ON GENERALIZED WEAK I-LIFTING MODULES

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Abstract. In this paper, the concept of $\mathcal{I}$-lifting modules is extended to weak $\mathcal{I}$-lifting and generalized weak $\mathcal{I}$-lifting modules. Some properties of these modules are investigate and some results about $\mathcal{I}$-lifting modules are extended.

Keywords: lifting module; $\mathcal{I}$-lifting module; semiregular ring; weak $\mathcal{I}$-lifting module.

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1. Introduction

Throughout this paper, R will denote an arbitrary associative ring with identity, M a unitary right R-module and $S = End_R(M)$ the ring of all R-endomorphisms of M. We will use the notation $N \ll M$ to indicate that N is small in M (i.e., $\forall L \lneq M, L + N \neq M$); $N \leq^e M$ to indicate that N is an essential submodule of M (i.e., $\forall 0 \neq L \leq M, L \cap N \neq 0$). The notation $N \leq^\oplus M$ denotes that N is a direct summand of M. $N \trianglelefteq M$ means that N is a fully invariant submodule of M (i.e., $\forall \phi \in End_R(M),\ \phi(N) \subseteq N$). For all $I \subseteq S$, the left and right annihilators of I in S are denoted by $\ell_S(I)$ and $r_S(I)$, respectively. We also denote $r_M(I) = \{x \in M \mid Ix = 0\}$, for $I \subseteq S$; $\ell_S(N) = \{\phi \in S \mid \phi(N) = 0\}$, for $N \subseteq M$. $\Delta(M) = \{f \in S \mid Kerf \leq^e M\}$ and $Z(M) = \{x \in M \mid xI = 0$ for some essential right ideal I of R $\}$. A ring R is called a *semiregular* ring if for each $a \in R$, there exists $e^2 = e \in aR$ such that $(1 - e)a \in J(R)$ [7]. A ring R is called *weak semiregular* if, for any $a \in R$, there exists $0 \neq b \in R$ and an idempotent $g \in abR$ such that $(1 - g)ab \in J(R)$. A ring R is called *I-weak semiregular* if, for any $a \in R$, there exist $0 \neq b \in R$ and $e^2 = e \in abR$ such that $(1 - e)ab \in I$ [4].

Lifting modules play important roles in rings and categories of modules, and have been studied extensively by many authors in recent years (see for example, [1], [3], [6]). A module M is called *lifting* if for every $A \leq M$, there exists a direct summand B of M such that $B \subseteq A$ and $A/B \ll M/B$ [6].

In [1], we introduced $\mathcal{I}$-lifting modules as a generalization of lifting modules. Following [1], a module M is called *$\mathcal{I}$-lifting* if for every $\phi \in S$ there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq \mathrm{Im}\phi$ and $M_2 \cap \mathrm{Im}\phi \ll M_2$. It is obvious that every lifting module is $\mathcal{I}$-lifting while the converse in not true (the $\mathbb{Z}$-module $\mathbb{Q}$ is $\mathcal{I}$-lifting but it is not lifting). It is easily checked that R_R is an $\mathcal{I}$-lifting module if and only if R is a semiregular ring.

In this paper, we call a module M *weak $\mathcal{I}$-lifting* if, for every $\phi \in S$, there exists $0 \neq b \in S$ such that $\phi bM = eM \oplus N$, where $e^2 = e \in S$ and $N \ll M$. We give an example which shows that these modules are non-trivial generalization of $\mathcal{I}$-lifting modules. We call a module M *generalized weak $\mathcal{I}$-lifting* if, for any element $\phi \in S = End_R(M)$, there exist $0 \neq b \in S$ and a decomposition $r_M\ell_S(\phi b) = A \oplus B$ such that $A \subseteq \phi bM$ and $\phi bM \cap B \ll M$. In this note our aim is to investigate and study some properties of these modules.

