ON SOME MORE q-SERIES IDENTITIES

S.N. Singh
Satya Prakash Singh

Department of Mathematics
T.D.P.G. College, Jaunpur-222002 (U.P.)
India
e-mails: snsp39@gmail.com
snsp39@yahoo.com

Abstract. In this paper we have discussed and also generalized some interesting identities found in the 'Lost' notebook of Ramanujan.

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1. Introduction, notations and definitions

The q-rising factorial $(a;q)_k$ is defined as,

$$(a,q)_k = \begin{cases} 1, & \text{if } k=0; \\ (1-a)(1-aq)(1-aq^2)\dots(1-aq^{k-1}), & \text{if } k \geq 1. \end{cases}$$

Similarly, the infinite q-rising factorial is defined by

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r), \quad \text{for} \quad |q|<1.$$

Ramanujan's Notebooks, specially the 'Lost' notebook contains a large number of q-series identities. Some of the identities found in the 'Lost' notebook are very interesting and noteworthy. Our aim is to discuss and also generalize few identities found in the 'Lost' notebook of Ramanujan.

2. Main results

In this section, we shall discuss following identities. Making use of the q-binomial theorem Ramanujan established following transformation

$$\sum_{m=0}^{\infty}\frac{(a;q^h)_m(b;q)_{mh}t^m}{(q^h;q^h)_m(c;q)_{mh}} = \frac{(b;q)_\infty(at;q^h)_\infty}{(c;q)_\infty(t;q^h)_\infty}\sum_{n=0}^{\infty}\frac{(c/b;q)_m(t;q^h)_m b^m}{(q;q)_m(at;q^h)_m}, \tag{2.1}$$

where $|t|<1, |b|<1$. [2; Theorem (1.2.1) p.6]

Further, he deduced a large number of identities as special cases of (2.1). One of the identities is,

$$\sum_{n=0}^{\infty}\frac{(a;q^2)_n(b;q)_n t^n}{(q^2;q^2)_n(c;q)_n}=\frac{(b;q)_\infty(at;q^2)_\infty}{(c;q)_\infty(t;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(c/b;q)_{2n}(t;q^2)_n b^{2n}}{(q;q)_{2n}(at;q^2)_n}+\frac{(b;q)_\infty(atq^2;q^2)_\infty}{(c;q)_\infty(tq;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(c/b;q)_{2n+1}(tq;q^2)_n}{(q;q)_{2n+1}(atq;q^2)_n}b^{2n+1}, \tag{2.2}$$

where $|t|<1, |b|<1$. [2; Theorem (1.2.2) p.7]

(a) The two variable generalization of (2.1) is

$$\sum_{m,n=0}^{\infty}\frac{(a;q^{h_1})_m(d;q^{h_2})_n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n}\frac{(b;q)_{mh_1+nh_2}}{(c;q)_{mh_1+nh_2}}x^m y^n =\frac{(b;q)_\infty(ax;q^{h_1})_\infty(dy;q^{h_2})_\infty}{(c;q)_\infty(x;q^{h_1})_\infty(y;q^{h_2})_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_r(x;q^{h_1})_r(y;q^{h_2})_r b^r}{(q;q)_r(ax;q^{h_1})_r(dy;q^{h_2})_r}, \tag{2.3}$$

where $|x|, |y|, |b|<1$.

Proof of (2.3). The left hand side of (2.3) is

$$=\frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m,n=0}^{\infty}\frac{(a;q^{h_1})_m(d;q^{h_2})_n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n}\frac{(cq^{mh_1+nh_2};q)_\infty}{(bq^{mh_1+nh_2};q)_\infty}x^m y^n$$

Applying q-binomial theorem [2; (1.22) p. 6].

$$=\frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m,n=0}^{\infty}\frac{(a;q^{h_1})_m(d;q^{h_2})_n x^m y^n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n}\sum_{r=0}^{\infty}\frac{(c/b;q)_r b^r q^{r(mh_1+nh_2)}}{(q;q)_r}.$$

