CLEAN MULTIPLICATIVE HYPERRINGS

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Abstract. In this work, we introduce the notation of clean multiplicative hyperrings and we will obtain some properties of them. In particular, we study some topological concepts to realize clean elements of a multiplicative hyperring by clopen subsets of its Zariski topology.

Keywords: multiplicative hyperring, hyperideal, clean element, clean multiplicative hyperring, Zariski topology.
AMS Mathematics Subject Classification: 20N20, 16Y99.

1. Introduction

The theory of hyperstructures has been introduced by Marty in 1934 during the 8^{th} Congress of the Scandinavian Mathematicians [22]. Marty introduced hypergroups as a generalization of groups. He published some notes on hypergroups, using them in different contexts as algebraic functions, rational fractions, non commutative groups and then many researchers have been worked on this new field of modern algebra and developed it. It was later observed that the theory of hyperstructures has many applications in both pure and applied sciences; for example, semi-hypergroups are the simplest algebraic hyperstructures that possess the properties of closure and associativity. The theory of hyperstructures has been widely reviewed [10], [11], [16], [39].

In [11], Corsini and Leoreanu-Fotea have collected numerous applications of algebraic hyperstructures, especially those from the last fifteen years to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence, and probabilities. The hyperrings were introduced and studied by Krasner [21], Nakasis [25], Massouros [23] and especially studied by Davvaz and Leoreanu-Fotea [16], Zahedi and Ameri [40], Ameri and Norouzi [3], [4]. The study on hyperrings in [38] ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems. A well-known type of a hyperring, called the Krasner hyperring [21]. Krasner hyperrings are essentially rings, with approximately modified axioms in which addition is a hyperoperation, while the multiplication is an operation. Then, this concept has been studied by a variety of authors. Some principal notions of hyperring theory can be found in [15], [16], [24], [37], [39]. The another type of hyperrings was introduced by Rota in 1982 which the multiplication is a hyperoperation, while the addition is an operation, and it is called it a multiplicative hyperring (for more details see [32], [33], [34], [35]) which was subsequently investigated by Olson and Ward [26] and many others. De Salvo [17] introduced hyperrings in which the additions and the multiplications are hyperoperations. Moreover, there exists another types of hyperrings that both the addition and multiplication are hyperoperations and instead associativity, commutativity and distributivity satisfy in weak associativity, weak commutativity and weak distributivity, which is called H_V-hyperrings, this type of hyperrings can be seen in [38], [39]. Also, there are other types of hyperrings which were completely studied in [15]. These hyperrings are studied by Rahnamai Barghi [31]. Procesi and Rota in [29] have studied ring of fractions in Krasner hyperrings and also they conceptualized in [30] the notion of primeness of hyperideal in a multiplicative hyperring, and in [12], Dasgupta extended the prime and primary hyperideals in multiplicative hyperrings.

A special equivalence relations which is called fundamental relations play important roles in the the theory of algebraic hyperstructures. The fundamental relations are one of the most important and interesting concepts in algebraic hyperstructures that ordinary algebraic structures are derived from algebraic hyperstructures by them. The fundamental relation β^* on hypergroups was defined by Koskas [20], mainly studied by Corsini [23], Freni [18], [19], Vougiouklis [39] (for more details about hyperrings and fundamental relations on hyperrings see [3], [4], [13], [15], [37], [39]). In [8], the authors studied the notation of regular multiplicative hyperrings and in [7], we studied Zariski topology on multiplicative hyperrings. In this paper, we introduce the notation of clean multiplicative hyperrings and we will obtain some properties of them. Also we study some topological concepts to realize clean elements of a multiplicative hyperring by clopen subsets of its Zariski topology.