In Section 2, we introduce weak $\mathcal{I}$-lifting and generalized weak $\mathcal{I}$-lifting modules. We give conditions under which a generalized weak $\mathcal{I}$-lifting module is weak $\mathcal{I}$-lifting (Proposition 2.7 and Corollary 2.2). We also prove the following:

Let $f_1+\cdots+f_n = 1$ in S, where f_i 's are orthogonal central idempotents. Then M is a generalized weak $\mathcal{I}$-lifting module if and only if each f_iM is generalized weak $\mathcal{I}$-lifting (see Corollary 2.4).

Let F be a submodule of an R-module M. A module M is called *F-$\mathcal{I}$-lifting* if for every $\phi \in S$ there exist a decomposition $M = A \oplus B$ such that $A \subseteq \mathrm{Im}\ \phi$ and $\mathrm{Im}\ \phi \cap B \leq F$. Let F be a submodule of an R-module M. A module M is called *weak F-$\mathcal{I}$-lifting* if for every $\phi \in S$ there exist $0 \neq b \in S$ and a decomposition $M = A \oplus B$ such that $A \subseteq \phi bM$ and $\phi bM \cap B \leq F$. These modules are non-trivial generalization of F-$\mathcal{I}$-lifting modules. In Section 3, we give a equivalent condition of semi-projective weak FI-$\mathcal{I}$-lifting modules. We also prove the following corollary (see Corollary 3.6):

Let M be a semi-projective retractable module. Then the following are equivalent:

(1) M is weak $Z(M)$-$\mathcal{I}$-lifting.

(2) S is weak $\Delta(M)$-semiregular.

(3) S is weak $Z_r(S)$-semiregular.

(4) M is weak $\Delta(M)M$-$\mathcal{I}$-lifting.

2. Generalized weak $\mathcal{I}$-lifting modules

Definition 2.1 Let M be a right R-module. M is called *weak $\mathcal{I}$-lifting* if, for every $\phi \in S$, there exists $0 \neq b \in S$ such that $\phi bM = eM \oplus N$, where $e^2 = e \in S$ and $N \ll M$.

It is clear that every $\mathcal{I}$-lifting module is weak $\mathcal{I}$-lifting. But the converse is not true as we see in the following example.

Example 2.2 Let $R = M = \{(x_1, x_2, ..., x_n, x, x, ...) \mid x_1, x_2, ..., x_n \in M_2(\mathbb{Z}_2),$ $x \in \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}\}$. Then M_R is a weak $\mathcal{I}$-lifting R-module but not $\mathcal{I}$-lifting.

A module M is called *semi-projective* if for any epimorphism $f : M \to N$, where N is a submodule of M, and for any homomorphism $g : M \to N$, there exists $h : M \to M$ such that $fh = g$.

Lemma 2.3 *Let M be a semi-projective module. Then the following are equivalent for an element $\phi \in S$:*

(1) *There exist $0 \neq b \in S$ and $e^2 = e \in \phi bS$ with $\phi bM \cap (1-e)M \ll M$.*

(2) *M is weak $\mathcal{I}$-lifting.*

Proof. (1) $\Rightarrow$ (2) Note that $\phi bM = eM \oplus [\phi bM \cap (1-e)M]$. The rest is clear.

(2) $\Rightarrow$ (1) By (2), there exists $0 \neq b \in S$ such that $\phi bM = eM \oplus N$ where $e^2 = e \in S$ and $N \ll M$. First we show that $e^2 = e \in \phi bS$. Consider the epimorphisms $\phi b : M \to \phi bM$ and $e : M \to eM$. Since M is semi-projective, there exists a homomorphism $g \in S$ such that $\phi bg = ie = e$, where $i : eM \to \phi bM$ is the inclusion map. Hence $e \in \phi bS$. Since $\phi bM = eM \oplus N$, $\phi bM \cap (1-e)M = N \cap (1-e)M \subseteq N$. As $N \ll M$, $\phi bM \cap (1-e)M \ll M$. ∎

