SYNTACTIC FUZZY MONOIDS

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Abstract. Fuzzy monoids and fuzzy congruences are introduced and the syntactic fuzzy monoid M_L associated to a subset L of a fuzzy monoid M is constructed. It is shown that M_L is minimal among all fuzzy epimorphisms $h : M \to M'$ whose kernel saturates L. The subset L is said to be fuzzy recognizable whenever M_L is finite. The so obtained class is closed under boolean operations and inverse morphisms.

Keywords: fuzzy sets, recognizability, monoid.

1. Introduction

Traditionally, a fuzzy subset F of a semigroup S is called a fuzzy subsemigroup of S if

$$F(ab) \geq F(a) \wedge F(b),$$

for all $a, b \in S$ [6], [7]. On the other hand, a nondeterministic semigroup is a set S endowed with an associative nondeterministic operation $M \times M \to \mathcal{P}(M)$ [1], [4], [5]. In the present paper we extend nondeterminism to the fuzzy setup. Thus a fuzzy semigroup in our sense is a set S equipped with an associative fuzzy operation $S \times S \to Fuz(S)$, which to every pair (s_1, s_2) of elements of S assigns a fuzzy subset $s_1 s_2$ of S.

Hence in the first approach in order to obtain a fuzzy algebraic structure one takes a fuzzy subset of a crisp algebraic structure while in our approach we construct a fuzzy structure by means of a fuzzy operation. Moreover, traditionally a fuzzy congruence on a crisp monoid S is a fuzzy equivalence $R : S \times S \to [0, 1]$ satisfying

$$R(sa, sb) \geq R(ab), \quad R(as, bs) \geq R(ab).$$

for all $a, b \in S$. Whereas, a congruence in our sense, is an equivalence $\sim$ on a fuzzy semigroup S such that $s_1 \sim s'_1$ and $s_2 \sim s'_2$ imply

$$\sup_{n \in C}, (s_1 s_2)(n) = \sup_{n \in C}(s'_1 s'_2)(n)$$

for all $\sim$-classes C. It is clear that the proposed fuzzy monoid notion significantly diverges from the traditional and that it fuzzifies the nondeterministic case.

Naturally, in the proposed setup, a fuzzy monoid M is a fuzzy semigroup together with a unit element. The quotient $M/\sim$ of a fuzzy monoid M by a fuzzy congruence $\sim$, admits a fuzzy monoid structure rendering the canonical function $m \mapsto [m]$ an epimorphism of fuzzy monoids. The classical Isomorphism Theorem of Algebra still holds in the fuzzy setup.

> For any epimorphism of fuzzy monoids $h : M \to M'$ and every fuzzy congruence $\sim$ on M' its inverse image $h^{-1}(\sim)$ defined by
>
> $$m_1 h^{-1}(\sim) m_2 \quad \text{iff} \quad h(m_1) \sim h(m_2),$$
>
> is again a fuzzy congruence and the quotient fuzzy monoids $M/h^{-1}(\sim)$ and $M'/\sim$ are isomorphic. In particular if $\sim$ is the equality, then $h^{-1}(=)$ is the kernel congruence of h (denoted by $\sim_h$)
>
> $$m_1 \sim_h m_2 \quad \text{iff} \quad h(m_1) = h(m_2),$$
>
> and the fuzzy monoids $M/\sim_h$ and M' are isomorphic.

We show that fuzzy congruences are closed under the join operation. This allows us to construct the greatest fuzzy congruence included in an equivalence $\sim$. It is the join of all fuzzy congruences on M included into $\sim$ and it is denoted by $\sim^{fuz}$. The quotient fuzzy monoid $M/\sim^{fuz}$ is denoted by M^{fuz} and has the following universal property.

> Given an epimorphism of fuzzy monoids $h : M \to M'$ whose kernel $\sim_h$ saturates the equivalence $\sim$ there exists a unique epimorphism of fuzzy monoids $h' : M' \to M^{fuz}$ such that $h' \circ h = h^{fuz}$, where $h^{fuz} : M \to M^{fuz}$ is the canonical epimorphism into the quotient.

