FUZZY ISOMORPHISM THEOREMS OF SOFT GROUPS

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Abstract. In this paper, we introduce the concept of fuzzy normal subgroups of soft groups. The first, second and third fuzzy isomorphism theorems of soft groups are established, respectively. In particular, some classes of quotient groups are characterized by their fuzzy normal subgroups.

Keywords: soft groups; fuzzy normal subgroups; quotient groups; fuzzy isomorphism theorems.

2000 Mathematics Subject Classification: 16Y60; 16Y99; 03E72.

1. Introduction

The fuzzy set theory, proposed by Zadeh [15], has been extensively applied to many scientific fields. This work opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists and many other researchers, so the field grew enormously and applications were found in many areas. In 1971, Rosenfeld [13] applied the concept to the theory of groupoids and groups, this job was the first fuzzification of algebraic structures. Since then, various fuzzy algebraic structures have been appeared in the literature [4], [9], [10]. Fang [4] introduced the fuzzy homomorphism and fuzzy isomorphism between two fuzzy groups by a natural way, and studied some of their properties. Liu [9], [10] investigated the various constructions of fuzzy quotient groups, fuzzy quotient rings and fuzzy isomorphisms, respectively.

Following the discovery of fuzzy sets, much attention has been paid to generalize the concept of soft sets. Molodtsov [11] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches in 1999. At present, the research on the soft set theory is developing rapidly. Several directions

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for the applications of soft sets have been pointed out by several researchers (see, e.g., [5], [6], [17], [18], and they also studied some operations on the theory of soft sets. Aktas et al. [1] researched the basic concepts of soft set theory and compared soft sets to fuzzy and rough sets. Ali et al. [2] introduced a new concept of sets with some order among the parameters and some properties of lattice ordered soft sets were given. The soft algebraic structures have been investigated by some researchers. Aktas et al. [1] applied the notion of soft sets to the theory of groups. Liu et al. [7], [8] proposed the definition of soft rings, investigated some properties of soft rings, and established three isomorphism theorems and three fuzzy isomorphism theorems, respectively. Zhan et al. [17], [18] applied the concepts of soft union sets to the hemirings, h-hemiregular and h-intrahemiregular hemirings, and researched some of their properties. In particular, Zhan investigated the ideal theory on hemirings, most of relevant conclusions have been already demonstrated in Zhan's book, which is referred to [16].

In this paper, we introduce the fuzzy normal subgroups of soft groups. Then, we use fuzzy normal subgroups to describe some classes of quotient soft groups. Finally, the first, second and third fuzzy isomorphism theorems of soft groups are given.

2. Preliminaries

Definition 2.1 ([13]) A fuzzy subset μ of a group G is said to be a fuzzy subgroup of G if

(1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,
(2) $\mu(x^{-1}) \geq \mu(x)$,

holds for all $x, y \in G$.

Equivalently, $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$.

Proposition 2.2 ([3]) *For a fuzzy subgroup μ of a group G, the following hold:*

(1) $\mu(e) \geq \mu(x)$ *for all* $x \in G$,
(2) $\mu(x^{-1}) = \mu(x)$ *for all* $x \in G$,
(3) $\mu(xy^{-1}) = \mu(e)$ *implies* $\mu(x) = \mu(y)$ *for all* $x, y \in G$.

Definition 2.3 ([3]) A fuzzy subset μ of a group G is said to be a fuzzy normal subgroup of G if the following axioms hold:

(1) $\mu(xyx^{-1}) \geq \mu(y)$ for all $x, y \in G$.

Equivalently, $\mu(xyx^{-1}) = \mu(y)$ for all $x, y \in G$, or $\mu(xy) = \mu(yx)$ for all $x, y \in G$.

Proposition 2.4 ([3]) *A fuzzy subset μ of a group G is a fuzzy normal subgroup of G if and only if for all $t \in [0, 1]$, $\mu_t = \{x \in G | \mu(x) \geq t\}$ is a normal subgroup of G.*

Definition 2.5 ([12]) Let $f : X \to Y$ be a mapping of sets, μ a fuzzy subset of X and ν a fuzzy subset of Y. Then, for any $x \in X$, the image $f(\mu)$ of μ and preimage $f^{-1}(\nu)$ of ν are both fuzzy sets defined respectively as follows:

$$\begin{aligned}(f(\mu))(y) &= \bigvee_{f(x)=y} \mu(x),\\ (f^{-1}(\nu))(x) &= \nu(f(x)).\end{aligned}$$

Molodtsov defined the soft set in the following way: let U be an initial universe and E a set of parameters. Denote the power set of U by $P(U)$ and consider $A \subset E$.

