A GENERALIZED COMMON FIXED POINT THEOREM FOR SIX SELF-MAPS

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Abstract. A recent common fixed point theorem of Kikina and Kikina (2011) has been extended to two triads of self-maps through the notions of weak compatibility and the property (EA), under an implicit-type relation and restricted completeness, namely orbital completeness of the space.

Keywords: property (EA), implicit relation, orbital completeness, weak compatibility, common fixed point.

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1. Introduction

Throughout this paper, (X, d) denotes a metric space. Given $x_0 \in X$ and f, g and h self-maps on X, the associated sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with the choice

$$x_{3n-2} = fx_{3n-3},\ x_{3n-1} = gx_{3n-2},\ x_{3n} = hx_{3n-1} \quad \text{for} \quad n = 1, 2, 3, ... \tag{1.1}$$

is an (f, g, h)-orbit at x_0. The metric space X is (f, g, h)-orbitally complete [6] if every Cauchy sequence in the (f, g, h)-orbit at each $x_0 \in X$ converges in X.

Kikina and Kikina [6] proved the following

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Theorem 1.1 *Let f, g and h be self-maps on X satisfying the three conditions:*

$$(1.2)\quad \begin{aligned}[1+pd(x,y)]d(fx,gy) &\le p[d(x,fx)d(y,gy)+d(x,gy)d(y,fx)]\\ &+q\max\left\{d(x,y),d(x,fx),d(y,gy),\frac{1}{2}[d(x,gy)+d(y,fx)]\right\},\end{aligned}$$

$$(1.3)\quad \begin{aligned}[1+pd(x,y)]d(gx,hy) &\le p[d(x,gx)d(y,hy)+d(x,hy)d(y,gx)]\\ &+q\max\left\{d(x,y),d(x,gx),d(y,hy),\frac{1}{2}[d(x,hy)+d(y,gx)]\right\},\end{aligned}$$

$$(1.4)\quad \begin{aligned}[1+pd(x,y)]d(hx,fy) &\le p[d(x,hx)d(y,fy)+d(x,fy)d(y,hx)]\\ &+q\max\left\{d(x,y),d(x,hx),d(y,fy),\frac{1}{2}[d(x,fy)+d(y,hx)]\right\}\end{aligned}$$

for all $x, y \in X$, *where*

$$p > -\frac{1}{\max\left\{d(x,y):\ x,y\in X\right\}} \quad \textit{with} \quad \max\{d(x,y):\ x,y\in X\} > 0$$

and $0 \le q < 1$. *If X is* (f,g,h)*-orbitally complete, then f, g and h will have a unique common fixed point.*

In this paper, we first extend the orbital completeness to two triads of self-maps and give an extended version of Theorem 1.1 using weak compatibility and property (EA) under generalized inequalities of (1.2)-(1.4) involving an implicit relation.

2. Preliminaries

Self-maps f and r on (X,d) are known to be commuting if $frx = rfx$ for all $x \in X$, where fr denotes the composition of f and r. As a weaker form of it, Sessa [14] introduced weakly commuting maps f and r on X with the choice:

$$(2.1)\quad d(frx, rfx) \le d(fx, rx) \quad \text{for all} \quad x \in X.$$

As a further weaker form of commuting mappings, Gerald Jungck [3] introduced the notion of compatibility as follows:

Definition 2.1 *Self-maps f and r on X are said to be compatible if there is a sequence* $\langle x_n\rangle_{n=1}^{\infty} \subset X$ *such that*

$$(2.2)\quad \lim_{n\to\infty} fx_n = \lim_{n\to\infty} rx_n = p \quad \textit{for some} \quad p \in X$$

implies that

$$(2.3)\quad \lim_{n\to\infty} d(frx_n, rfx_n) = 0.$$

It is obvious that every commuting pair is weakly commuting and every weakly commuting pair is compatible. However neither reverse implication is true. For examples, one can refer to [3] and [14].

We observe that both compatibility and noncompatibility of (f, r) guarantee the existence of a sequence $\langle x_n\rangle_{n=1}^{\infty} \subset X$ with the choice (2.2). Self-maps f and r on X satify the property (EA) [1] if (2.2) holds good for some $\langle x_n\rangle_{n=1}^{\infty} \subset X$. The class of compatible maps is contained in the class of *weakly compatible* maps [4] which commute at their coincidence points. However, weak compatibility and property (EA) are independent of each other [9]. Various form of compatibility and weak compatibility and their interrelation was presented in [2]. Weak compatibility have nice applications in dynamical programming (See [10]).

