A-NUMERICAL RADIUS OF A-NORMAL OPERATORS IN SEMI-HILBERTIAN SPACES

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Abstract. Let $\mathcal{H}$ be a Hilbert space and A be a positive bounded linear operator on $\mathcal{H}$. The semi-inner product $\langle h, k\rangle_A = \langle Ah, k\rangle$, $h, k \in \mathcal{H}$, induces a seminorm for a bounded linear operator T, which is defined by

$$\|T\|_A = \sup\{\|Th\|_A/\|h\|_A : \|h\| \neq 0\}.$$

The main purpose of this paper is to prove that $\|T\|_A$ equals the A-numerical radius of T, when T is an A-normal operator. This generalizes the similar result for a normal operator on a Hilbert space.

Keywords and phrases: A-normal operator, A-adjoint operator, A-numerical radius.

2010 Mathematics Subject Classification: 46C05, 47A05.

1. Introduction

Throughout this paper, $\mathcal{H}$ denotes a complex Hilbert space with the inner product $\langle ., .\rangle$ and $\mathcal{L}(\mathcal{H})$ is the Banach algebra of all bounded linear operators on $\mathcal{H}$, and A always stands for a positive linear operator in $\mathcal{L}(\mathcal{H})$.

We are going to consider an additional semi-inner product $\langle ., .\rangle_A$ on $\mathcal{H}$ given by $\langle h, k\rangle_A = \langle Ah, k\rangle$, $(h, k \in \mathcal{H})$ which defines a seminorm $\|.\|_A$ on $\mathcal{H}$. This makes $\mathcal{H}$ into a semi-Hilbertian space. Then we replace the operator norm with

$$\|T\|_A = \sup\{\|Th\|_A/\|h\|_A : \|h\|_A \neq 0\}.$$

The previously defined concepts about operators can be generalized using this seminorm. For some references, the reader can see [1], [2], [7], [5].

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Our purpose in this paper, is to study the relationship between the new concept of normality of operators, called A-*normality*, and the numerical radius of them, called the A-*numerical radius*, when we consider $\|.\|_A$ instead of the initial norm. A-normal operators are defined in [6], in which some results related to $\|.\|_A$ and A-numerical radius of them are obtained. A question posed in the mentioned paper, states that if $\|T\|_A$ equals the A-numerical radius of T, when T is an A-normal operator. In this direction, Section 2 is devoted to collect some facts about $\|.\|_A$ and the relevant concepts. It is well-known that the numerical radius of a normal operator on a Hilbert space equals its norm [3]. Similar to this fact, the last section is dedicated to proving the same result for operators defined on a semi-Hilbertian space.

2. Preliminaries

The positive operator $A \in \mathcal{L}(\mathcal{H})$ define a positive semi-definite sesquilinear form $\langle .,. \rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ given by $\langle h, k\rangle_A = \langle Ah, k\rangle$. Note that $\langle .,. \rangle_A$ defines a semi-inner product on $\mathcal{H}$, and the seminorm induced by it is given by

$$\|h\|_A = \langle h, h\rangle_A^{1/2}.$$

It is easily seen that $\|.\|_A$ is a norm on $\mathcal{H}$ if and only if A is injective.

The above seminorm induces a seminorm on the subspace $\mathcal{L}^A(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ consisting of all $T \in \mathcal{L}(\mathcal{H})$ so that for some $c > 0$ and all $h \in \mathcal{H}$, $\|Th\|_A \leq c\|h\|_A$. Indeed, if $T \in \mathcal{L}^A(\mathcal{H})$, then

$$\|T\|_A := \sup\left\{\frac{\|Th\|_A}{\|h\|_A} : h \in \overline{R(A)}, h \neq 0\right\} < \infty.$$

Operators in $\mathcal{L}^A(\mathcal{H})$ are called A-bounded operators.

Some equivalent statements for $\|.\|_A$ are summarized in the next two results.

Proposition 1. *If* $T \in \mathcal{L}^A(\mathcal{H})$, *then*

$$\begin{aligned}\|T\|_A &= \sup\{\|Th\|_A/\|h\|_A : h \notin kerA\} \\ &= \sup\{\|Th\|_A : \|h\|_A = 1\} \\ &= \sup\{|\langle Th, k\rangle_A| : h, k \in \mathcal{H}, \|h\|_A \leq 1,\ \|k\|_A \leq 1\}.\end{aligned}$$

(*The last equality is stated in* [7].)