Since $|x|, |y|, |b|<1$, both series are convergent. So changing the order of summations we have

$$=\frac{(b;q)_\infty}{(c;q)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_r b^r}{(q;q)_r}\sum_{m=0}^{\infty}\frac{(a;q^{h_1})_m(xq^{rh_1})^m}{(q^{h_1};q^{h_1})_m}\sum_{n=0}^{\infty}\frac{(d;q^{h_2})_n(yq^{rh_2})^n}{(q^{h_2};q^{h_2})_n}$$

Again, applying q-binomial theorem

$$=\frac{(b;q)_\infty}{(c;q)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_r b^r}{(q;q)_r}\frac{(axq^{h_1r};q^{h_1})_\infty}{(xq^{h_1r};q^{h_1})_\infty}\frac{(dyq^{h_2r};q^{h_2})_\infty}{(yq^{h_2r};q^{h_2})_\infty}$$

$$=\frac{(b;q)_\infty(ax;q^{h_1})_\infty(dy;q^{h_2})_\infty}{(c;q)_\infty(x;q^{h_1})_\infty(y;q^{h_2})_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_r(x;q^{h_1})_r(y;q^{h_2})_r b^r}{(q;q)_r(ax;q^{h_1})_r(dy;q^{h_2})_r},$$

which is precisely the right hand side of (2.3).

Proceeding in the above manner we have the multi-variables generalization of (2.1) as

$$\sum_{m_1,m_2,\ldots m_r=0}^{\infty}\frac{(a_1;q^{h_1})_{m_1}(a_2;q^{h_2})_{m_2}\ldots(a_r;q^{h_r})_{m_r}(b;q)_{m_1h_1+m_2h_2+\ldots+m_rh_r}x_1^{m_1}x_2^{m_2}\ldots x_r^{m_r}}{(q^{h_1};q^{h_1})_{m_1}(q^{h_2};q^{h_2})_{m_2}\ldots(q^{h_r};q^{h_r})_{m_r}(c;q)_{m_1h_1+m_2h_2+\ldots+m_rh_r}}$$

$$=\frac{(b;q)_\infty(a_1x_1;q^{h_1})_\infty\ldots(a_rx_r;q^{h_r})_\infty}{(c;q)_\infty(x_1;q^{h_1})_\infty\ldots(x_r;q^{h_r})_\infty}\sum_{s=0}^{\infty}\frac{(c/b;q)_s(x_1;q^{h_1})_s\ldots(x_r;q^{h_r})_s b^s}{(q;q)_s(a_1x_1;q^{h_1})_s\ldots(a_rx_r;q^{h_r})_s}, \tag{2.4}$$

where $|x_1|, |x_2|, \ldots, |x_r|, |b|<1$.

(b) Two variables generalization of (2.2) is given by,

$$\sum_{m,n=0}^{\infty}\frac{(a;q^2)_m(d;q^2)_n(b;q)_{m+n}}{(q^2;q^2)_m(q^2;q^2)_n(c;q)_{m+n}}x^my^n$$

$$= \frac{(b;q)_\infty(ax;q^2)_\infty(dy;q^2)_\infty}{(c;q)_\infty(x;q^2)_\infty(y;q^2)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_{2r}(x;q^2)_r(y;q^2)_r b^{2r}}{(q;q)_{2r}(ax;q^2)_r(dy;q^2)_r} \tag{2.5}$$

$$+\frac{(b;q)_\infty(axq;q^2)_\infty(dyq;q^2)_\infty}{(c;q)_\infty(xq;q^2)_\infty(yq;q^2)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_{2r+1}(xq;q^2)_r(yq;q^2)_r b^{2r+1}}{(q;q)_{2r+1}(axq;q^2)_r(dyq;q^2)_r},$$

where $|x|, |y|, |b| < 1$.