This result states that h^{fuz} is minimal among all epimorphisms saturating $\sim$. Let M be a fuzzy monoid and $L \subseteq M$. Denote by $\sim_L$ the greatest congruence of M included in the partition (equivalence) $\{L, M-L\}$, i.e., $\sim_L = \{L, M-L\}^{fuz}$. The quotient fuzzy monoid $M_L = M/\sim_L$ will be called the syntactic fuzzy monoid of L and it is characterized by the following universal property.

> For every fuzzy monoid M and every epimorphism $h : M \to M'$ verifying $h^{-1}(h(L)) = L$, there exists a unique epimorphism $h' : M' \to M_L$ such that $h' \circ h = h_L$ where $h_L : M \to M_L$ is the canonical projection into the quotient.

A subset L of a fuzzy monoid M is fuzzy recognizable if there exist a finite fuzzy monoid M' and a morphism $h : M \to M'$ such that $h^{-1}(h(L)) = L$. By taking into account the previous result we get that L is recognizable if and only if its syntactic fuzzy monoid is finite. Moreover fuzzy recognizable subsets are closed under boolean operations and inverse morphisms.

2. Fuzzy monoids

We first recall that for every $a \in [0,1]$ and every fuzzy set $F : M \to [0,1]$, the fuzzy set $a \wedge F : M \to [0,1]$ is defined by

$$(a \wedge F)(m) = a \wedge F(m),$$

for all $m \in M$. It is easy to verify the rules

$$(a_1 \wedge a_2) \wedge F = a_1 \wedge (a_2 \wedge F), \quad a \wedge (F_1 \wedge F_2) = (a \wedge F_1) \wedge F_2$$

for all $a_1, a_2 \in [0,1]$ and all fuzzy sets F, F_1, F_2 of M.

Every fuzzy set $F : M \to [0,1]$ is written

$$F = \sup_{m \in M} F(m) \wedge \hat{m}$$

where $\hat{m} : M \to [0,1]$ denotes the singleton function: $\hat{m}(\eta) = 1$, if $\eta = m$ and $\hat{m}(\eta) = 0$ if $\eta \neq m$. Indeed, for every $\eta \in M$ we have

$$\left(\sup_{m \in M} F(m) \wedge \hat{m}\right)(\eta) = \sup_{m \in M} F(m) \wedge \hat{m}(\eta) = F(\eta) \wedge \hat{\eta}(\eta) = F(\eta) \wedge 1 = F(m)$$

Convention 1 *From now on, we identify $\hat{m}$ with the element m itself.*

First recall that a non deterministic operation on a set M is a function

$$\triangle : M \times M \to \mathcal{P}(M) = \{0,1\}^M$$

which to every pair (m_1, m_2) of elements of M assigns a subset $m_1 \triangle m_2$ of m [1], [2], [3], [4].

More generally, a fuzzy operation on a set M is a function

$$\triangle : M \times M \to \mathit{Fuzzy}(M) = [0,1]^M$$

which to every pair (m_1, m_2) of elements of M assigns a fuzzy subset

$$m_1 \triangle m_2 : M \to [0,1].$$

The operation $\triangle$ is extended to the fuzzy subsets of M via the formula

$$F_1 \triangle F_2 = \sup_{m_1, m_2 \in M} F_1(m_1) \wedge F_2(m_2) \wedge (m_1 \triangle m_2) \tag{2.1}$$

for all $F_1, F_2 : M \to [0,1]$.

An operation $\triangle$ is associative whenever

$$m_1 \triangle (m_2 \triangle m_3) = (m_1 \triangle m_2) \triangle m_3, \tag{2.2}$$

for all $m_1, m_2, m_3 \in M$. Let us analyze the above equality, by taking into account (2.1) the left hand side member of (2.2) is written

$$\begin{aligned} m_1 \triangle (m_2 \triangle m_3) &= \sup_{\eta_1, \eta_2 \in M} m_1(n_1) \wedge (m_2 \triangle m_3)(\eta_2) \wedge (\eta_1 \triangle \eta_2) \\ &= \sup_{\eta \in M} (m_2 \triangle m_3)(\eta) \wedge (m_1 \triangle \eta) \end{aligned}$$

and, similarly,

$$\begin{aligned}(m_1\triangle m_2)\triangle m_3 &= \sup_{\eta_1,\eta_2\in M} (m_1\triangle m_2)(n_1)\wedge m_3(\eta_2)\wedge(\eta_1\triangle\eta_2)\\ &= \sup_{\eta\in M}(m_1\triangle m_2)(\eta)\wedge(\eta\triangle m_3).\end{aligned}$$

Thus associativity is explicitly expressed as

$$\sup_{\eta\in M}(m_2\triangle m_3)(\eta)\wedge(m_1\triangle\eta) = \sup_{\eta\in M}(m_1\triangle m_2)(\eta)\wedge(\eta\triangle m_3).$$

Remark 1 Other variants of associativity can be also formulated but they are unsuitable for the syntactic theory treated in this paper.