Proposition 2.4 *Let M be a projective weak $\mathcal{I}$-lifting module. Then $Rad(M)$ is small in M.*

Proof. Let $N \subseteq M$ be any submodule with $N + Rad(M) = M$. If $g : M \to M/N$ is the natural map, then there exists $f : M \to Rad(M)$ with $gf = g$. Then $g = gf^2$. Note that if $f = 0$, then $g = 0$ and so $M = N$. Suppose that $f \neq 0$. Since M is weak $\mathcal{I}$-lifting, there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq \mathrm{Im}f^2$ and $M_2 \cap \mathrm{Im}f^2 \ll M_2$. Note that $M_1 \subseteq \mathrm{Im}f^2 \subseteq \mathrm{Im}f \subseteq Rad(M)$. By [9, 22.3], $M_1 = 0$ and so $\mathrm{Im}f^2 \ll M$. Hence $f^2 \in \nabla = Jac(S)$, thus $g = 0$ and so $N = M$. This shows that $Rad(M) \ll M$. ∎

Recall that a projective module is *semiperfect* if every homomorphic image has a projective cover [9].

Corollary 2.1 *If R is a semiperfect ring, then the following are equivalent for a projective R-module M:*

(1) *M is semiperfect;*

(2) *$End_R(M)$ is semiregular;*

(3) *$Rad(M) \ll M$;*

(4) *M is $\mathcal{I}$-lifting;*

(5) *M is weak $\mathcal{I}$-lifting.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) By [7, Corollary 3.7].

(2) $\Rightarrow$ (4) Since M is projective, it is well known that, $J(S) = \nabla(M)$ where $\nabla(M) = \{\alpha \in S \mid \mathrm{Im}\alpha \ll M\}$. Assume that $f \in S$, then there exists an idempotent $e \in S$ such that $eS \subseteq fS$ and $(1-e)fS \subseteq J(S) = \nabla(M)$. Therefore $M = eM \oplus (1-e)M$, $eM \subseteq fM$ and $(1-e)fM \ll M$. Hence M is $\mathcal{I}$-lifting.

(4) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (3) By Proposition 2.4. ∎

Definition 2.5 Let M be a right R-module. M is called *generalized weak $\mathcal{I}$-lifting* if, for every $\phi \in S$, there exist $0 \neq b \in S$ and a decomposition $r_M\ell_S(\phi b) = A \oplus B$ such that $A \subseteq \phi bM$ and $B \cap \phi bM \ll M$.

Proposition 2.6 *Let M be a weak $\mathcal{I}$-lifting module. Then M is generalized weak $\mathcal{I}$-lifting.*

Proof. Let $\phi \in S$. Then there exist $0 \neq b \in S$ and $e^2 = e \in \phi bS$ such t hat $\phi bM \cap (1-e)M \ll M$. Thus $M = eM \oplus (1-e)M$, where $eM \subseteq \phi bM$ and $\phi bM \cap (1-e)M \ll M$. Since $\phi bM \subseteq r_M\ell_S(\phi b)$, it follows by the modular law that $r_M\ell_S(\phi b) = r_M\ell_S(\phi b) \cap (eM \oplus (1-e)M) = eM \oplus (r_M\ell_S(\phi b) \cap (1-e)M)$ and $\phi bM \cap (r_M\ell_S(\phi b) \cap (1-e)M) = \phi bM \cap (1-e)M \ll M$. Hence M is generalized weak $\mathcal{I}$-lifting. ∎

Proposition 2.7 *Let M be a semi-projective generalized weak $\mathcal{I}$-lifting module with $RadM \ll M$. If there exists $e^2 = e \in S$ such that $\ell_S(\phi) = \ell_S(e)$ for any $\phi \in S$, then M is weak $\mathcal{I}$-lifting.*