Proof of (2.5). The left hand side of (2.5) is

$$= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m,n=0}^{\infty}\frac{(a;q^2)_m(d;q^2)_n(cq^{m+n};q)_\infty}{(q^2;q^2)_m(q^2;q^2)_n(dq^{m+n};q)_\infty}x^my^n$$

Applying q-binomial theorem

$$= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m,n=0}^{\infty}\frac{(a;q^2)_m(d;q^2)_n x^my^n}{(q^2;q^2)_m(q^2;q^2)_n}\sum_{r=0}^{\infty}\frac{(c/b;q)_r b^r q^{mr+nr}}{(q;q)_r}$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m,n=0}^{\infty}\frac{(a;q^2)_m(d;q^2)_n x^my^n}{(q^2;q^2)_m(q^2;q^2)_n}\times$$

$$\times\left\{\sum_{r=0}^{\infty}\frac{(c/b;q)_{2r}b^{2r}q^{2mr+2nr}}{(q;q)_{2r}}+\sum_{r=0}^{\infty}\frac{(c/b;q)_{2r+1}b^{2r+1}q^{m(2r+1)}q^{n(2r+1)}}{(q;q)_{2r+1}}\right\}$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_{2r}b^{2r}}{(q;q)_{2r}}\sum_{m=0}^{\infty}\frac{(a;q^2)_m(xq^{2r})^m}{(q^2;q^2)_m}\sum_{n=0}^{\infty}\frac{(d;q^2)_n(yq^{2r})^n}{(q^2;q^2)_n}$$

$$+\frac{(b;q)_\infty}{(c;q)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_{2r+1}b^{2r+1}}{(q;q)_{2r+1}}\sum_{m=0}^{\infty}\frac{(a;q^2)_m(xq^{2r+1})^m}{(q^2;q^2)_m}\sum_{n=0}^{\infty}\frac{(d;q^2)_n(yq^{2r+1})^n}{(q^2;q^2)_n}.$$

Applying again q-binomial theorem

$$= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_{2r}b^{2r}(axq^{2r};q^2)_\infty(dyq^{2r};q^2)_\infty}{(q;q)_{2r}(xq^{2r};q^2)_\infty(yq^{2r};q^2)_\infty}$$

$$+\frac{(b;q)_\infty}{(c;q)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_{2r+1}b^{2r+1}(axq^{2r+1};q^2)_\infty(dyq^{2r+1};q^2)_\infty}{(q;q)_{2r+1}(xq^{2r+1};q^2)_\infty(yq^{2r+1};q^2)_\infty}.$$

$$= \frac{(b;q)_\infty(ax;q^2)_\infty(dy;q^2)_\infty}{(c;q)_\infty(x;q^2)_\infty(y;q^2)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_{2r}(x;q^2)_r(y;q^2)_r b^{2r}}{(q;q)_{2r}(ax;q^2)_r(dy;q^2)_r}$$

$$+\frac{(b;q)_\infty(axq;q^2)_\infty(dyq;q^2)_\infty}{(c;q)_\infty(xq;q^2)_\infty(yq;q^2)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_{2r+1}(xq;q^2)_r(yq;q^2)_r b^{2r+1}}{(q;q)_{2r+1}(axq;q^2)_r(dyq;q^2)_r},$$

which is the right hand side of (2.5).

(c) One noteworthy identity given in the 'Lost' notebook of Ramanujan [2; (1.4.20) p. 19] is

$$(2.6)\quad (-bq^n;q^n)_\infty \sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)/2}}{(q;q)_m(-bq^n;q^n)_m} = (-aq;q)_\infty \sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(q^n;q^n)_m(-aq;q)_{nm}}.$$

In order to establish (2.6) let us consider (2.1).
Putting t/a for t and then taking $a \to \infty$ and b=0 in (2.1) we get,

$$\sum_{m=0}^{\infty} \frac{q^{hm(m-1)/2}(-)^m t^m}{(q^h;q^h)_m(c;q)_{mh}} = \frac{(t;q^h)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(-)^m c^m q^{m(m-1)/2}}{(q;q)_m(t;q^h)_m},$$

which yields (2.6) if we put h=n, $t = -bq^n$ and c=-aq.

