FINITE GROUPS WITH SOME QUASINORMAL AND SELFNORMALIZING SUBGROUPS

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Abstract. A subgroup H of a finite group G is called quasinormal in G if $HK = KH$ holds for every subgroup K of G. In this paper, we mainly give the structure of finite nonabelian simple groups in which every cyclic subgroups of order 2 and order 4 of every second maximal subgroups is either quasinormal or selfnormalizing.

Key words: quasinormal subgroups; selfnormalizing subgroups; second maximal subgroups; minimal subgroups.

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1. Introduction

All groups considered in this paper will be finite. A group G is called a PN-group if all minimal subgroups of G are normal in G. Gaschütz and Itô [6, IV, 5.7 Satz] showed that if G is a PN-group, then G is solvable, and the commutator subgroup G' of G has a normal Sylow 2-subgroup with nilpotent factor. In 1970, Buckley [2] proved that a PN-group of odd order is supersolvable. The PN-groups have been generalized by many authors(see [1], [5] [9], [11] and [10]).

Definition 1.1 Let G be a group. A subgroup H of G is said to be quasinormal in G if $HK = KH$ holds for every subgroup K of G.

Obviously, the quasinormality is a generalization of the normality. Hence the concept of selfnormalizing subgroup is, in a sense, opposite of that of a quasinormal subgroup of G.

In this paper, we will investigate the structure of a group G in which every cyclic subgroups of order 2 and order 4 of every second maximal subgroups is either quasinormal or selfnormalizing. In the light of Feit-Thompson's Theorem,

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all groups of odd order are solvable, and so, the involutions should pose something of special interest. However, if we only give the conditions on the subgroups of order 2 of G, we could not expect a detailed structure. On the other hand, if we assume that every cyclic subgroup of a group G of order 2 and 4 is either quasinormal or selfnormalizing in G, then we are able to get some interesting results.

For convenience, we list the following condition.

($\mathcal{P}$) Every cyclic subgroup of a group G of order 2 and 4 is either quasinormal or selfnormalizing in G.

We first classify the nonsolvable groups all of whose second maximal subgroups satisfy condition ($\mathcal{P}$). In a sense, these groups can be viewed as a generalization of PN-groups. Next, we prove that a group G is solvable if every nonnilpotent subgroup of G is either quasinormal or selfnormalizing in G.

The notation and terminology used in this paper are standard. Particularly, we denote by $A \rtimes B$ the semidirect product of A and B; G_p always denotes a Sylow p-subgroup of G and $\pi(G)$ denotes the set of all prime divisors of $|G|$.

2. Preliminaries

In this section, we give some basic properties and collect some results that are needed in the sequel.

Lemma 2.1 *Let G be a dihedral group of order $2n$, where n is an odd integer. Then, G satisfies condition ($\mathcal{P}$).*

Proof. Let G be a dihedral group of order $2n$, where n is odd. Let a be any involution of G. Then, $N_G(\langle a\rangle) = \langle a\rangle$, and so, G satisfies condition ($\mathcal{P}$). ∎

Lemma 2.2 *Let G be a dihedral group of order $4n$, where n is an odd integer. Then, G does not satisfy condition ($\mathcal{P}$). Moreover, if n is not a prime, then there exist some maximal subgroups of G do not satisfy condition ($\mathcal{P}$).*

Proof. Let G be a dihedral group of order $4n$, where n is odd. We may write

$$G = \langle a, b \mid a^2 = b^{2n} = 1, a^{-1}ba = b^{-1}\rangle.$$

Then $N_G(\langle a\rangle) = \langle a\rangle \times \langle b^n\rangle$. It implies that the involution a is neither quasinormal nor selfnormalizing in G. Hence the dihedral group of order $4n$ does not satisfy condition ($\mathcal{P}$).

If there is an odd prime t such that $t < n$ and $t|n$, then G may have a maximal subgroup of order $\frac{4n}{t}$ which is dihedral. By what has been said above, such a subgroup does not satisfy condition ($\mathcal{P}$). ∎

Lemma 2.3 [7, Theorem B] *Let G be a nonsolvable group. Suppose that solvable subgroups of G are either 2-nilpotent or minimal non-nilpotent. Then G is one of the following groups:*

(1) $PSL(2, 2^r)$*, where* $2^r - 1$ *is a prime;*

(2) $PSL(2, q)$*, where* q *is odd and* $q \equiv 3,\ 5(mod\ 8)$*;*

(3) $SL(2, q)$*, where* q *is odd and* $q \equiv 3,\ 5(mod\ 8)$*.*

It is easy to see that if the exponent of Sylow 2-subgroups of G is not more than 4, then any quasinormal subgroup of order 2 of G is normal. We will use this fact freely in our following proof.