Proof. Let $\phi \in S$. Then there exist $0 \neq b \in S$ and a decomposition $r_M\ell_S(\phi b) = A \oplus B$ such that $A \subseteq \phi bM$ and $B \cap \phi bM \ll M$. Since $\ell_S(\phi b) = \ell_S(e)$ where $e^2 = e \in S$, we have $r_M\ell_S(\phi b) = r_M\ell_S(e) = eM$ and so $eM = A \oplus B$. As A and B are direct summands of M, we can write $eM = fM \oplus gM$ for some $f^2 = f \in S$, $g^2 = g \in S$. By [9, 18.4], we get $Hom_R(M, eM) = Hom_R(M, fM) + Hom_R(M, gM)$. Since M is semi-projective, $eS = fS + gS$. As $A \cap B = 0$, $fS \cap gS = 0$. Thus $eS = fS \oplus gS$. Since $fM \subseteq \phi bM$ and $gM \cap \phi bM \ll M$, $fS \subseteq \phi bS$ and $gS \cap \phi bS \subseteq Hom_R(M, RadM)$. Since $\ell_S(\phi b) = \ell_S(e)$, $r_S\ell_S(\phi b) = r_S\ell_S(e) = eS$ and so $\phi b = e\phi b$. Let $e = \alpha + \beta$, where $\alpha = \phi bh \in fS$ and $\beta \in gS$. Then $\phi b = e\phi b = \phi bh\phi b + \beta\phi b$ and $\phi bh = \phi bh\phi bh + \beta\phi bh$. As $\phi bh - \phi bh\phi bh = \beta\phi bh \in gS \cap fS = 0$, ϕbh is an idempotent. Moreover, we have $(1-\phi bh)\phi b = \phi b - \phi bh\phi b = \beta\phi b \in gS \cap \phi bS \subseteq Hom_R(M, RadM)$, hence $(1-\phi bh)\phi bM \subseteq RadM \ll M$. Therefore M is weak $\mathcal{I}$-lifting. ∎

Corollary 2.2 *Let $r_M\ell_S(\phi b)$ is a direct summand of a semi-projective module M for any $\phi \in S$. If M is a generalized weak $\mathcal{I}$-lifting module with $RadM \ll M$, then M is weak $\mathcal{I}$-lifting.*

Proof. Let $\phi \in S$. By assumption, $r_M\ell_S(\phi b) = eM$ for some $e^2 = e \in S$. Then $\ell_S(\phi b) = \ell_S(e)$ and so M is weak $\mathcal{I}$-lifting by Proposition 2.7. ∎

A ring R is called *left Rickart* if for every $a \in R$ there exists an idempotent $e \in R$ such that $\ell_R(a) = Re$ [2].

Corollary 2.3 *Let S be a left Rickart ring. If M is a finitely generated semi-projective generalized weak $\mathcal{I}$-lifting module, then M is weak $\mathcal{I}$-lifting.*

Proposition 2.8 *Let e be a central idempotent of S. If M is a generalized weak $\mathcal{I}$-lifting module, then eM is generalized weak $\mathcal{I}$-lifting.*