The idea of contraction type conditions involving an *implicit relation* was first introduced by Popa [13] which covers several contractive conditions and has the ability to unify several fixed point theorems.

For instance, [5] utilized one such implicit relation $\psi : \mathbb{R}_+^6 \to \mathbb{R}$ which is lower semicontinuous and satisfies the following conditions:

(C_1) ψ is nondecreasing in the fifth and sixth coordinate variables,

(C_2) For every $l \geq 0, m \geq 0$, there is a constant $0 \leq \omega < 1$ such that

$$\min\{\psi(l, m, m, l, l + m, 0), \psi(l, m, l, m, 0, l + m)\} \leq 0 \;\Rightarrow\; l \leq \omega m, \tag{2.4}$$

(C_3) $\psi(l, l, 0, 0, l, l) > 0$ for all $l > 0$.

In this paper, we first extend the property (EA) and orbital completeness to two triads of self-maps and then obtain an extended generalization of Theorem 1.1 involving weak compatibility and an implicit-type relation which does not require the choice (C_1), unlike in [5].

3. Main result

We begin with orbital completeness involving two triads of self-maps as follows:

Given $x_0 \in X$ and two triads (f, g, h) and (r, s, t) of self-maps on X, if there exist points $x_1, x_2, x_3, \ldots$ in X such that

$$\begin{aligned} y_{3n-2} = fx_{3n-3} = rx_{3n-2},\; y_{3n-1} = gx_{3n-2} = sx_{3n-1}, \\ y_{3n} = hx_{3n-1} = tx_{3n} \quad \text{for } n = 1, 2, 3, \ldots, \end{aligned} \tag{3.1}$$

then the associated sequence $\langle y_n\rangle_{n=1}^{\infty}$ is an (f, g, h)-orbit relative (r, s, t) at x_0.

The space X is (f, g, h)-orbitally complete at x_0 relative (r, s, t) if every Cauchy sequence in an (f, g, h)-orbit at x_0 relative (r, s, t) converges in X, and X is (f, g, h)-orbitally complete relative (r, s, t) if it is (f, g, h)-orbitally complete relative to (r, s, t) at each $x_0 \in X$.

It may be noted that if $r = s = t = i$, the identity map, then we get the orbital completeness as discussed in [6].

Similarly, if $g = h = f$ and $t = s = r$ in this notion, we get the f-orbital completeness relative to r as given in [11].

The property (EA) was extended to two pairs of self-maps in [7]. In fact, self-maps (f, r) and (g, s) share the common property (EA) on X if there exist sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ in X such that

$$(3.2) \qquad \lim_{n\to\infty} fx_n = \lim_{n\to\infty} rx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} sy_n = u \quad \text{for some} \quad u \in X.$$

We improve the notion of [7] to a pair of triads of maps as follows:

Let f, g, h, r, s and t be self-maps on X. We say that the triads (f, g, h) and (r, s, t) share the common property (EA) if we can find sequences $\langle x_n \rangle_{n=1}^{\infty}$, $\langle y_n \rangle_{n=1}^{\infty}$ and $\langle z_n \rangle_{n=1}^{\infty}$ in X such that

$$(3.3) \qquad \begin{aligned} \lim_{n\to\infty} fx_n &= \lim_{n\to\infty} gy_n = \lim_{n\to\infty} hz_n = \lim_{n\to\infty} rx_n \\ &= \lim_{n\to\infty} sy_n = \lim_{n\to\infty} tz_n = u \quad \text{for some } u \in X. \end{aligned}$$

If $r = s = t = i$ in this notion, we get the property (EA) on the collection $\{f, g, h\}$ as given in [8].