Two variables generalization of (2.6) can be easily deduced from (2.3) in the following form

$$(2.7)\quad \begin{aligned} &(-aq;q)_\infty \sum_{m,n=0}^{\infty} \frac{q^{h_1 m(m+1)/2+h_2 n(n+1)/2}\alpha^m\beta^n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n(-aq;q)_{mh_1+nh_2}} \\ &= (-\alpha q^{h_1};q^{h_1})_\infty(-\beta q^{h_2};q^{h_2})_\infty \sum_{r=0}^{\infty} \frac{a^r q^{r(r+1)/2}}{(q;q)_r(-\alpha q^{h_1};q^{h_1})_r(-\beta q^{h_2};q^{h_2})_r}. \end{aligned}$$

(d) Another interesting identity due to Ramanujan is

$$(2.8)\quad (-bq;q)_\infty \sum_{n=0}^{\infty} \frac{(-\lambda/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n} = (-aq;q)_n \sum_{n=0}^{\infty} \frac{(-\lambda/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}.$$

[1; (6.2.9) p.146]

Identity (2.8) is noteworthy for several reasons.
It contains large number of identities as its special cases.
If we take $\lambda = 0$ in (2.8) we get the identity,

$$(2.9)\quad (-bq;q)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n} = (-aq;q)_\infty \sum_{n=0}^{\infty} \frac{b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n},$$

which is n=1, case of (2.6).

Taking $\lambda = ab$ in (2.8), we get

$$(2.10)\quad (-b;q)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n(1+bq^n)} = (-a;q)_\infty \sum_{n=0}^{\infty} \frac{b^n q^{n(n+1)/2}}{(q;q)_n(1+aq^n)}.$$

If we put $\lambda = -ac$ in (2.8) we get the new form of the identity as,

$$(-bq;q)_\infty \sum_{n=0}^{\infty} \frac{(c;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n} = (-aq;q)_\infty \sum_{n=0}^{\infty} \frac{(ac/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}. \quad (2.11)$$

From (2.11) it is easy to have

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n-1)/2}}{(q;q)_n(-bq;q)_n}} = \frac{\displaystyle\sum_{n=0}^{\infty}\frac{(ac/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}}{\displaystyle(1+a)\sum_{n=0}^{\infty}\frac{(ac/bq;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-a;q)_n}}$$

$$(2.12)\qquad = \frac{1}{1+}\,\frac{a(1-c)/(1+bq)}{1-}\,\frac{aq(c+bq)/(1+bq)(1+bq^2)}{1+}\,\frac{aq(1-cq)/(1+bq^2)(1+bq^3)}{1-}\,\frac{aq^3(c+bq^2)/(1+bq^3)(1+bq^4)}{1+}\,\frac{aq^2(1-cq^2)/(1+bq^4)(1+bq^5)}{1-}\cdots\quad .$$

Proof of (2.12).

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n-1)/2}}{(q;q)_n(-bq;q)_n}} = \cfrac{1}{1+\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n-1)/2}(1-q^n)}{(q;q)_n(-bq;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}}$$

$$= \cfrac{1}{1+\cfrac{a(1-c)/(1+bq)}{\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq^2;q)_n}}}}$$

$$= \cfrac{1}{1+\cfrac{a(1-c)/(1+bq)}{1+\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n(n+1)/2}}{(q;q)_n}\left\{\frac{(c;q)_n}{(-bq;q)_n}-\frac{(cq;q)_n}{(-bq^2;q)_n}\right\}}{\displaystyle\sum_{n=0}^{\infty}\frac{a^n(cq;q)_n q^{n(n+1)/2}}{(q;q)_n(-bq^2;q)_n}}}}$$

$$= \cfrac{1}{1+\cfrac{a(1-c)/(1+bq)}{1-\cfrac{aq(c+bq)/(1+bq)(1+bq^2)}{\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(cq;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq^2;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(cq;q)_n a^n q^{n(n+3)/2}}{(q;q)_n(-bq^3;q)_n}}}}}$$

Proceeding in the same way we get the continued fraction for the above ratio.

3. A theorem of Ramanujan

In this section, we shall give a simple proof of the following theorem of Ramanujan.

Theorem. *Let a and b be any complex numbers and* $|xy| < 1$.