Proof. Let $\phi \in End_R(eM) = eSe$. Then there exists $0 \neq b \in S$ such that $\phi b \neq 0$ and $r_M\ell_S(\phi b) = P \oplus L$ where $P \subseteq \phi bM$ and $L \cap \phi bM \ll M$. Note that $\phi b = e\phi b = \phi eb$ since $\phi \in eSe$. We claim that $r_{eM}\ell_{eSe}(\phi b) = eP \oplus eL$. Since $\phi \in eSe$, $1 - e \in \ell_S(\phi) \subseteq \ell_S(\phi b)$. Thus for every $t \in L$, we have $(1 - e)t = 0$, which implies that $eL = L$. Similarly, $eP = P$. Take any $y \in eP \subseteq e\phi bM$, where $y = ey_1$, $y_1 \in P \subseteq r_M\ell_S(\phi b)$. Then for every $\psi \in \ell_{eSe}(\phi b) \subseteq \ell_S(\phi b)$, $\psi y_1 = 0$. As $y_1 \in P \subseteq \phi bM$, $y_1 = \phi bm_1$ for some $m_1 \in M$. Thus we have $\psi y = \psi ey_1 = \psi e\phi bm_1 = \psi\phi bm_1 = \psi y_1 = 0$. Hence $y \in r_{eM}\ell_{eSe}(\phi b)$ and $eP \subseteq r_{eM}\ell_{eSe}(\phi b)$. Similarly, $eL \subseteq r_{eM}\ell_{eSe}(\phi b)$. On the other hand, let $x \in r_{eM}\ell_{eSe}(\phi b)$. Then for every $f \in \ell_S(\phi b)$, we have $efe\phi bM = ef\phi bM = 0$. Thus $efe \in \ell_{eSe}(\phi b)$ and so $efex = 0$ which gives $fx = fex = efex = 0$ since $x \in eM$. Hence $r_{eM}\ell_{eSe}(\phi b) \subseteq r_M\ell_S(\phi b)$. Take $x = x_1 + x_2$, where $x_1 \in P$ and $x_2 \in L$. Then $x = ex = ex_1 + ex_2 \in eP + eL$. This shows that $r_{eM}\ell_{eSe}(\phi b) = eP \oplus eL$. Since $eP \subseteq e\phi bM = \phi beM$, it is enough to show that $eL \cap \phi beM \ll eM$. Note that $eL \cap \phi beM = L \cap \phi beM \ll M$. Thus $eL \cap \phi beM \ll eM$ since $eM \leq^{\oplus} M$. Therefore eM is generalized weak $\mathcal{I}$-lifting. ∎

Theorem 2.9 *Let e and f be orthogonal central idempotents of S. If eM and fM are generalized weak $\mathcal{I}$-lifting modules, then $gM = eM \oplus fM$ is generalized weak $\mathcal{I}$-lifting.*

Proof. Let $\phi \in End_R(gM) \cong gSg = gS$. Then $e\phi \in eS$ and $f\phi \in fS$. By assumption, there exists $0 \neq b \in eS$ such that $e\phi b \neq 0$ and $r_{eM}\ell_{eS}(e\phi b) = P_e \oplus L_e$, where $P_e \subseteq e\phi beM = \phi beM$ and $\phi beM \cap L_e \ll eM$. Similarly, there exists $0 \neq t \in fS$ such that $f\phi t \neq 0$ and $r_{fM}\ell_f S(f\phi t) = P_f \oplus L_f$, where $P_f \subseteq \phi tfM$ and $L_f \cap \phi tfM \ll fM$. Note that $g = e + f$ is central idempotent and $\phi be + \phi tf = \phi(be + tf) \neq 0$. Let $h = be + tf \in gS$. Then $\phi h \neq 0$. We claim that $r_{gM}\ell_{gS}(\phi h) = P_e \oplus L_e \oplus P_f \oplus L_f$. Take any $x \in r_{gM}\ell_{gS}(\phi h)$. Then for any $\psi \in \ell_{eS}(e\phi b)$, we have $\psi e\phi b = 0$ and so $\psi\phi h = \psi\phi(be + tf) = \psi\phi be + \psi\phi tf = \psi\phi tf = \psi e\phi tf = 0$. Hence $g\psi\phi h = 0$ and $g\psi \in \ell_{gS}(\phi h)$. Thus $\psi(x) = g\psi(x) = 0$ and so $\psi ex = e\psi(x) = 0$, hence $ex \in r_{eM}\ell_{eS}(e\phi b) = P_e \oplus L_e$. Similarly, $fx \in r_{fM}\ell_{fS}(f\phi t) = P_f \oplus L_f$. Then $x = gx = ex + fx \in P_e \oplus P_f \oplus L_e \oplus L_f$ since e and f are orthogonal. Hence $r_{gM}\ell_{gS}(\phi h) \subseteq P_e \oplus L_e \oplus P_f \oplus L_f$. On the other hand, $P_e \oplus P_f \subseteq \phi beM \oplus \phi tfM \subseteq \phi hgM$. Let $x \in L_e$. For any $\psi \in \ell_{gS}(\phi h)$, we have $\psi\phi h = 0$, and so $e\psi e\phi h = e\psi\phi h = 0$. Thus $e\psi e\phi b = e\psi\phi b = 0$ and $e\psi \in \ell_{eS}(e\phi b)$. As $L_e \subseteq r_{eM}\ell_{eS}(e\phi b)$, $e\psi x = 0$. Note that $L_e \subseteq eM \subseteq gM$ and $\psi x = e\psi x = \psi ex = 0$. Hence $L_e \subseteq r_{gM}\ell_{gS}(\phi h)$. Similarly, $L_f \subseteq r_{gM}\ell_{gS}(\phi h)$. This shows that $r_{gM}\ell_{gS}(\phi h) = P_e \oplus P_f \oplus L_e \oplus L_f$. It is easily checked that $(L_e \oplus L_f) \cap \phi hgM \subseteq (L_e \oplus L_f) \cap (\phi beM + \phi tfM) \subseteq (L_e \cap \phi beM) \oplus (L_f \cap \phi tfM) \ll eM \oplus fM = gM$. Therefore gM is generalized weak $\mathcal{I}$-lifting. ∎