2. Preliminaries

A hyperoperation $''.''$ on H is a mapping of $H \times H$ into the family of non-empty subsets of H. Let $''.''$ be a hyperoperation on H. Then, $(H, .)$ is called a hypergroupoid. A hypergroup is a hypergroupoid $(H, .)$, that satisfies:

(1.) $\forall a, b, c \in H : a(bc) = (ab)c;$

(2.) $\forall a \in H, aH = H = Ha.$

In the above definition, if $A, B \subseteq H$ and $h \in H$, then

$$AB = \bigcup_{a \in A, b \in B} ab, \quad Ah = A\{h\}, hB = \{h\}B.$$

A non-empty set R with two hyperoperations $+$ and $.$ is said to be a hyperring if $(R, +)$ is a canonical hypergroup, $(R, .)$ is a semihypergroup with $r.0 = 0.r = 0$ for all $r \in R$ (0 as a bilaterally absorbing element) and the hyperoperation $''.''$ is distributive over $+$, i.e., for every $a, b, c \in R; a(b+c) = ab+ac$ and $(a+b)c = ac+bc$.

A multiplicative hyperring is an additive commutative group $(R, +)$ endowed with a hyperoperation $''.''$ which satisfies the following conditions:

(1) $\forall a, b, c \in R : a(bc) = (ab)c$;

(2) $\forall a, b, c \in R : (a + b)c \subseteq ac + bc, a(b + c) \subseteq ab + ac$;

(3) $\forall a, b \in R : (-a)b = a(-b) = -(ab)$.

If, in (2), we have equalities instead of inclusions, then we say that the multiplicative hyperring is strongly distributive.

Let $(R, +, .)$ be a hyperring. We define the relation γ as follows:

$a\gamma b$ if and only if $\{a, b\} \subseteq U$ where U is a finite sum of finite products of elements of R, i.e.,

$$a\gamma b \Longleftrightarrow \exists z_1, ..., z_n \in R \text{ such that } \{a, b\} \subseteq \sum_{j \in J} \prod_{i \in I_j} z_i; \quad I_j, J \subseteq \{1, ..., n\}.$$

We denote the transitive closure of γ by γ^*. The relation γ^* as the smallest equivalence relation on a multiplicative hyperring $(R, +, .)$ such that the quotient R/γ^*, the set of all equivalence classes, is a fundamental ring. Let $\mathcal{U}$ be the set of all finite sums of products of elements of R we can rewrite the definition of γ^* on R as follows:

> $a\gamma^* b \Longleftrightarrow \exists z_1, ..., z_{n+1} \in R$ with $z_1 = a, z_{n+1} = b$ and $u_1, ..., u_n \in \mathcal{U}$ such that $\{z_i, z_{i+1}\} \subseteq u_i$ for $i \in \{1, ..., n\}$.

Suppose that $\gamma^*(a)$ is the equivalence class containing $a \in R$. Then, both the sum $\oplus$ and the product $\odot$ in R/γ^* are defined as follows:$\gamma^*(a) \oplus \gamma^*(b) = \gamma^*(c)$ for all $c \in \gamma^*(a) + \gamma^*(b)$ and $\gamma^*(a) \odot \gamma^*(b) = \gamma^*(d)$ for all $d \in \gamma^*(a).\gamma^*(b)$. Then R/γ^* is a ring, which is called fundamental ring of R (for more details see [37] and [38]).

Definition 2.1. Let R be a multiplicative hyperring. Then

(i) an element $e \in R$ is said to be a left (resp. right) identity if $a \in e.a$ (resp. $a \in a.e$) for $a \in R$. An element e is called an *identity element* if it is both left and right identity element.

(ii) an element $e \in R$ is said to be a left (resp. right) scalar identity if $a = e.a$ (resp., $a = a.e$) for $a \in R$. An element e is called an *scalar identity element* if it is both left and right scalar identity element.

(iii) Let R be a a multiplicative hyperring with an identity e. An element A is called a left (right) invertible (with respect to e), if there exists $x \in R$, such that $e \in xa(e \in ax)$ and a is called invertible if it is both a left and a right invertible.

A multiplicative hyperring R is called a *left (right) invertible* if every element of R has a left (right) invertible and R is called *invertible* if it is both a left and a right invertible. Denote the set of all invertible elements in R by $U(R)$ (with respect to the identity e by $U_e(R)$).

Definition 2.2. Let R be a multiplicative hyperring. We say that $a \in R$ is regular, if there exists $x \in R$ such that $0 \in a(1 - xa)$. Also, we say that R is regular multiplicative hyperring, if all of elements in R are regular elements. Denote the set of all regular elements in R by $V(R)$.