If $\Phi(a,x,y) = \sum_{n=0}^{\infty} \frac{a^n x^{n(n+1)/2}}{(xy;xy)_n}$, *then*

$$\Phi(a,x,y)\Phi(b,y,x) = \sum_{n=0}^{\infty} \frac{(ax+by^n)(ax^2+by^{n-1})...(ax^n+by)}{(xy;xy)_n} \tag{3.1}$$

Proof. It is easy to show that right hand side of (3.1) is

$$= \sum_{n=0}^{\infty} \frac{b^n y^{n(n+1)/2}(-axy^{-n}/b;xy)_n}{(xy;xy)_n} = \sum_{n=0}^{\infty} \frac{b^n y^{n(n+1)/2}(-axy^{-n}/b;xy)_\infty}{(xy;xy)_n(-ax^{n+1}/b;xy)_\infty}$$

Applying q-binomial theorem

$$= \sum_{n=0}^{\infty} \frac{b^n y^{n(n+1)/2}}{(xy;xy)_n} \sum_{r=0}^{\infty} \frac{(x^{-n}y^{-n};xy)_r}{(xy;xy)_r} (-)^r \frac{a^r}{b^r} x^{r(r+1)}$$

Applying the identity [4; lemma 1(2) p.100] we get

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{b^{n+r} y^{nr+n(n+1)/2+r(r+1)/2}(x^{-n-r}y^{-n-r};xy)_r}{(xy;xy)_{n+r}(xy;xy)_r} \left(-\frac{a}{b}\right)^r x^{nr+r(r+1)}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{b^{n+r} y^{nr+n(n+1)/2+r(r+1)/2}}{(xy;xy)_{n+r}(xy;xy)_r} \left(-\frac{a}{b}\right)^r x^{nr+r(r+1)} \frac{(x^{-n-r}y^{-n-r};xy)_\infty}{(x^{-n}y^{-n};xy)_\infty}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-)^r a^r b^r y^{nr+n(n+1)/2+r(r+1)/2} x^{nr+r(r+1)}}{(xy;xy)_{n+r}(xy;xy)_r(x^{-n}y^{-n};xy)_{-r}}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-)^r a^r b^r y^{nr+n(n+1)/2+r(r+1)/2} x^{nr+r(r+1)}(-x^{-n}y^{-n})^r}{(xy;xy)_{n+r}(xy;xy)_r(xy)^{r(r+1)/2}} (x^{n+1}y^{n+1};xy)_r$$

which on simplification gives

$$= \sum_{n=0}^{\infty} \frac{b^n y^{n(n+1)/2}}{(xy;xy)_n} \sum_{r=0}^{\infty} \frac{a^r x^{r(r+1)/2}}{(xy;xy)_r} = \Phi(b,y,x)\Phi(a,x,y).$$

This is precisely the left hand side of (3.1) if we put y=x and b=a in (3.1). We get

$$\left\{\sum_{n=0}^{\infty} \frac{a^n x^{n(n+1)/2}}{(x^2;x^2)_n}\right\}^2 = \sum_{n=0}^{\infty} \frac{a^n x^{n(n+1)/2}(-x^{-n+1};x^2)_n}{(x^2;x^2)_n} \tag{3.2}$$

4. Special cases

In this section, we shall deduce certain special cases of the results established in previous sections.

Taking $h_1 = h_2 = 1$ in (2.3), we get

$$\begin{aligned}\Phi_D[b:a,d;c:q;x,y] &= \sum_{m,n=0}^{\infty} \frac{(b;q)_{m+n}(a;q)_m(d;q)_n x^m y^n}{(c;q)_{m+n}(q;q)_m(q;q)_n} \\ &= \frac{(b;q)_\infty(ax;q)_\infty(dy;q)_\infty}{(c;q)_\infty(x;q)_\infty(y;q)_\infty}\ {}_3\Phi_2\left[\begin{matrix} c/b, x, y; q; b \\ ax, dy \end{matrix}\right],\end{aligned} \tag{4.1}$$

which is n=2 case of a known result [3; Theorem 5, p.207].

Putting x/a for x, y/d for y and then taking $a, d \to \infty$ in (2.3) we get

$$\begin{aligned}&\sum_{m,n=0}^{\infty} \frac{(-)^{m+n} q^{h_1 m(m-1)/2 + h_2 n(n-1)/2}(b;q)_{mh_1+nh_2} x^m y^n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n(c;q)_{mh_1+nh_2}} \\ &= \frac{(b;q)_\infty(x;q^{h_1})_\infty(y;q^{h_2})_\infty}{(c;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_r b^r}{(q;q)_r(x;q^{h_1})_r(y;q^{h_2})_r},\end{aligned} \tag{4.2}$$

provided $|b| < 1$.