Corollary 2.4 *Let $f_1 + \cdots + f_n = 1$ in S, where f_i 's are orthogonal central idempotents. Then M is a generalized weak $\mathcal{I}$-lifting module if and only if each f_iM is generalized weak $\mathcal{I}$-lifting.*

A ring R is called *abelian* if every idempotent is central, that is, $ae = ea$ for any $a, e^2 = e \in R$.

Corollary 2.5 *If S is an abelian ring, then any finite direct sum of generalized weak $\mathcal{I}$-lifting modules is generalized weak $\mathcal{I}$-lifting.*

3. Weak F-$\mathcal{I}$-lifting modules

Definition 3.1 Let F be a submodule of an R-module M. A module M is called *weak F-$\mathcal{I}$-lifting* if for every $\phi \in S$ there exist $0 \neq b \in S$ and a decomposition $M = A \oplus B$ such that $A \subseteq \phi bM$ and $\phi bM \cap B \leq F$.

It is clear that every weak $\mathcal{I}$-lifting module is weak $Rad(M)$-$\mathcal{I}$-lifting.

The following example shows that weak F-$\mathcal{I}$-lifting modules need not be F-$\mathcal{I}$-lifting.

Example 3.2 Let C be a commutative Von Neumann regular ring with no minimal ideal and J a maximal ideal. Let $M_1 = \begin{pmatrix} C & C \\ J & C \end{pmatrix}$, $M_2 = Z_2[[X]]$ be the formal power series ring over Z_2, and $M = R = M_1 \oplus M_2$. Then M_R is a weak $Rad(M)$-$\mathcal{I}$-lifting module but not $Rad(M)$-$\mathcal{I}$-lifting.

Lemma 3.3 *Let F be a fully invariant submodule of a semiprojective module M. Then the following are equivalent for an element $\phi \in S$:*

(1) *There exist $0 \neq b \in S$ and $e^2 = e \in \phi bS$ with $\phi bM \cap (1-e)M \subseteq F$.*

(2) *M is weak F-$\mathcal{I}$-lifting.*

Proof. (1) $\Rightarrow$ (2) Note that $\phi bM = eM \oplus [\phi bM \cap (1-e)M]$. The rest is clear.