Definition 2.6 [11] A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.7 ([1],[14])

(1) Let (F, A) be a soft set over group G. Then (F, A) is said to be a soft group over G if and only if $F(x) < G$ for all $x \in A$.

(2) Let G be a group and (F, A) be a non-null soft set over G. Then (F, A) is called a normalistic soft group over G if $F(x)$ is a normal subgroup of G for all $x \in \mathrm{Supp}(F, A)$.

Let G be an algebra and A a non-empty set. Now, we use ρ to refer an arbitrary binary relation between an element of A and an element of G. Thus, a set-valued function $F : A \to P(G)$ can be defined by $F(x) = \{y \in G | (x, y) \in \rho, x \in A\}$.

Definition 2.8 ([7],[8])

(1) Let (F, A) be a soft set over an algebra G. Then (F, A) is said to be a soft algebra over G if and only if $F(x)$ is a subalgebra of G for all $x \in A$. For our convenience, the empty set $\emptyset$ is regarded as a subalgebra of G.

(2) Given soft groups (F, A) and (H, B) over G and K respectively, a soft homomorphism $(f, g) : (F, A) \to (H, B)$ comprises a map $g : A \to B$ and a group homomorphism $f : G \to K$. If f is an isomorphism from G to K and g is a bijective mapping, then (f, g) is called a soft isomorphism from (F, A) to (H, B), denoted by $(F, A) \simeq (H, B)$.

3. Fuzzy normal subgroups of soft groups

Definition 3.1 Let (F, A) be a soft group over a group G and μ a fuzzy subset in G. Then μ is said to be a fuzzy subgroup of (F, A) if, for any $t \in A$ and $x, y \in F(t)$, the following conditions hold:

(1) $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all x, y of G,

(2) $\mu(x^{-1}) \geq \mu(x)$ for all x of G.

Equivalently, $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$.

Definition 3.2 Let (F, A) be a soft group over a group G and μ a fuzzy subset in G. Then μ is said to be a fuzzy normal subgroup of (F, A) if

(1) $\mu(xyx^{-1}) \geq \mu(y)$ for all $x, y \in G$.

Equivalently, $\mu(xyx^{-1}) = \mu(y)$ for all $x, y \in G$ or $\mu(xy) = \mu(yx)$ for all $x, y \in G$.

Remark 3.3

(1) Let (F, A) be a soft group over a group G and μ a fuzzy subset in G. Then μ is a fuzzy subgroup (normal subgroup) of (F, A) if and only if μ is a fuzzy subgroup (normal subgroup) of $F(t)$ for any $t \in A$;

(2) Let (F, A) be a soft group over a group G. If μ is a fuzzy subgroup (normal subgroup) of G, then μ is a fuzzy subgroup (normal subgroup) of (F, A), but the converse is not true.

Example 3.4 Suppose that $G = S_3 = \{e, (12), (13), (23), (123), (132)\}$ and (F, A) be a soft set over G, where $A = \{e, (12)\}$ and $F : A \to P(G)$ is defined by $F(x) = \{y \in G | x\rho y \Leftrightarrow y = x^n, n \in N\}$ for all $x \in A$. Then, $F(e) = \{e\}$, $F(12) = \{e, (12)\}$ and (F, A) is a soft group of G.

Let μ be a fuzzy subset in G, where μ is defined by

$$\mu(x) = \begin{cases} 0.8 & \text{if} \quad t \in \{e, (123)\}, \\ 0.1 & \text{if} \quad t \in \{(12), (13)\}, \\ 0.5 & \text{if} \quad t \in \{(23), (132)\}. \end{cases}$$

We can verify that for any $t \in A$ and $x, y \in F(t)$, μ is a fuzzy normal subgroup of (F, A). But μ is not a fuzzy normal subgroup of G, because $\mu((123)(23)(123)^{-1}) = \mu(12) < \mu(23)$.

Proposition 3.5 *Let (F, A) be a soft group over a group G and μ a fuzzy subset in G. Then μ is a fuzzy subgroup(normal subgroup) of (F, A) if and only if, for any $x \in A$ and $t \in [0, 1]$, $U(t, x) = \{r \in F(t) | \mu(r) \geq t\}$ is a subgroup (normal subgroup) of $F(x)$.*

Proof. It is straightforward. ∎

Let μ be a fuzzy normal subgroup of a group G. For any $x, y \in G$, define a binary relation $\sim$ on G by $x \sim y$ if and only if $\mu(xy^{-1}) = \mu(e)$, where e is the unit element of G.

Lemma 3.6 ([10]) $\sim$ is a congruence relation of G.

The following lemma is similar to fuzzy ideals in [7]:

Lemma 3.7 *Let N be a group G and μ a fuzzy normal subgroup of G. We have*

(1) *if μ is restricted to N, written μ_N, then μ_N is fuzzy normal subgroup of N;*

(2) *N/μ_N is a normal subgroup of G/μ.*

For our convenience, we can denote μ instead of μ_N.