3. The main results and proofs

Theorem 3.1 *If a group G satisfies condition $(\mathcal{P})$, then G is 2-nilpotent.*

Proof. Assume that G is not 2-nilpotent. Then, G has a minimal non-2-nilpotent subgroup K. By a result of Itô [6, III, 5.4 Satz], G is a minimal nonnilpotent group and $K = K_2 \rtimes K_p$ with $\exp K_2 \leq 4$.

Let a be any nontrivially element of K_2. Then $\langle a \rangle$ is quasinormal in G by hypothesis. Hence $\langle a \rangle K_p$ is a subgroup of K. If $\langle a \rangle K_p < K$, then $\langle a \rangle K_p$ is nilpotent. It implies that K is nilpotent, a contradiction. Hence we have $\langle a \rangle K_p = K$ and $K_2 = \langle a \rangle$ is cyclic. Therefore K is 2-nilpotent, which implies that K is nilpotent, a contradiction. Thus, G is 2-nilpotent. ∎

Theorem 3.2 *Let G be a group. If every maximal subgroup of G satisfies condition $(\mathcal{P})$, then one of the following holds:*

(1) *G is 2-nilpotent;*

(2) *$G = PQ$ is a minimal nonnilpotent group, where P is an elementary abelian 2-group and Q is a cyclic q-group ($q > 2$), or P is a quaternion group of order 8 and Q is a cyclic 3-group.*

Proof. Assume that G is not 2-nilpotent. Then, G has a minimal non-2-nilpotent subgroup K. By our hypothesis and Theorem 3.1, we know that $G = K$. By [6, III, 5.4 Satz], G is a minimal nonnilpotent group. According to a result due to Schmidt [6, III, 5.2 Satz], we have $G = PQ$, where P is a normal Sylow 2-subgroup of G with $\exp P \leq 4$, and Q is a cyclic Sylow q-subgroup ($q > 2$). Furthermore, if P is abelian, then P is elementary abelian; if P is not abelian, then $Z(P) = \Phi(P) = P'$. In the case P is abelian, it is clear that every maximal subgroup of G satisfies condition $(\mathcal{P})$. Assume that P is not abelian, and let $x \in P$ with $x^4 = 1$. By the hypothesis, we have $\langle x \rangle$ is quasinormal in P. Let $Z = \langle x^2 \rangle$, then $\langle xZ \rangle$ is quasinormal in P/Z. Hence $\langle xZ \rangle$ is normal in P/Z, which implies that $\langle x \rangle$ is normal in P. Therefore, P is a Hamiltonian group. It follows that $P = Q_8 \times A$, where Q_8 is the quaternion group of order 8, and A is an elementary abelian 2-group. However, $A \leq Z(P) = P' \leq Q_8$. Hence $A = 1$, and thus $P = Q_8$. Since $Aut(Q_8) \cong S_4$, the symmetry group of degree 4, we have $q = 3$. ∎

Theorem 3.3 *Let G be a nonsolvable group. If every second maximal subgroup of G satisfies condition $(\mathcal{P})$, then G is one of the following groups:*

(1) $PSL(2, 2^r)$, *where r is a prime such that $2^r - 1$ is a prime;*

(2) $PSL(2, p)$, *where $p > 3$ is a prime with $p \equiv 3, 5(mod\ 8)$ and $p^2 \not\equiv 1(mod\ 5)$. Moreover, if $p \pm 1 = 4s$, then s is an odd prime;*

(3) $PSL(2, 3^r)$, *where r is an odd prime and $3^r \equiv 3, 5(mod\ 8)$. Moreover, if $3^r \pm 1 = 4s$, then s is an odd prime;*

(4) $SL(2, p)$, *where $p > 3$ is a prime with $p \equiv 3, 5(mod\ 8)$ and $p^2 \not\equiv 1(mod\ 5)$;*

(5) $SL(2, 3^r)$, *where r is an odd prime and $3^r \equiv 3, 5(mod\ 8)$.*

Proof. Let M be any maximal subgroup of G. By the hypothesis and Theorem 3.2, M is 2-nilpotent or a minimal nonnilpotent group. In particular, M is solvable. Then, by Lemma 2.3, G is one of the following groups:

(i) $PSL(2, 2^r)$, where $2^r - 1$ is a prime such that $2^r - 1$ is a prime;

(ii) $PSL(2, q)$, where q is odd and $q \equiv 3,\ 5(mod\ 8)$;

(iii)) $SL(2, q)$, where q is odd and $q \equiv 3,\ 5(mod\ 8)$.