Definition 2.3. Let R be a multiplicative hyperring. An element $e \in R$ is an idempotent, if there exists $e' \in R$ such that $0 \in ee'$ and $e + e' = 1$. Also, we say that a set A of R is idempotent if every element in A is an idempotent. Denote the set of all idempotent elements of R by $Idem(R)$.

Remark 2.4. In throughout of the paper we define $A^2 = \bigcup\limits_{a \in A} a^2$.

Definition 2.5. We say that I is a hyperideal of multiplicative hyperring $(R, +, .)$ if it satisfies the following conditions:

(1) $I - I \subseteq I$,

(2) $\forall x \in I$, $r \in R$, $xr \cup rx \subseteq I$.

Definition 2.6. An element a of R is nilpotent if there exists an n such that $0 \in a^n$ and denote the set of all nilpotent elements of R by $nil_\omega(R)$. Also, we define $rad(I) = \{r \in R | r^n \subseteq I, n \in \mathbb{N}\}$.

Definition 2.7. ([12]) A hyperideal $I(\neq R)$ of a multiplicative hyperring $(R, +, \circ)$ is a prime hyperideal of R, if for all $a, b \in R$, $a \circ b \subseteq I$ then $a \in I$ or $b \in I$.

A hyperideal $I(\neq R)$ of a multiplicative hyperring R is maximal if for any hyperideal J of R, $I \subsetneq J \subseteq R$ then $J = R$.

Definition 2.8. ([12])Let $\mathcal{C}$ be the class of all finite products of elements of a multiplicative hyperring $(R, +, \circ)$, i.e.,

$$\mathcal{C} = \left\{ \prod_{i=1}^{n} r_i | r_i \in R, i \in \mathbb{N} \right\} \subseteq P^*(R).$$

A hyperideal I of R is said to be a $\mathcal{C}$-ideal of R if, for any $A \in \mathcal{C}, A \cap I \neq \emptyset$, then $A \subseteq I$.

Definition 2.9. ([16]) A homomorphism (resp. good homomorphism) between two multiplicative hyperrings $(R, +, \circ)$ and $(R', +', \circ')$ is a map ϕ from R into R' such that for all $a, b \in R$,

$$f(a+b) = f(a) +' f(b) \quad \text{and} \quad \phi(a \circ b) \subseteq \phi(a) \circ' \phi(b),$$

(resp. $\phi(a \circ b) = \phi(a) \circ' \phi(b)$).

Therefore, let $(R, +, \circ)$ be a multiplicative hyperring and consider the canonical projection $\phi : R \to R/\gamma^*$. The heart of R is the set $\omega_R = \{r \in R | \phi(r) = \bar{0}\}$.

Definition 2.10. Let R be a multiplicative hyperring. An element $a \in R$ is ω-regular, if there exists $x \in R$ such that $a - axa \subseteq \omega_R$. Also, we say that R is ω-regular multiplicative hyperring, if all of elements in R are regular elements. Denote the set of all regular elements in R by $V_\omega(R)$.

Definition 2.11. Let R be a multiplicative hyperring. An element $e \in R$ is said to be ω-idempotent, if there exists $e - e^2 \subseteq \omega_R$. Denote the set of all ω-idempotent elements of R by $Idem_\omega(R)$.

Definition 2.12. An element a of R is ω-nilpotent if there exists a positive integer n such that $a^n \subseteq \omega_R$ and denote the set of all ω-nilpotent elements of R by $nil_\omega(R)$.

3. Clean Multiplicative Hyperrings

Definition 3.1. Let R be a commutative multiplicative hyperring with a unity. Then we say that an element $a \in R$ is clean if there exist $e \in Idem(R)$ and $u \in U(R)$ such that $a = e + u$ and we denote R is clean, if every element in R is clean. Also, we say that an element $a \in R$ is ω-clean if there exist $e \in Idem(R)$ and $u \in U(R)$ such that $a - (e + u) \in \omega_R$ and we denote R is ω-clean, if every element in R is ω-clean.