Our main result is

Theorem 3.1 *Let f,g, h, r, s and t be self-maps on X sharing the common property (EA) and satisfying the following inequalities:*

$$(3.4) \quad \psi(d(fx, gy), d(rx, sy), d(rx, fx), d(sy, gy), d(rx, gy), d(sy, fx)) < 0,$$

$$(3.5) \quad \psi(d(gx, hy), d(sx, ty), d(sx, gx), d(ty, hy), d(sx, hy), d(ty, gx)) < 0,$$

$$(3.6) \quad \psi(d(hx, fy), d(tx, ry), d(tx, hx), d(ry, fy), d(tx, fy), d(ry, hx)) < 0,$$

for all $x, y \in X$. Suppose that r, s and t are onto. If (f, r), (g, s) and (h, t) are weakly compatible, then all the six maps f, g, h, r, s and t will have a unique common fixed point.

Proof. Suppose f, g, h, r, s and t share the common property. Then we can find sequences $\langle x_n \rangle_{n=1}^{\infty}$, $\langle y_n \rangle_{n=1}^{\infty}$ and $\langle z_n \rangle_{n=1}^{\infty}$ in X with the choice (3.3). Since r is onto, we have

$$(3.7) \qquad u = rp \quad \text{for some} \quad p \in X.$$

Writing $x = p$ and $y = x_n$ in (3.4), we get

$$\psi(d(fp, gx_n), d(rp, sx_n), d(rp, fp), d(sx_n, gx_n), d(rp, gx_n), d(sx_n, fp)) < 0.$$

Applying the limit as $n \to \infty$ and then using (3.3) and (3.7), this yields

$$\psi(d(fp, rp), 0, d(rp, fp), 0, 0, d(rp, fp)) < 0,$$

which is (2.4) with $l = d(fp, rp)$ and $m = 0$. So, by (C_2) we get $d(fp, rp) \leq 0$, that is $fp = rp = u$. Since (f, r) is weakly compatible, f and r commute at the coincidence point p, that is $frp = rfp$, we get

$$fu = ru. \tag{3.8}$$

Now s is onto implies that

$$u = sq \quad \text{for some} \quad q \in X. \tag{3.9}$$

Then writing $x = q$ and $y = y_n$ in (3.5), we get

$$\psi(d(gq, hy_n), d(sq, ty_n), d(sq, gq), d(ty_n, hy_n), d(sq, hy_n), d(ty_n, gq)) < 0.$$

Applying the limit as $n \to \infty$ and then using (3.3) and (3.9), this yields

$$\psi(d(gq, sq), 0, d(sq, gq), 0, 0, d(sq, gq)) < 0,$$

which is (2.4) with $l = d(gq, sq)$ and $m = 0$. So, by (C_2) we get $d(gq, sq) \leq 0$, that is $gq = sq = u$. By the weak compatibility of, g and s, we get $gsq = sgq$ or

$$gu = su. \tag{3.10}$$

Finally, t is onto implies that

$$u = tw \quad \text{for some} \quad w \in X. \tag{3.11}$$

Then (3.6) with $x = w$ and and $y = w_n$ gives

$$\psi(d(hw, fz_n), d(tw, rz_n), d(tw, hw), d(rz_n, fz_n), d(tw, fz_n), d(rz_n, hw)) < 0.$$

Applying the limit as $n \to \infty$ and then using (3.3) and (3.11), this yields

$$\psi(d(hw, tw), 0, d(tw, hw), 0, 0, d(tw, hw)) < 0.$$

This in view of (C_2) with $l = d(tw, hw)$ and $m = 0$ finally gives $d(tw, hw) \leq 0$, that is $tw = hw = u$. By the weak compatibility of, h and t, we get $thw = htw$ or

$$tu = hu. \tag{3.12}$$

But from (3.4) with $x = y = u$, it follows that

$$\psi(d(fu, gu), d(fu, gu), 0, 0, d(fu, gu), d(gu, fu)) < 0,$$

which is nothing but (2.4) with $l = d(fu, gu)$ and $m = 0$. Hence by (C_2), we get $d(fu, gu) \leq 0$ so that $fu = gu$.

On one hand, from (3.5) with $x = u = y$ it follows that

$$\psi(d(gu, hu), d(ru, ru), d(ru, gu), d(ru, hu), d(ru, hu), d(ru, gu)) < 0,$$

or that $\psi(d(gu,hu),0,0,d(fu,hu),d(ru,hu),0)<0$.