A fuzzy monoid is a set M equipped with an associative fuzzy operation $\triangle$ which admits a neutral element $e\in M$

$$m\triangle\eta = m = \eta\triangle m,\quad \text{for all } m\in M.$$

Example 1 Any ordinary monoid can be viewed as a fuzzy monoid.

Example 2 A nondeterministic monoid (ND-monoid) is a set M equipped with a nondeterministic operation which is associative and unitary. Every ND-monoid can be viewed as a fuzzy monoid in the obvious way.

In the present study it is necessary to have a congruence notion. First let us remark that every equivalence relation $\sim$ on a set M induces an equivalence relation $\approx$ on $Fuz(M)$ as follows: for all $F_1, F_2\in Fuz(M)$

$$F_1\approx F_2 \quad\text{iff}\quad \sup_{m\in C}F_1(m) = \sup_{m\in C}F_2(m),$$

for all $\sim$-classes C. An equivalence relation $\sim$ on M is termed a left (resp. right) fuzzy congruence if

$$\begin{aligned}m_1\sim m_2,&\quad\text{implies}\quad m_1m\approx m_2m,\quad\text{for all } m\in M\\ (\text{resp. } m_1\sim m_2,&\quad\text{implies}\quad mm_1\approx mm_2,\quad\text{for all } m\in M).\end{aligned}$$

A left and right fuzzy congruence is called a fuzzy congruence on M.

Lemma 1 *The equivalence $\sim$ on M is a fuzzy congruence on M if and only if*

$$m_1\sim m_1', m_2\sim m_2',\quad\textit{implies}\quad m_1m_2\approx m_1'm_2'.$$

Proof. One direction is clear. Conversely, assume that $m_1\sim m_1'$ and $m_2\sim m_2'$. Then

$$m_1m_2\approx m_1'm_2\approx m_1'm_2'.$$ ∎

The condition of the above lemma can be explicitly described in the following way: $m_1\sim m_1'$ and $m_2\sim m_2'$ imply

$$\sup_{m\in C}(m_1m_2)(m) = \sup_{m\in C}(m_1'm_2')(m)$$

for all $\sim$-classes C.

Proposition 1 *The quotient set $M/\sim$ is structured into a fuzzy monoid by defining its fuzzy multiplication via the formula*

$$([m_1][m_2])([n]) = \sup_{m\in[n]} (m_1m_2)(m).$$

Proof. First observe that the above multiplication is well defined. Next for every $\sim$-class $[b]$ we have

$$\begin{aligned}(([m_1][m_2])\,[m_3])\,([b]) &= \sup_{[n]\in M/\sim} ([m_1][m_2])([n]) \wedge ([n][m_3])([b]) \\ &= \sup_{[n]\in M/\sim}\ \sup_{n_1\in[n]} (m_1m_2)(n_1) \wedge \Big(\sup_{b'\in[b]} (nm_3)(b')\Big)\end{aligned}$$

since $n \sim n_1$ we get

$$\begin{aligned}&= \sup_{[n]\in M/\sim}\ \sup_{n_1\in[n]} (m_1m_2)(n_1) \wedge \Big(\sup_{b'\in[b]} sup(n_1m_3)(b')\Big) \\ &= \sup_{[n]\in M/\sim}\ \sup_{b'\in[b]}\ \sup_{n_1\in[n]} (m_1m_2)(n_1) \wedge \big((n_1m_3)(b')\big) \\ &= \sup_{b'\in[b]}\ \sup_{n_1\in M} (m_1m_2)(n_1) \wedge \big((n_1m_3)(b')\big)\end{aligned}$$

and by taking into account the associativity of M we obtain

$$\begin{aligned}&= \sup_{b'\in[b]}\ \sup_{n_1\in M} (m_2m_3)(n_1) \wedge \big((m_1n_1)(b')\big) \\ &= ([m_1]([m_2][m_3]))([b]).\end{aligned}$$