Example 3.2. (see [8]) Let $(R, +, .)$ be clean commutative ring with a unity 1. Let $A \in P^*(R) = P(R) - \{\emptyset\}, |A| \geq 2$, and $1 \in A$, define a multiplicative hyperring $(R_A, +, \circ)$, where $R_A = R$ and for all $x, y \in R_A$, $x \circ y = \{xay \mid a \in A\}$. Then $(R_A, +, \circ)$ is a clean multiplicative hyperring.

Example 3.3. (see [8]) Let $(R, +, .)$ be a non-zero clean ring and for all $a, b \in R$ define a hyper-operation $a \circ b = \{a.b, 2a.b, 3a.b, \dots\}$. Then $(R, +, \circ)$ is a clean multiplicative hyperring, which is not strongly distributive.

Theorem 3.4. *Let R be a commutative multiplicative hyperring with a scalar identity. Then we have the following statements:*

(i) *If $e \in Idem(R)$, then $1 - e \in Idem(R)$,*

(ii) *If $e \in Idem(R)$, then $e \in Idem_\omega(R)$,*

(iii) *If* $e \in Idem(R)$*, then* $\gamma^*(e) \in Idem(R/\gamma^*)$*,*

(iv) *If* $e \in Idem_\omega(R)$*, then* $\gamma^*(e) \in Idem(R/\gamma^*)$*,*

(v) *If* $\gamma^*(e) \in Idem(R/\gamma^*)$*, then* $e \in Idem_\omega(R)$*,*

(vi) *If* $u \in U(R)$*, then* $\gamma^*(u) \in U(R/\gamma^*)$*.*

(vii) $a \in R$ *is clean (*ω*-clean) if and only if* $1 - a$ *is clean (*ω*-clean).*

Proof. Straightforward. ∎

Theorem 3.5. *Let* R *be a commutative multiplicative hyperring with a scalar identity. Then the following statements hold:*

(i) *If* R *is clean, then* R/γ^* *is clean,*

(ii) *If* R/γ^* *is clean then* R *is* ω*-clean.*

Proof. (i) Assume that R is clean, then for all $\gamma^*(a) \in R/\gamma^*$, we have $a \in R$. Thus there exist $e \in Idem(R)$ and $u \in U(R)$ such that $a = e + u$. Therefore $\gamma^*(a) = \gamma^*(e) + \gamma^*(u)$. Now, by 3.4(iii)(vi), we have $\gamma^*(a)$ is clean. Hence R/γ^* is clean.

(ii) It is straightforward. ∎

Lemma 3.6. *Let* R *be a commutative multiplicative hyperring with a scalar identity. If* $u + nil(R)$ *is a unit in* $R/nil(R)$ *(resp.,* $R/nil_\omega(R)$*) and* $nil(R)$ *is* $\mathcal{C}$*-ideal, then we have the following statements:*

(i) $u + \omega_R \in U(R/\omega_R)$*.*

(ii) *if* $|\omega_R| = 1$*, then* $u \in U(R)$*.*

Proof. (i) Assume that $u + nil(R)$ is an unit in $R/nil(R)$, then there exists $\ell + nil(R)$ in $R/nil(R)$ such that $1 + nil(R) \in (u + nil(R))(\ell + nil(R))$. Thus $(1 - u\ell) \cap nil(R) \neq \emptyset$, and since $nil(R)$ is $\mathcal{C}$-ideal, then $1 - u\ell \subseteq nil(R)$. So, there exists $n \in \mathbb{N}$ such that $0 \in (1 - u\ell)^n$. Therefore, $\gamma^*(0) = \gamma^*((1 - u\ell)^n) = \gamma^*(1 - uv)$, where $v = \sum_{i=1}^{n} \binom{n}{i}(-1)^i u^{i-1}\ell^i$. Now, we have $1 - uv \subseteq \omega_R$, then $1 + \omega_R = (u + \omega_R)(v + \omega_R)$. Hence $u + \omega_R \in U(R/\omega_R)$.