If $d(gu,hu)>0$, from (C_3), we see a contradiction that

$$\psi(d(gu,hu),0,0,d(fu,hu),d(ru,hu),0)>0.$$

Hence we must have $fu=gu$.

Now, writing $x=y=u$ in (3.5) and then using (3.8), (3.10), (3.12) and $fu=gu$, we get

$$\psi(d(gu,hu),d(gu,hu),0,0,d(gu,hu),d(hu,gu))<0$$

which, due to (C_3), implies that $gu=hu$.

In other words, u is a common coincidence point of all the six maps, that is

$$fu=gu=hu=ru=su=tu. \tag{3.13}$$

Further, we see from (3.4) with $x=u$ and $y=x_n$ becomes

$$\psi(d(fu,gx_n),d(ru,rx_n),d(ru,fu),d(rx_n,gx_n),d(ru,gx_n),d(rx_n,fu))<0.$$

Proceeding the limit as $n\to\infty$, in this and using (3.13), we obtain

$$\psi(d(fu,u),d(fu,u),0,0,d(fu,u),d(fu,u))<0, \tag{3.14}$$

which would be a contradiction to $(C3)$ if $d(fu,u)>0$, proving that u is a fixed point of f and hence a common fixed point of f,g,h and r. ■

Remark 3.1 Writing $s=t=r$ in this main result, recently it has been shown in [15] that any two of the three inequalities (3.4)-(3.6) are sufficient to obtain a common fixed point for the four self-maps f,g,h and r, using the weak compatibility of one of the three pairs $(f,r),(g,r)$ and (h,r).

Now we write

$$\psi(l_1,l_2,l_3,l_4,l_5,l_6)=(1+pl_2)l_1-p[l_3l_4+l_5l_6]-q\max\left\{l_2,l_3,l_4,\frac{1}{2}[l_5+l_6]\}\right\},$$

where p and q have the same choice as given in Theorem 1.1.

Corollary 3.1 *Let f, g, h, r, s and t be self-maps on X satisfying following conditions:*

$$\begin{aligned}[1+pd(rx,sy)]d(fx,gy)&\le p[d(rx,fx)d(sy,gy)+d(rx,gy)d(sy,fx)]\\&+q\max\{d(rx,sy),d(rx,fx),d(sy,gy),\frac{1}{2}[d(rx,gy)+d(sy,fx)]\},\end{aligned} \tag{3.15}$$

$$\begin{aligned}[1+pd(sx,ty)]d(gx,hy)&\le p[d(sx,gx)d(ty,hy)+d(sx,hy)d(ty,gx)]\\&+q\max\{d(sx,ty),d(sx,gx),d(ty,hy),\frac{1}{2}[d(sx,hy)+d(ty,gx)]\},\end{aligned} \tag{3.16}$$

$$\begin{aligned}(3.17)\quad &[1+pd(tx,ry)]d(hx,fy)\le p[d(tx,hx)d(ry,fy)+d(tx,fy)d(ry,hx)]\\ &+q\max\{d(tx,ry),d(tx,hx),d(ry,fy),\frac{1}{2}[d(tx,fy)+d(ry,hx)]\}\end{aligned}$$

for all $x, y \in X$*, where*

$$p > -\frac{1}{\max\{d(x,y) : x,y\in X\}}$$

with $\max\{d(x,y) : x,y \in X\} > 0$ *and* $0 \le q < 1$*. Suppose that*

$$f(X) \subset r(X), g(X) \subset s(X), h(X) \subset t(X). \tag{3.18}$$

Suppose that X *is* (f, g, h)*-orbitally complete relative to* (r, s, t)*, and* r, s *and* t *are onto. If* (f, r)*,* (g, s) *and* (h, t) *are weakly compatible, then all the six maps* f*,* g*,* h*,* r*,* s *and* t *will have a unique common fixed point.*

Proof. Let $p_0 \in X$. In view of (3.18), there exists an (f, g, h)-orbit relative (r, s, t) at p_0 with the choice (3.1). By a routine iteration procedure just similar to that of Theorem 1.1, one can prove that (f, g, h)-orbit relative (r, s, t) at p_0 is Cauchy, and hence converges to some $u \in X$, since X is (f, g, h)-orbitally complete at p_0. That is

$$\begin{aligned}\lim_{n\to\infty} fp_{3n-3} &= \lim_{n\to\infty} gp_{3n-2} = \lim_{n\to\infty} hp_{3n-1} = \lim_{n\to\infty} rp_{3n-2}\\ &= \lim_{n\to\infty} sp_{3n-1} = \lim_{n\to\infty} tp_{3n} = u \quad \text{for some } u\in X.\end{aligned}$$