As $b \to 0$, (4.2) yields

$$\begin{aligned}&\sum_{m,n=0}^{\infty} \frac{(-)^{m+n} q^{h_1 m(m-1)/2 + h_2 n(n-1)/2} x^m y^n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n(c;q)_{mh_1+nh_2}} \\ &= \frac{(x;q^{h_1})_\infty(y;q^{h_2})_\infty}{(c;q)_\infty} \sum_{r=0}^{\infty} \frac{(-)^r c^r q^{r(r-1)/2}}{(q;q)_r(x;q^{h_1})_r(y;q^{h_2})_r}\end{aligned} \tag{4.3}$$

Taking $c = -aq, x = -\alpha q^{h_1}$ and $y = \beta q^{h_2}$ in (4.3), we get

$$\begin{aligned}&\sum_{m,n=0}^{\infty} \frac{q^{h_1 m(m+1)/2 + h_2 n(n+1)/2}\alpha^m \beta^n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n(c;q)_{mh_1+nh_2}} \\ &= \frac{(-\alpha q^{h_1};q^{h_1})_\infty(-\beta q^{h_2};q^{h_2})_\infty}{(-aq;q)_\infty} \sum_{r=0}^{\infty} \frac{a^r q^{r(r+1)/2}}{(q;q)_r(-\alpha q^{h_1};q^{h_1})_r(-\beta q^{h_2};q^{h_2})_r}.\end{aligned} \tag{4.4}$$

Putting c=bq in (2.5) we have,

$$\begin{aligned}\sum_{m,n=0}^{\infty} \frac{(a;q^2)_m(d;q^2)_n x^m y^n}{(q^2;q^2)_m(q^2;q^2)_n(1-bq^{m+n})} &= \frac{(ax;q^2)_\infty(dy;q^2)_\infty}{(x;q^2)_\infty(y;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(x;q^2)_r(y;q^2)_r b^{2r}}{(ax;q^2)_r(dy;q^2)_r} \\ &+ \frac{(axq;q^2)_\infty(dyq;q^2)_\infty}{(xq;q^2)_\infty(yq;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(xq;q^2)_r(yq;q^2)_r b^{2r+1}}{(axq;q^2)_r(dyq;q^2)_r}\end{aligned} \tag{4.5}$$

Taking b=1 and a=-1 in (2.9) we get,

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{(q^2;q^2)_n} = (q;q)_\infty(q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n^2}. \tag{4.6}$$

Putting c=0 in (2.12), we get

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n(n-1)/2}}{(q;q)_n(-bq;q)_n}} = \frac{\displaystyle\sum_{n=0}^{\infty}\frac{b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}}{\displaystyle(1+a)\sum_{n=0}^{\infty}\frac{b^n q^{n(n+1)/2}}{(q;q)_n(-a;q)_n}} \tag{4.7}$$

$$= \frac{1}{1+}\frac{a/(1+bq)}{1-}\frac{abq^2/(1+bq)(1+bq^2)}{1+}$$

$$\frac{aq/(1+bq^2)(1+bq^3)}{1-}\frac{abq^5/(1+bq^3)(1+bq^4)}{1+}\cdots \quad .$$

Putting c=q in (2.12) we find,

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n(n+1)/2}}{(-bq;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n(n-1)/2}}{(-bq;q)_n}} = \frac{\displaystyle\sum_{n=0}^{\infty}\frac{(aq/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}}{\displaystyle(1+a)\sum_{n=0}^{\infty}\frac{(a/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-a;q)_n}} \tag{4.8}$$

$$= \frac{1}{1+}\frac{a(1-q)/(1+bq)}{1-}\frac{aq^2(1+b)/(1+bq)(1+bq^2)}{1+}$$

$$\frac{aq(1-q^2)/(1+bq^2)(1+bq^3)}{1-}\frac{aq^4(1+bq)/(1+bq^3)(1+bq^4)}{1+}$$

$$\frac{aq^2(1-q^3)/(1+bq^4)(1+bq^5)}{1-}\cdots \quad .$$

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