(ii) It follows immediately from part (i). ∎

Theorem 3.7. *Let* R *be a commutative multiplicative hyperring with a scalar identity* 1*. If* $a + nil_\omega(R)$ *is clean element in* $R/nil_\omega(R)$*, then the following statements hold:*

(i) a *is* ω*-clean.*

(ii) *If* $|\omega_R| = 1$*, then* a *is clean in* R*.*

Proof. (i) Since $a+nil_\omega(R)$ is clean, then there exist $u+nil_\omega(R) \in U(R/nil_\omega(R))$ and $s+nil_\omega(R) \in Idem(R/nil_\omega(R))$ such that $a+nil_\omega(R) = (u+s)+nil_\omega(R)$. Thus, by Lemma 3.6(i) and Lemma 4.13 of [7], there exists $u+\omega_R \in U(R/\omega_R)$ and $e \in Idem_\omega(R)$ such that $u+nil_\omega(R) = (u+\omega_R)+nil_\omega(R)$ and $s+nil_\omega(R) = (e+\omega_R)+nil_\omega(R)$. Therefore it's enough to set $a+\omega_R = (u+e)+\omega_R$, then $a-(u+e) \subseteq \omega_R$, i.e., a is ω-clean.

(ii) Obvious from part (i). ∎

Corollary 3.8. *Let R be a commutative multiplicative hyperring with a scalar identity. Then we have the following statements:*

(i) *If $R/nil_\omega(R)$ is clean, then R/ω_R is so.*

(ii) *If $|\omega_R| = 1$ and $R/nil_\omega(R)$ is clean, then R is so.*

Proof. It follows immediately from Theorem 3.7. ∎

Theorem 3.9. *Let R be a regular commutative multiplicative hyperring with a scalar identity 1. Then there exist $u+\omega_R \in U(R/\omega_R)$ and $A \subseteq A^2$ such that $a-uA \subseteq \omega_R$.*

Proof. Since for all $a \in R$, there exists $y \in R$ such that, $0 \in a(1-ya)$, so we set $A = ya$, then $A \subseteq A^2$. Now, we can put $U = u+\omega_R$, where $u = aA+(1-A)$. Thus by getting $V = \nu+\omega_R$, where $\nu = yA+(1-A)$, we have $1-u\nu \subseteq \omega_R$. Hence $a-uA \subseteq \omega_R$, where $u+\omega_R \in U(R/\omega_R)$ and $A \subseteq A^2$. ∎

Theorem 3.10. *Let R be a regular commutative multiplicative hyperring with a scalar identity 1. Then there exist $u+\omega_R \in U(R/\omega_R)$ and $A \subseteq A^2$ such that $a-(u+A) \subseteq \omega_R$.*

Proof. By Theorem 3.9, there exist $u+\omega_R \in U(R/\omega_R)$ and $A \subseteq A^2$ such that $a-uA \subseteq \omega_R$. Now, we take $\ell = uA+(1-A)$ and $\mathfrak{m} = 1-A$. Thus $\ell+\omega_R \in U(R/\omega_R)$ and $\mathfrak{m} \in Idem_\omega(R)$ and $a-(\ell+\mathfrak{m}) \subseteq \omega_R$. ∎

Let R be a multiplicative hyperring with identity 1. Define

$$\mathfrak{M} = \{(a_{ij})_{n\times n} \mid 0 \in a_{ij}a_{jj}a_{jk};\ 1 \le i,j,k \le n,\ i \ne j \ne k\}.$$

Now, we have the following properties:

Lemma 3.11. *Let for all $(a_{ij})_{n\times n} \in \mathfrak{M}$, $a_{jj} \in U(R)$. Then $\mathfrak{M} \subseteq U(M_n(R))$.*

Proof. Assume that $A = (a_{ij})_{n\times n} \in \mathfrak{M}$ and $u_{jj} = a_{jj} \in U(R)$ and u'_{jj} is inverse of u_{jj}. Then by some calculation we obtain that