Case 1. $G \cong PSL(2, 2^r)$, where r is a prime such that $2^r - 1$ is also a prime.

In this case, any maximal subgroup of G is one of the following groups:

(i) a minimal nonabelian group of order $2^r(2^r - 1)$;

(ii) dihedral groups of order $2(2^r - 1)$ or $2(2^r + 1)$.

By Lemma 2.1, G satisfies condition $(\mathcal{P})$. That is G is of type (1).

Case 2. $G \cong PSL(2, q)$, where $q \equiv 3, 5(mod\ 8)$.

Let $q = p^n$ and let $p > 3$. Since $PSL(2, p^n)$ contains the nonsolvable subgroup $PSL(2, p)$, we have $n = 1$.

If $p^2 \equiv 1(mod\ 5)$, then $PSL(2, p)$ contains the subgroup A_5, and therefore, $p = 5$, which is a contradiction. Hence, $p^2 \not\equiv 1(mod\ 5)$.

By [6, II, Theorem 8.27], G has only three kinds of maximal subgroups:

(i) Frobenius group N and $N = P \rtimes C$ is a minimal nonabelian group, where P is an elementary abelian group and C is a cyclic group of order $\frac{p-1}{2}$;

(ii) dihedral groups of order $p - 1$ or $p + 1$;

(iii) A_4, the alternating group of degree 4.

Clearly, all maximal subgroups of (i) are abelian. All maximal subgroups of A_4 are satisfy condition $(\mathcal{P})$. Now we consider (ii). Since $p = q \equiv 3, 5(mod\ 8)$, we get $p-1 = 2s$ or $4s$ and $p+1 = 2s$ or $4s$, where s is an odd integer. If $p \pm 1 = 2s$, then the groups of type (ii) satisfy condition $(\mathcal{P})$ by Lemma 2.1. If $p \pm 1 = 4s$, then the groups of type (ii) satisfy condition $(\mathcal{P})$ if and only if s is a prime by Lemma 2.2. That is G is of type (2).

When $p = 3$, if n is even, then $PSL(2,9) \leq PSL(2,3^n)$, and therefore, $PSL(2,3^n)$ contains a proper subgroup A_5, which is a contradiction. Hence, n is odd. If n is an odd composite, let $n = st$, where t is a prime with $t < n$. By [6, II, Theorem 8.27], we know $PSL(2,3^n)$ contains a nonsolvable proper subgroup $PSL(2,3^t)$, which is a contradiction. Thus n is an odd prime. Now by the same argument as above, we get G is of type (3) in this case.

Case 3. $G \cong SL(2,q)$, where $q \equiv 3, 5(mod\ 8)$.

Let $q > 5$ and $q = p^n$. It is well known that $SL(2,q)$ has only one element of order 2, say, z, and $SL(2,q)/\langle z\rangle \cong PSL(2,q)$. By the same arguments as in Case 2, we get $n = 1$ and $p^2 \not\equiv 1(mod\ 5)$ if $p > 3$, and n is an odd prime for $p = 3$. By [3, Theorem 6.17], we know that any maximal subgroup of G is one of the following groups:

(i) Frobenius group N and $N = P \rtimes C$ is a minimal nonabelian group, where P is an elementary abelian group and C is a cyclic group of order $p^n - 1$;

(ii) the group M defined by the presentation $\langle x, y | x^m = y^2, y^{-1}xy = x^{-1}\rangle$, where $2m = p^n - 1$ or $p^n + 1$;

(iii) $SL(2,3)$.

We consider these groups in turn.

Since all maximal subgroups of (i) are abelian, the groups of type (i) satisfy condition $(\mathcal{P})$. For the groups of type (ii), we get $m = 2s$ or $2s \pm 1$, where s is odd. In the later case, M is the generalized quaternion group of order $4(2s \pm 1)$. It is well known that the cyclic subgroup of order 2 of M is normal in M, and the cyclic subgroup of order 4 of M is self-normalizing in M. Hence M satisfies condition $(\mathcal{P})$. For the case $m = 2s$, the maximal subgroups of M are

$$M_1 = \langle x^2, y\rangle, M_2 = \langle x^2, xy\rangle, M_3 = \langle x\rangle.$$

Since both M_1 and M_2 are the generalized quaternion group, and M_3 is cyclic, we know that M satisfies condition $(\mathcal{P})$. Since the maximal subgroups of $SL(2,3)$ are the quaternion group of order 8 and the abelian group $Z_2 \times Z_3$. Hence, we get conclusions (4) and (5). The proof of the theorem is now complete. ∎

A group G is called a $\mathcal{QNS}$-group if every minimal subgroup of G is either quasinormal in G or self-normalizing. It is proved that every $\mathcal{QNS}$-group is solvable(see [5]). We now give a slight improvement of this result.