Example 3 Congruences on an ordinary monoid M coincide with fuzzy congruences when M is viewed as a fuzzy monoid. On the other hand an equivalence $\sim$ on an ND-monoid M is a congruence whenever $m_1 \sim m_1'$ and $m_2 \sim m_2'$ imply

$$m_1m_2 \cap C \neq \emptyset \Rightarrow m_1'm_2' \cap C \neq \emptyset.$$

for all $\sim$-classes C. The quotient set $M/\sim$ is structured into an ND-monoid by means of the operation

$$[m_1][m_2] = \{C \mid m_1m_2 \cap C \neq \emptyset\}.$$

The first question arising is whether a fuzzy congruence is a good algebraic notion. This is checked by the validity of the known isomorphism theorems in their fuzzy variant.

Given fuzzy monoids M and M', a strict morphism from M to M' is a function $h : M \to M'$ preserving fuzzy multiplication and units, i.e.,

$$\bar{h}(m_1m_2) = h(m_1)h(m_2), \quad h(e) = e',$$

for all $m_1, m_2 \in M$, where e, e' are the units of M, M' respectively, and $\bar{h} : Fuz(M) \to Fuz(M')$ the canonical extension of h defined by

$$\bar{h}(F) = \sup_{m\in M} (F(m) \wedge h(m)).$$

Theorem 1 *Given an epimorphism of fuzzy monoids $h : M \to M'$ and a fuzzy congruence $\sim$ on M', its inverse image $h^{-1}(\sim)$ defined by*

$$m_1 h^{-1}(\sim) m_2 \quad \textit{if} \quad h(m_1) \sim h(m_2)$$

is also a fuzzy congruence and the fuzzy quotient monoids $M/h^{-1}(\sim)$ and $M'/\sim$ are isomorphic.

Proof. Assume that

$$m_1 h^{-1}(\sim) m_1' \text{ and } m_2 h^{-1}(\sim) m_2'$$

that is

$$h(m_1) \sim h(m_1') \text{ and } h(m_2) \sim h(m_2').$$

Then

$$\bar{h}(m_1 m_2) = h(m_1)h(m_2) \approx h(m_1')h(m_2') = \bar{h}(m_1' m_2').$$

That is for all $C \in M'/\sim$, we have

$$\sup_{c \in C} \bar{h}(m_1 m_2)(c) = \sup_{c \in C} \bar{h}(m_1' m_2')(c)$$

but

$$\begin{aligned} \sup_{c \in C} \bar{h}(m_1 m_2)(c) &= \sup_{c \in C} \sup_{m \in M} (m_1 m_2)(m) \wedge h(m)(c) \\ &= \sup_{m \in M} (m_1 m_2)(m) \wedge \sup_{c \in C} h(m)(c) \\ &= \sup_{m \in h^{-1}(C)} (m_1 m_2)(m). \end{aligned}$$

Recall that all $h^{-1}(\sim)$-classes are of the form $h^{-1}(C)$, $C \in M'/\sim$. Consequently,

$$\sup_{m \in h^{-1}(C)} (m_1 m_2)(m) = \sup_{m \in h^{-1}(C)} (m_1' m_2')(m)$$

which shows that $h^{-1}(\sim)$ is indeed a congruence of the fuzzy monoid M. The desired isomorphism $\hat{h} : M/h^{-1}(\sim) \to M'/\sim$ is given by

$$\hat{h}([m]_{h^{-1}(\sim)}) = [h(m)]_{\sim}.$$

■

Corollary 1 *Let $h : M \to M'$ be an epimorphism of fuzzy monoids. Then the kernel equivalence*

$$m_1 \sim_h m_2 \textit{ if } h(m_1) = h(m_2)$$

is a congruence on M and the fuzzy quotient monoid $M/\sim_h$ is isomorphic to M'.