Then, using the inequalities (3.15)-(3.17), it is not difficult to see that

$$\begin{aligned}(3.19)\quad \lim_{n\to\infty} fp_{3n-3} &= \lim_{n\to\infty} rp_{3n-3} = \lim_{n\to\infty} gp_{3n-2} = \lim_{n\to\infty} sp_{3n-2}\\ &= \lim_{n\to\infty} hp_{3n-1} = \lim_{n\to\infty} tp_{3n-1} = u \text{ for some } u\in X.\end{aligned}$$

Here we note that (3.19) is a special case of (3.3) with $x_n = p_{3n-3}, y_n = p_{3n-2}$ and $z_n = x_{3n-1}$. Therefore, the common fixed point follows from Theorem 3.1. ■

When $r = s = t = I$, the identity map on X in Corollary 3.1, we find that (1.2), (1.3) and (1.4)are particular cases of the implicit relations of Theorem 3.1. Also it is well-known that I is onto and commutes with evrey map and hence is weakly compatible with the maps f, g and h, and hence a common fixed point follows directly from Theorem 3.1. Thus Theorem 1.1 is a stronger version of Corollary 3.1.

Finally, writing $h = g = f$ and $r = s = t$ in Theorem 3.1, we get a particular case of each of (3.4)-(3.6) as

$$\psi(d(fx,fy),d(rx,ry),d(rx,fx),d(ry,fy),d(rx,fy),d(ry,fx)) < 0 \tag{3.20}$$

for all $x, y \in X$. Also the space X reduces to *f-orbitly complete* relative to r in the sense that every Cauchy sequence in the (f, r)-orbit $O_{f,r}(x_0)$ at each x_0 converges in X, where $O_{f,r}(x_0)$ has the choice

$$fx_{n-1} = rx_n \quad \text{for} \quad n = 1, 2, 3, ... \tag{3.21}$$

Then we have

Corollary 3.2 *Let f and r be self-maps on X satisfying the property E. A. and the inequality* (3.20)*. If $r(X)$ is f-orbitally complete relative to r, then f and r will have a coincidence point. Further if (f, r) is weakly compatible, then f and r will have a unique common fixed point.*

The following example shows that f-orbital completeness of $r(X)$ is necessary in Corollary 3.2:

Example 3.1 Let

$$\psi(l_1, l_2, l_3, l_4, l_5, l_6) = l_1^2 - al_2^2 - \frac{bl_5l_6}{l_3^2 + l_4^2 + 1},$$

where $a = 1/2$ and $b = 1/4$. Set $X = \left\{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ...\right\}$ with the usual metric d. Define $f, r : X \to X$ by

$$f0 = \frac{1}{2^2}, f\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^{n+1}} \quad \text{and} \quad r0 = \frac{1}{2}, r\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^n},$$

for $n = 1, 2, 3, ...$ Then (f, r) satisfies the property E.A. and

$$r(X) = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ...\right\}.$$

For $x_0 = 0$, choose $x_1 = \frac{1}{2}, x_2 = \frac{1}{2^2}, x_3 = \frac{1}{2^3}, ...$ so that

$$O_{f,r}(x_0) = \left\{\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, ...\right\}$$

while for $x_0 = \frac{1}{2^{n-1}}$, we have

$$O_{f,r}(x_0) = \left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, ...\right\}$$

for each $n = 1, 2, 3, ...$ In either case, $O_{f,r}(x_0)$ converges to $0 \notin r(X)$.

Thus $r(X)$ is not orbitally complete at each x_0. As such the maps f and r do not have a coincidence point, though X is complete.

Since every complete metric space is orbitally complete at each of its points, we have

Corollary 3.3 (Theorem 3.1, [5]) *Let f and r be self-maps on X satisfying the property E. A. and the inequality* (3.20)*. If $r(X)$ is complete, then f and r will have a coincidence point. Further, f and r will have a unique common fixed point, provided (f, r) is weakly compatible.*

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