(2) $\Rightarrow$ (1) By (2), there exists $0 \neq b \in S$ such that $\phi bM = eM \oplus N$ where $e^2 = e \in S$ and $N \subseteq F$. First we show that $e^2 = e \in \phi bS$. Consider the epimorphisms $\phi b : M \to \phi bM$ and $e : M \to eM$. Since M is semi-projective, there exists a homomorphism $g \in S$ such that $\phi bg = ie = e$, where $i : eM \to \phi bM$ is the inclusion map. Hence $e \in \phi bS$. Since $\phi bM = eM \oplus N$, $\phi bM \cap (1-e)M = N \cap (1-e)M \subseteq N$. As $N \subseteq F$, $\phi bM \cap (1-e)M \subseteq F$. ■

An R-module M is called *retractable* if $Hom_R(M, N) \neq 0$ for all nonzero submodules N of M.

Lemma 3.4 *Let M be a semi-projective module. Consider the following conditions for $\phi \in S$:*

(1) *There exists $0 \neq b \in S$ such that $\phi bM = eM \oplus N$ where $e^2 = e \in S$ and N is a singular submodule of M.*

(2) *There exists $0 \neq b \in S$ such that $\phi bS = eS \oplus B$ where $e^2 = e \in S$ and $B \subseteq \Delta(M)$ is a right ideal of S.*

Then (1) $\Rightarrow$ (2) holds and, if moreover M is a retractable module, then (2) $\Rightarrow$ (1) holds.

Proof. (1) $\Rightarrow$ (2) Suppose that $\phi bM = eM \oplus N$ as in (1). First we show that $N = \phi bhM$ for some $h \in S$. Consider the homomorphism $\phi b : M \to \phi bM$. Since M is semi-projective, there exists a homomorphism $h : M \to M$ such that $\phi bh = i\pi\phi b$, where $i : N \to \phi bM$ and $\pi : \phi bM \to N$ are injection and projection maps respectively. Hence $\phi bhM = \pi(\phi bM) = N$. Now, by [9, 18.4], we have $Hom_R(M, \phi bM) = Hom(M, eM) + Hom(M, \phi bhM)$. Since M is semi-projective, $\phi bS = eS + \phi bhS$. As $eM \cap N = 0$, $eS \cap \phi bhS = Hom_R(M, eM) \cap Hom_R(M, \phi bhM) = Hom_R(M, eM \cap \phi bhM) = 0$. Thus $\phi bS = eS \oplus \phi bhS$. Finally, since $N = \phi bhM$ is singular and $\phi bhM \cong \frac{M}{\mathrm{Ker}_{\phi bh}}$, $\mathrm{Ker}\phi bh \leq^e M$ by [8, Lemma 2.1]. So $\phi bh \in \Delta(M)$.

(2) $\Rightarrow$ (1) Let $\phi bS = eS \oplus B$ as in (2). Clearly, $\phi bM = eM + BM$. Since $eS \cap B = 0$ and M is semi-projective, we have $Hom_R(M, eM) \cap Hom_R(M, BM) = 0$. Therefore, $Hom_R(M, eM \cap B) = 0$. Hence $eM \cap BM = 0$ by retractability. It follows that $\phi bM = eM \oplus BM$ and $BM \subseteq \Delta(M)M \subseteq Z(M)$. ∎

Corollary 3.6 *Let M be a semi-projective retractable module. Then the following are equivalent:*

(1) *M is weak $Z(M)$-$\mathcal{I}$-lifting.*

(2) *S is weak $\Delta(M)$-semiregular.*

(3) *S is weak $Z_r(S)$-semiregular.*

(4) *M is weak $\Delta(M)M$-$\mathcal{I}$-lifting.*

Proof. (1) $\Leftrightarrow$ (2) By Lemma 3.4. (2) $\Leftrightarrow$ (3) By [5, Proposition 2.4]. (2) $\Leftrightarrow$ (4) Similar to the proof of Lemma 3.4. ∎

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