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a_{21}u'_{11} & 1 & 0 & \cdots & 0 \\ -a_{31}u'_{11} & -a_{32}u'_{22} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1}u'_{11} & -a_{n2}u'_{22} & -a_{n3}u'_{33} & \cdots & 1 \end{pmatrix} A \begin{pmatrix} 1 & -u'_{11}a_{12} & -u'_{11}a_{13} & \cdots & -u'_{11}a_{1n} \\ 0 & 1 & -u'_{22}a_{23} & \cdots & -u'_{22}a_{2n} \\ 0 & 0 & 1 & \cdots & -u'_{33}a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

is containing the diagonal matrix B such that its diagonal entries are u_{jj}. Therefore, since B is in $U(M_n(R))$, then A is in $U(M_n(R))$. ∎

Theorem 3.12. *Let R be a multiplicative hyperring with identity 1. If R is clean, then $\mathfrak{M}$ is clean, so.*

Proof. Assume that R is clean. For all $A = (a_{ij})_{n\times n} \in \mathfrak{M}$ we have $a_{jj} = e_{jj} + u_{jj}$, where $e_{jj} \in Idem(R)$ and $u_{jj} \in U(R)$. Now we have $A = E + B$ where:

$$E = \begin{pmatrix} e_{11} & 0 & \dots & 0 \\ 0 & e_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_{nn} \end{pmatrix} \quad and \quad B = \begin{pmatrix} u_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & u_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & u_{nn} \end{pmatrix}.$$

Therefore $E \in Idem(M_n(R))$ and by Lemma 3.11, $B \in U(M_n(R))$. Hence $\mathfrak{M}$ is clean. ∎

4. Characterizations of clean multiplicative hyperrings

By $Spec(R)$ and $Max(R)$ we mean the spectrum of prime ideals and maximal ideals of R, respectively. For any hyperideal I of R and $a \in R$, we set

$$V(a) = \{P \in Spec(R) | a \in P\} \text{ and } V(I) = \{P \in Spec(R) | I \subseteq P\}.$$

Then, the set $V(I) = \bigcap_{a\in I} V(a)$, where I is an hyperideal of R, satisfies the axioms for the closed sets of a topology on $Spec(R)$, called the Zariski topology (see [7]). For any subset S of $Spec(R)$, we consider the topology of S as subspace topology of $Spec(R)$. Also we define, $V_S(a) = S \bigcap V(a)$ and $V_S(I) = S \bigcap V(I)$.

Theorem 4.1. *Let R be a commutative multiplicative hyperring with an identity 1 such that $nil(R) = \{0\}$ and $Max(R) \subseteq S$. Then R is a clopen subset of S if and only if $A = V(e)$ for an idempotent $e \in R$.*

Proof. Assume that A is a clopen subset of S and $I = \bigcap A$ and $J = \bigcap A^c$. Clearly, $A = \bar{A} = V_S(\bigcap A) = V_S(I)$ and $A^c = V_S(J)$ and $V_S(I) \bigcap V_S(J) = \emptyset$. Therefore $I + J = R$ by the assumption. Since $1 \in R$, there exist $e \in I$ and $e' \in J$ such that $e + e' = 1$. Also, since $V_S(e) \bigcup V_S(e') = S$, we have $V(ee') = S$ and by [[7], Theorem 3.6(ii)] and it follows that $ee' \subseteq nil(R) = \{0\}$, i.e., $ee' = \{0\}$. Hence e is an idempotent and $A = V_S(I) = V_S(e)$. The converse is straightforward. ∎

Theorem 4.2. *Let R be a commutative multiplicative hyperring with an identity 1 such that $nil(R) = \{0\}$. Then $a \in R$ is clean if and only if there exists U in $Spec(R)$ such that $V(a-1) \subseteq U \subseteq Spec(R) \setminus V(a)$.*