Given fuzzy monoids $M_1, \ldots, M_k$ the fuzzy multiplication

$$[(m_1, \ldots, m_k) \cdot (m_1', \ldots, m_k')](n_1, \cdots, n_k) = (m_1 m_1')(n_1) \wedge \cdots \wedge (m_k m_k')(n_k)$$

structures the set $M_1 \times \cdots \times M_k$ into a fuzzy monoid so that the canonical projection

$$\pi_i : M_1 \times \cdots \times M_k \to M_i, \quad \pi_i(m_1, \ldots, m_k) = m_i$$

becomes a morphism of fuzzy monoids. Notice that the above multiplication is fuzzy because

$$\begin{aligned}\sup_{1\le i\le k}\ \sup_{n_i\in M_i}(m_1m_1')(n_1) & \wedge\cdots\wedge(m_km_k')(n_k)\\ &= \sup_{n_1\in M_1}(m_1m_1')(n_1)\wedge\cdots\wedge\sup_{n_k\in M_k}(m_km_k')(n_k)\\ &= 1\cdots 1 = 1.\end{aligned}$$

Theorem 2 *Let $\sim_i$ be a fuzzy congruence on the fuzzy monoid M_i $(1\le i\le k)$. Then $\sim_1\times\cdots\times\sim_k$ is a fuzzy congruence on the fuzzy monoid $M_1\times\cdots\times M_k$ and the fuzzy monoids $M_1\times\cdots\times M_k/\sim_1\times\cdots\times\sim_k$ and $M_1/\sim_1\times\cdots\times M_k/\sim_k$ are isomorphic.*

3. The greatest fuzzy congruence saturating an equivalence

First observe that, due to the symmetric property which an equivalence relation satisfies, the sumability condition in the definition of a congruence can be replaced by the weaker condition: $m_1\sim m_1'$ and $m_2\sim m_2'$ implies

$$\sup_{m\in C}(m_1m_2)(m)\le\sup_{m\in C}(m_1'm_2')(m)$$

for all $\sim$-classes C. Equivalently, we may state: $m\sim m'$, implies

$$\sup_{b\in C}(mn)(b)\le\sup_{b\in C}(m'n)(b)\quad\text{and}\quad\sup_{b\in C}(nm)(b)\le\sup_{b\in C}(nm')(b).$$

Next, we demonstrate that fuzzy congruences are closed under the join operation. We recall that the join $\bigvee_{i\in I}\sim_i$ of a family of equivalences $(\sim_i)_{i\in I}$ on a set A is the reflexive and transitive closure of their union:

$$\bigvee_{i\in I}\sim_i=\left(\bigcup_{i\in I}\sim_i\right)^*.$$

Theorem 3 *If $(\sim_i)_{i\in I}$ is a family of fuzzy congruences on M, then their join $\bigvee_{i\in I}\sim_i$ is also a fuzzy congruence.*

Proof. Let $\sim_1,\sim_2$ be two congruences on M and $\sim=\sim_1\vee\sim_2$. First we show that $m\sim_1 m'$ implies

$$\sup_{b\in C}(mn)(b)\le\sup_{b\in C}(m'n)(b),$$

for all $\sim$-classes C. From the inclusion $\sim_1\subseteq\sim$ we get that C is the disjoint union

$$C=\bigcup_{j\in J}C_j^1$$

where C_j^1 denote $\sim_1$-classes and J is an index set. Then

$$\sup_{b\in C}(mn)(b)=\sup_{j\in J}\ \sup_{b\in C_j^1}(mn)(b)\le\sup_{j\in J}\ \sup_{b\in C_j^1}(m'n)(b)=\sup_{b\in C}(m'n)(b).$$

By a similar argument we show that $m \sim_2 m'$ implies

$$\sup_{b\in C}(mn)(b) \leq \sup_{b\in C}(m'n)(b),$$

for all $\sim$-classes C. Now, if $m \sim m'$, without any loss we may assume that

$$m \sim_1 m_1 \sim_2 m_2 \sim_1 \cdots \sim_1 m_{2\lambda-1} \sim_2 m'$$

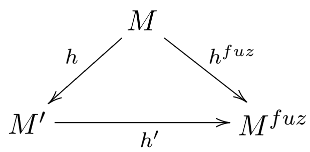
for some elements $m_1, \ldots, m_{2\lambda-1} \in M$. Applying successively the previous facts, we obtain

$$\sup_{b\in C}(mn)(b) \leq \sup_{b\in C}(m_1n)(b) \leq \cdots \leq \sup_{b\in C}(m_{2\lambda-1}n)(b) \leq \sup_{b\in C}(m'n)(b).$$

