, we get $J_\beta = A$ and this is a contradiction. Therefore $J \neq A$ and then $J \in C$. Now, the proof follows by Zorn's Lemma. ∎

Proposition 4.16. *Let $f : A_1 \to A_2$ be an epimorphism of hyper hoops. Then,*

(i) *if F is a maximal filter of A_1 which contains $kerf$, then $f(F)$ is a maximal filter of A_2,*

(ii) *if F is a maximal filter of A_2, then $f^{-1}(F)$ is a maximal filter of A_1 which contains $kerf$,*

(iii) *the map $f : F \to f(F)$ is one to one corresponding between the maximal filters of A_1 containing $kerf$ and maximal filters of A_2.*

Proof. (i) Let $f(F) = A_2$. Since $F \neq A_1$, there exists $x \in A_1$ such that $x \notin F$. So $f(x) \in A_2 = f(F)$. Thus there exists some $a \in F$ such that $f(x) = f(a)$ and then $1 = f(1) \in f(a) \to f(x) = f(a \to x)$. So there exists $t \in a \to x$ such that $f(t) = 1$, i.e., $t \in kerf \subseteq F$. Then $F \ll a \to x$. Thus by Theorem 3.7, $x \in F$ which is a contradiction. Hence $f(F) \neq A_2$. Since F is a filter of A_1 which contains $kerf$, then $f(F)$ is a filter of A_2 by Theorem 4.10 (v). Let J be a filter of A_2 such that $f(F) \subseteq J \subseteq A_2$. By Lemma 4.10 (ii), $f^{-1}(J)$ is a filter of A_1 such that $F \subseteq f^{-1}(J) \subseteq f^{-1}(A_2) = A_1$. Since F is a maximal filter of A_1, we get $f^{-1}(J) = F$ or $f^{-1}(J) = A_1$. Hence $J = f(F)$ or $J = f(A_1) = A_2$, i.e., $f(F)$ is a maximal filter of A_2.
(ii) Let $f^{-1}(F) = A_1$. Since $F \neq A_2$, there exists $y \in A_2$ such that $y \notin F$. So $f^{-1}(y) \in A_1 = f^{-1}(F)$. Thus there exists some $x \in A_1$ such that $y = f(x) \in F$ which is a contradiction. Hence $f^{-1}(F) \neq A_1$. Similarly to the part(i), we can show that $f^{-1}(F)$ is a maximal filter of A_1.

(iii) The proof is straightforward by (i) and (ii). ∎

Definition 4.17. The hyper hoop A is called *simple*, if it has only two filters $\{1\}$ and A.

Example 4.18. Let $A = \{1, a, b, c\}$. Define the hyperoperations $\odot$ and $\to$ on A as follows,

$\odot$	1	a	b	c
1	$\{1\}$	$\{1, a\}$	$\{1, b\}$	$\{1, a, b, c\}$
a	$\{1, a\}$	$\{1, a, b, c\}$	$\{1, a, b\}$	$\{1, a, b, c\}$
b	$\{1, b\}$	$\{1, a, b\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$
c	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$

$\rightarrow$	1	a	b	c
1	$\{1,a,b,c\}$	$\{a,c\}$	$\{b,c\}$	$\{c\}$
a	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{a,b,c\}$	$\{a,c\}$
b	$\{1,a,b,c\}$	$\{a,b,c\}$	$\{1,a,b,c\}$	$\{b,c\}$
c	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$

Then $(A,\odot,\rightarrow,1)$ is a simple hyper hoop.

Remark 4.19. The hyper hoop A is simple if and only if $\{1\}$ is a maximal filter of A .

Theorem 4.20. *Let $\boldsymbol{R}$ be a regular relation on A and $F=[1]$. Then $\frac{A}{\boldsymbol{R}}$ is simple if and only if F is a maximal filter of A.*

Proof. Let $\frac{A}{\mathbf{R}}$ be simple. Then by Remark 4.19, $\{[1]\}$ is a maximal filter of $\frac{A}{\mathbf{R}}$ and so $\{[1]\}\neq\frac{A}{\mathbf{R}}$. Then there exists $[x]\in\frac{A}{\mathbf{R}}$ such that $[x]\neq[1]$ and so $x\notin F$. Hence $F\neq A$. Suppose that F is not a maximal filter of A. By Proposition 4.15, there exists a maximal filter J of A such that $F\subset J$. Consider canonical epimorphism $\varphi:A\rightarrow\frac{A}{\mathbf{R}}$ in which $ker(\varphi)=F$. Using Proposition 4.16(i), $\frac{J}{\mathbf{R}}$ is a maximal filter of $\frac{A}{\mathbf{R}}$. Also, we have $\{[1]\}=\{[x]|x\in F\}\subseteq\{[x]|x\in J\}=\frac{J}{\mathbf{R}}$. So $\{[1]\}=\frac{J}{\mathbf{R}}$. Then for all $x\in J$, we have $x\mathbf{R}1$, i.e., $x\in F$. Hence $J\subseteq F$ which is a contradiction.

Conversely, let F be a maximal filter of A. By contrary, suppose that $\frac{A}{\mathbf{R}}$ is not simple. Then by Remark 4.19, $\{[1]\}$ is not a maximal filter of $\frac{A}{\mathbf{R}}$. By Proposition 4.15, there exists a maximal filter K of $\frac{A}{\mathbf{R}}$ such that $\{[1]\}\subseteq K$. By Proposition 4.16(ii), $\varphi^{-1}(K)$ is a maximal filter of A such that $F\subseteq\varphi^{-1}(K)$. So $F=\varphi^{-1}(K)$. Hence $\{[1]\}=\varphi(F)=K$ which is a contradiction. ∎

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Accepted: 8.06.2015