Theorem 3.4 *Let G be a group. If every nonnilpotent proper subgroup of G is a $\mathcal{QNS}$-group, then G is solvable.*

Proof. Assume that the theorem is false, and let G be a minimal counterexample. By [5, Theorem 3.1], we know that the $\mathcal{QNS}$-groups are solvable, and therefore, $G/\Phi(G)$ is a minimal simple group, where $\Phi(G)$ is the Frattini subgroup of G. By Thompson's classification of minimal simple groups (see [12]), we know that $G/\Phi(G)$ is isomorphic to one of the following groups:

(1) $PSL(2,p)$, where p is a prime with $p > 3$ and $5 \nmid p^2 - 1$;

(2) $PSL(2,2^q)$, where q is a prime;

(3) $PSL(2,3^q)$, where q is a prime;

(4) $PSL(3,3)$;

(5) The Suzuki group $Sz(2^q)$, where q is an odd prime.

Let H be the 2-complement of $\Phi(G)$. Then $H \trianglelefteq G$ and H is nilpotent. We have

(1) $H \leq Z(G)$.

Indeed, let $P \in Syl_p(H)$, where p is a prime in $\pi(H)$. Then $P \trianglelefteq G$. Let X be any minimal subgroup of P. Since G is nonsolvable, there exists some nonnilpotent maximal subgroup M of G such that $P \leq M$. By the hypothesis, M is a $\mathcal{QNS}$-group. Then, X is a subnormal subgroup of M. Thus, X is normal in M, and therefore, normal in G. Hence $G/C_G(X) = N_G(X)/C_G(X) \lesssim Aut(X) \cong C_{p-1}$. If $C_G(X)$ is a proper subgroup G, then $C_G(X)$ is solvable and G is hence solvable, a contradiction. Thus $C_G(X) = G$, i.e., $X \leq Z(G)$. Let $G_2 \in Syl_2(G)$ and $K = G_2P$. Applying Ito's Lemma, we see that K is p-nilpotent and so K is nilpotent. Then we have that $G_2 \leq C_G(P) \trianglelefteq G$. Using the simplicity of $G/\Phi(G)$, we conclude that $H \leq Z(G)$. Thus (1) holds.

(2) $H = 1$.

Set $\overline{G} = G/\Phi(G)_2$, where $\Phi(G)_2 \in Syl_2(\Phi(G))$. Then by (1), $\overline{G}/Z(\overline{G}) \cong G/\Phi(G)$ and $\overline{G}$ is a quasisimple group with the center of odd order. Hence in order to prove $H = 1$, i.e., $Z(\overline{G}) = 1$, it will suffice to show that the Schur multiplier of each of the minimal simple groups is a 2-group. Indeed, this is true by checking the table on the Schur multipliers of the known simple groups (see [3, p. 302]).

(3) Every subgroup of order $2^m p$ (p an odd prime) of $\overline{G} = G/\Phi(G)$ is 2-nilpotent.

By (2), $\Phi(G)$ is a 2-group. Assume $L/\Phi(G)$ is a proper subgroup of order $2^m p$. Then L is a proper subgroup of order $2^n p$ for some natural number n. Let $G_p \in Syl_p(L)$. Then $|G_p| = p$ and hence G_p is quasinormal in L. Since G_p is subnormal in L, we have that G_p is normal in L, that is, L is 2-nilpotent. Thus $L/\Phi(G)$ is 2-nilpotent.

(4) Final contradiction.

We know that $\overline{G}$ is isomorphic to one of the above simple groups. Suppose that $\overline{G} \cong PSL(2,p)$, $PSL(2,3^q)$ or $PSL(3,3)$. Indeed, each of $PSL(2,p)$, $PSL(2,3^q)$ and $PSL(3,3)$ contains a subgroup which is isomorphic to A_4, the alternating group of degree 4, by (3) we conclude that $\overline{G}$ cannot be any one of $PSL(2,p), PSL(2,3^q)$ and $PSL(3,3)$. Suppose that $\overline{G} \cong PSL(2,2^q)$ or $Sz(2^q)$. Then $\overline{G}$ is a Zassenhaus group of odd degree and the stabilizer of a point is a Frobenius group with kernel a 2-group. So $\overline{G}$ cannot be any one of $PSL(2,2^q)$ and $Sz(2^q)$ as well. Thus the proof is complete. ∎

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