For an arbitrary set of congruences we proceed in a similar way. ■

The above fact leads us to introduce the greatest fuzzy congruence included into an equivalence $\sim$ of M. It is the join of all fuzzy congruences on M included into $\sim$ and it is denoted by $\sim^{fuz}$. The quotient fuzzy monoid $M/\sim^{fuz}$ is denoted by M^{fuz} and has the following universal property

Theorem 4 *Given an epimorphism of fuzzy monoids $h : M \to M'$ whose kernel $\sim_h$ saturates k the equivalence $\sim$ there exists a unique epimorphism of fuzzy monoids $h' : M' \to M^{fuz}$ rendering commutative the triangle*



where $h^{fuz} : M \to M^{fuz}$ is the canonical projection $m \mapsto [m]_{fuz}$ sending every element $m \in M$ on its $\sim^{fuz}$-class.

Proof. By virtue of the Isomorphism Theorem the fuzzy monoid M' is isomorphic to the quotient $M/\sim_h$. Since by assumption $\sim_h \subseteq \sim^{fuz}$, h' is the following composition

$$M' \xrightarrow{\sim} M/\sim_h \xrightarrow{f} M/\sim^{fuz} = M^{fuz}$$

with $f([m]_h) = [m]_{fuz}$, $[m]_h$ being the $\sim_h$-class of m. ■

The previous result states that h^{fuz} is minimal among all epimorphisms saturating $\sim$.

4. Syntactic fuzzy monoids

Let M be a fuzzy monoid and $L \subseteq M$. Denote by $\sim_L$ the greatest congruence of M included in the partition (equivalence) $\{L, M - L\}$, i.e.,

$$\sim_L = \{L, M - L\}^{fuz}.$$

The quotient fuzzy monoid $M_L = M/\sim_L$ will be called the *syntactic fuzzy monoid* of L and it is characterized by the following universal property.

Theorem 5 *For every fuzzy monoid M and every epimorphism $h : M \to M'$ verifying $h^{-1}(h(L)) = L$, there exists a unique epimorphism $h' : M' \to M_L$ rendering commutative the triangle*

$$\begin{array}{ccc} & M & \\ {}^{h}\swarrow & & \searrow^{h_L} \\ M' & \xrightarrow[h']{} & M_L \end{array}$$

where h_L is the canonical morphism sending every element $m \in M$ to its $\sim_L$-class.

Proof. The hypothesis $h^{-1}(h(L)) = L$ means that $\sim_h$ saturates L and so the statement follows immediately by Theorem 4. ∎

Given fuzzy monoids M, M' we write $M' < M$ if there is a fuzzy monoid $\bar{M}$ and a situation

$$M' \xleftarrow{h} \bar{M} \xrightarrow{i} M$$

where i (resp. h) is a monomorphism (resp. epimorphism).

Theorem 6 *Given subsets L_1, L_2, L of a fuzzy monoid M it holds*

(i) $M_{L_1 \cap L_2} < M_{L_1} \times M_{L_2}$,

(ii) $M_L = M_{\bar{L}}$, *where $\bar{L}$ designates the set theoretic complement of L,*

(iii) $M_{L_1 \cup L_2} < M_{L_1} \times M_{L_2}$,

(iv) *If $h : M \to N$ is an epimorphism of ND-monoids and $L \subseteq N$, then $M_{h^{-1}(L)} = M_L$.*

Proof. The proof follows by applying Theorem 5. ∎

A subset L of a fuzzy monoid M is *fuzzy recognizable* if there exist a finite fuzzy monoid M' and a morphism $h : M \to M'$ such that $h^{-1}(h(L)) = L$. The class of fuzzy recognizable subsets of M is denoted by $FRec(M)$. By taking into account Theorem 5 we get

Proposition 2 *$L \subseteq M$ is recognizable if and only if its syntactic fuzzy monoid is finite, i.e., $card(M_L) < \infty$.*

Putting this result together with Theorem 6 we yield

Proposition 3 *The class $FRec(M)$ is closed under boolean operations and inverse morphisms.*

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