Proof. Assume that $a \in R$ is clean, then there exist $e \in U(R)$ and $e \in Idem(R)$ such that $a = u + e$. Thus for all $P \in V(a-1)$ we have $P \notin V(1-e)$. So $P \in V(e)$ and we have $V(a-1) \subseteq V(e) \subseteq Spec(R) \setminus V(a)$. For the converse, let there exists U in $Spec(R)$ such that $V(a-1) \subseteq U \subseteq Spec(R) \setminus V(a)$, then for some $e \in Idem(R)$ we have $U = V(e)$, by Theorem 4.1. Set $u = a - e$ and we prove that u is in $U(R)$. Assume that $u \notin U(R)$. Then there exists $P \in Spec(R)$ such that $u \in P$. But we know that either $P \in V(e)$ or $P \in V(1-e)$, because in otherwise if $P \in V(e)$ and $P \in V(1-e)$ we have $1 \in P$ and if $P \notin V(e)$ and $P \notin V(1-e)$ we have $0 \notin P$ which both cases are incorrect. Therefore either $P \in V(a) \bigcap V(e) = \emptyset$ or $P \in V(a-1) \bigcap V(e-1) = \emptyset$, which is a contradiction. Hence $u \in U(R)$ and $a \in R$ is clean. ∎

Corollary 4.3. *Let R be a commutative multiplicative hyperring with an identity 1 such that $nil(R) = \{0\}$. Then for all $a \in R$, we have the following statements:*

(i) *a^2 is clean if and only if a and $-a$ are clean.*

(ii) *for a positive integer n, a^n is clean, then a is clean.*

(iii) *If I is a clean hyperideal of R, then $rad(I)$ is so.*

Proof. (i) Suppose that a^2 is clean. Then there exists a clopen subset U of R such that $V(a-1) \bigcup V(a+1) = V(a^2-1) \subseteq U \subseteq Spec(R) \setminus V(a^2) = Spec(R) \setminus V(a) = Spec(R) \setminus V(-a)$. Therefore a and $-a$ are clean. The converse is similarly.

(ii) Assume that a^n is clean. Then, there exists U if $Spec(R)$ such that $V(a-1) \subseteq V(a^n-1) \subseteq U \subseteq Spec(R) \setminus V(a^n) = Spec(R) \setminus V(a)$. Hence, by Theorem 4.2, a is clean.

(iii) It follows immediately from part (ii). ∎

Lemma 4.4. *Let R be a commutative multiplicative hyperring with identity 1. If every prime hyperideal of R is $\mathcal{C}$-ideal, then for $a \in R$ and $e \in Idem(R)$, $V(ae-1) = V(a-1) \bigcap V(e-1)$.*

Proof. Let $P \in V(ae-1)$, then $e \notin P$ and $e-1 \in P$. Since $(a-1)e \subseteq (ae-1)+(e-1) \subseteq P$, then $a-1 \in P$. Therefore, $P \in V(a-1) \bigcap V(e-1)$. Conversely, suppose $P \in V(a-1) \bigcap V(e-1)$. Then $a-1 \in P$ and $e-1 \in P$. Since $(ae-1) \supseteq a(e-1)+(a-1)$ and $a(e-1)+(a-1) \subseteq P$ then $(ae-1) \bigcap P \neq \emptyset$ and since P is $\mathcal{C}$-ideal, $(ae-1) \subseteq P$, i.e., $P \in V(ae-1)$. ∎

Theorem 4.5. *Let R be a commutative multiplicative hyperring with identity 1 such that $nil(R) = \{0\}$ and every prime hyperideal of R is $\mathcal{C}$-ideal. If $a \in R$ is clean and $e \in Idem(R)$, then ae is clean.*

Proof. Since a is clean then by Theorem 4.1, there exists $e' \in Idem(R)$ such that $V(a-1) \subseteq V(e') \subseteq Spec(R) \setminus V(a)$. Now $U = V(e') \bigcap V(e-1)$ is a clopen and by Lemma 4.4 we have $V(ae-1) = V(a-1) \bigcap V(e-1) \subseteq U \subseteq (Spec(R) \setminus V(a)) \bigcap (Spec(R) \setminus V(e)) = Spec(R) \setminus V(ae)$. Hence ae is a clean. ∎

Acknowledgements. The first author is partially supported by Center of Excellence of Algebraic Hyperstructures and its Applications of Tarbiat Modares University (CEAHA) and Research Center in Algebraic Hyperstructures and Fuzzy Mathematics, University of Mazandaran, Babolsar, Iran.

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Accepted: 19.10.2015