Proof. Suppose that $h \notin kerA$. Then h can be represented as $h = h_1 + h_2$ in which $h_1 \in \overline{R(A)}$ and $h_2 \in N(A^{1/2})$. Note that $h_1 \neq 0$, thanks to the fact that $A^{1/2}h \neq 0$. Thus,

$$\frac{\|Th\|_A}{\|h\|_A} = \frac{\|A^{1/2}Th_1 + A^{1/2}Th_2\|}{\|A^{1/2}h_1 + A^{1/2}h_2\|}.$$

Since $kerA^{1/2}$ is an invariant subspace for T, $Th_2 \in kerA^{1/2}$, and so the above equality implies that

$$\frac{\|Th\|_A}{\|h\|_A} = \frac{\|Th_1\|_A}{\|h_1\|_A}.$$

Thus,

$$\sup\{\|Th\|_A/\|h\|_A : h \in \mathcal{H}, A^{1/2}h \neq 0\} = \|T\|_A.$$

The second inequality is obvious, and from which it can be easily deduced that

$$\|T\|_A = \sup\{\|Th\|_A : \|h\|_A \leq 1\}. \tag{1}$$

To prove the last equality, suppose that $h, k \in \mathcal{H}$ are so that $\|h\|_A \leq 1$ and $\|k\|_A \leq 1$. Then

$$|\langle Th, k\rangle| \leq \|T\|_A,$$

and so

$$\alpha := \sup\{|\langle Th, k\rangle_A| : h, k \in \mathcal{H}, \|h\|_A \leq 1, \|k\|_A \leq 1\} \leq \|T\|_A.$$

Now suppose that $\|h\| \leq 1$. For $\varepsilon > 0$, let $k = Th/(\|Th\|_A + \varepsilon)$. Then

$$|\langle Th, k\rangle_A| = \|Th\|/(\|Th\|_A + \varepsilon) \leq \alpha.$$

Now, letting $\varepsilon \longrightarrow 0$, we observe that $\|Tx\|_A \leq \alpha$ for each $h \in \mathcal{H}$ with $\|h\|_A \leq 1$. This, coupled with (1), shows that $\|T\|_A \leq \alpha$. Hence $\|T\|_A = \alpha$. ∎

Corollary 1. *If $T \in \mathcal{L}^A(\mathcal{H})$, then*

$$\|T\|_A = \sup\{|\langle Th, k\rangle_A| : h, k \in \mathcal{H}, \|h\|_A = \|k\|_A = 1\}.$$

Proof. Suppose that $\|h\|_A = 1$, and $\|Th\|_A \neq 0$. Then

$$\begin{aligned} \|Th\|_A &= |\langle Th, T(\frac{h}{\|Th\|_A})\rangle_A| \\ &\leq \sup\{|\langle Th, k\rangle| : \|k\|_A = 1\}. \end{aligned}$$

So, in light of Proposition 1, the result holds. ∎

An operator $S \in \mathcal{L}(\mathcal{H})$ is called an A-adjoint of an operator $T \in \mathcal{L}(\mathcal{H})$, if $\langle Th, k\rangle_A = \langle h, Sk\rangle_A$, for every $h, k \in \mathcal{H}$, or equivalently, $AT = S^*A$. If T is an A-adjoint of itself, then T is called an A-selfadjoint operator. It is possible that an operator T does not have an A-adjoint, and if S is an A-adjoint of T we may find many A-adjoints; in fact, if $AV = 0$ for some $V \in \mathcal{L}(\mathcal{H})$, then $S + V$ is an A-adjoint of T. The set of all A-bounded operators which admit an A-adjoint is denoted by $\mathcal{L}_A(\mathcal{H})$. By Douglas' theorem [4], $\mathcal{L}_A(\mathcal{H})$ consists of all operators T such that $R(T^*A) \subseteq R(A)$. If $T \in \mathcal{L}_A(\mathcal{H})$, the reduced solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which is denoted by $T^\sharp$. Note that $T^\sharp = A^\dagger T^* A$ in which $A^\dagger$ is the Moore-Penrose inverse of A. For more details see [1] and [2].

Definition 1. An operator $T \in \mathcal{L}_A(\mathcal{H})$ is an A-normal operator, if $T^\sharp T = TT^\sharp$.

3. A-numerical radius

The A-numerical radius of an operator $T \in \mathcal{L}(\mathcal{H})$, denoted by $w_A(T)$ is defined as

$$w_A(T) = \sup\{|\langle Th, h\rangle_A| : h \in \mathcal{H}, \|h\|_A = 1\}.$$

It is a generalization of the concept of numerical radius of an operator. Clearly, w_A defines a seminorm on $\mathcal{L}(\mathcal{H})$.

Furthermore, for every $h \in \mathcal{H}$,

$$|\langle Th, h\rangle_A| \le w_A(T).\|h\|_A{}^2.$$

Taking Corollary 1 into consideration, it is obvious that

$$w_A(T) \le \|T\|_A. \tag{2}$$

From the proof of part (2) of Corollary 3.2 of [6] we can derive the following lemma.

Lemma 1. *If $T \in \mathcal{L}_A(\mathcal{H})$ is an A-normal operator, then $\|T^n\|_A = \|T\|_A{}^n$ for each positive integer n.*

As it is known [3], for a normal operator T, $w(T) = \|T\|$, where $w(T)$ denotes the numerical radius of T.

In the next theorem, we establish a similar result for A-normal operators.

Theorem 1. *Suppose that $T \in \mathcal{L}_A(\mathcal{H})$ is an A-normal operator. Then*

$$\|T\|_A = w_A(T).$$

Proof. Take $h, k \in \mathcal{H}$ so that $\|h\|_A = \|k\|_A = 1$.

For an arbitrary operator $S \in \mathcal{L}_A(\mathcal{H})$,

$$\begin{aligned} |2\langle Sh, k\rangle_A + 2\langle Sk, h\rangle_A| &= |\langle S(h+k), h+k\rangle_A - \langle S(h-k), h-k\rangle_A| \\ &\le w_A(S)\|h+k\|_A{}^2 + w_A(S)\|h-k\|_A{}^2 \\ &= 2w_A(S)(\|h\|_A{}^2 + \|k\|_A{}^2) \\ &= 4w_A(S). \end{aligned}$$

If $\|Sh\|_A \ne 0$, substitute k by $\|Sh\|_A{}^{-1}Sh$ in the above computations to get

$$\frac{1}{\|Sh\|_A}|\langle Sh, Sh\rangle_A + \langle S^2h, h\rangle_A| \le 2w_A(S).$$

Consequently,

$$|\|Sh\|_A{}^2 + \langle S^2h, h\rangle_A| \le 2w_A(S)\|Sh\|_A. \tag{3}$$

Choose θ so that

$$e^{2i\theta}\langle T^2h, h\rangle_A = |\langle T^2h, h\rangle_A|.$$

Then, applying (3) for $S = e^{i\theta}T$ implies that

$$|\|Th\|_A{}^2 + \langle e^{2i\theta}T^2h, h\rangle_A| \leq 2w_A(T)\|Th\|_A.$$

Therefore,

$$\|Th\|_A{}^2 + |\langle T^2h, h\rangle_A| \leq 2w_A(T)\|Th\|_A, \tag{4}$$

which implies that

$$\|Th\|_A \leq 2w_A(T).$$

Note that this is true, even if $\|Th\|_A = 0$. Consequently,

$$\|T\|_A \leq 2w_A(T). \tag{5}$$

Besides, (4) shows that

$$\begin{aligned} 0 &\leq 2w_A(T)\|Th\|_A - \|Th\|_A{}^2 - |\langle T^2h, h\rangle_A| \\ &\leq w_A(T)^2 - |\langle T^2h, h\rangle_A|, \end{aligned}$$

and so

$$|\langle T^2h, h\rangle_A| \leq w_A(T)^2.$$

Hence

$$w_A(T^2) \leq w_A(T)^2.$$

Now using a mathematical induction, we observe that for every positive integer number k,

$$w_A(T^{2^k}) \leq w_A(T)^{2^k}.$$

This coupled with (5) and Lemma (1) shows that

$$\|T\|_A{}^{2^k} \leq 2w_A(T)^{2^k},$$

and so

$$\|T\|_A \leq \lim_{k\longrightarrow\infty} 2^{2^{-k}}w_A(T) = w_A(T).$$

Now, taking (2) into account, the equality holds. ∎

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Accepted: 06.05.2015