Let μ be a fuzzy normal subgroup of a group G and (F, A) a normalistic soft group over G. Now, we restrict μ to $F(x)$ for all $x \in A$; then, for all $x \in A$, μ is fuzzy normal subgroup of $F(x)$ and $F(x)/\mu$ is a normal subgroup of G/μ. Thus, we can define a set-valued function $F/\mu : A \to P(G/\mu)$ by $(F/\mu)(x) = F(x)/\mu$.

By Lemma 3.7, we can very easily deduce the following theorem.

Theorem 3.8 *If μ is a fuzzy normal subgroup of a group G and (F, A) is a normalistic soft group over G, then*

(1) *μ is a fuzzy normal subgroup of (F, A);*

(2) *$(F/\mu, A)$ is a normalistic soft group over G/μ.*

Notation 3.9 The $(F/\mu, A)$ is called the quotient soft group of (F, A) induced by fuzzy normal subgroup μ.

The following result is similar to fuzzy subgroups in [10]:

Lemma 3.10 *If μ, ν are two fuzzy normal subgroups of a group G, then $\mu \cap \nu$ is also a fuzzy normal subgroup of G.*

By Lemma 3.10, we can very easily deduce the following proposition.

Proposition 3.11 *If (F, A) is a soft group over a group G and μ, ν are two fuzzy normal subgroups of (F, A), then $\mu \cap \nu$ is also a fuzzy normal subgroup of (F, A).*

Let μ be be a fuzzy subgroup of a group G. We denote $G_\mu = \{x \in G | \mu(x) = \mu(e)\}$. Clearly, G_μ is a subgroup of G. Furthermore, if μ is a fuzzy normal subgroup of G, then G_μ is a normal subgroup of G.

Proposition 3.12 *If μ is a fuzzy subgroup of a group G and (F, A) is a soft group over G, let $F_\mu(t) = \{x \in F(t) | \mu(x) = \mu(e)\}$ for all $t \in A$. Then (F_μ, A) is a soft group over G_μ.*

Proof. The proof is easy and is omitted. ■

In a similar way, we have the following proposition.

Proposition 3.13 *If μ is a fuzzy subgroup of a group G and (F, A) is a normalistic soft group over G, let $F_\mu(t) = \{x \in F(t) | \mu(x) = \mu(e)\}$ for all $t \in A$. Then (F_μ, A) is a normalistic soft group over G_μ.*

Lemma 3.14 *Let μ be a fuzzy normal subgroup of a group G. Then the fuzzy subset μ^* of G/G_μ, defined by $\mu^*(xG_\mu) = \mu(x)$ for all $x \in G$, is a fuzzy normal subgroup of G/G_μ.*

Theorem 3.15 *If μ is a fuzzy normal subgroup of a group G and (F, A) is a soft group over G with $F(t) \supset G_\mu$ for all $t \in A$, then*

(1) *$(F/G_\mu, A)$ is a soft group of G/G_μ, where $(F/G_\mu)(t) = F(t)/G_\mu$ for all $t \in A$,*

(2) *the fuzzy subset μ^* of G/G_μ, defined in Lemma 3.14, is a fuzzy normal subgroup of the soft group $(F/G_\mu, A)$.*

Proof. (1) Since μ is a fuzzy normal subgroup of a group G and (F, A) is a soft group over G with $F(t) \supset G_\mu$ for all $t \in A$, $F(t)$ is a subgroup of G and G_μ is a subgroup of $F(t)$ for all $t \in A$. Let $xG_\mu, yG_\mu \in F(t)/\mu$, where $x, y \in F(t)$, it follows that $y^{-1} \in F(t)$, then $(xG_\mu)(y^{-1}G_\mu) = (xy^{-1})G_\mu \in F(t)/\mu$. Hence, $(F/G_\mu, A)$ is a soft group of G/G_μ.

(2) By Lemma 3.14, μ^* is a fuzzy normal subgroup of G/G_μ, then μ^* is a fuzzy normal subgroup of the soft group $(F/G_\mu, A)$. ∎

4. Fuzzy isomorphism theorems of soft groups

In this section, we give the First, Second and Third Fuzzy Isomorphism Theorems of soft groups by means of fuzzy normal subgroups.

Lemma 4.1 ([10]) *Let $f : G \to K$ be an epimorphism of groups and μ a fuzzy normal subgroup of G, then $f(\mu)$ is a fuzzy normal subgroup of K.*

Lemma 4.2 ([10]) *Let $f : G \to K$ be a homomorphism of groups, μ a fuzzy normal subgroup of G and ν a fuzzy normal subgroup of K, then*

(1) *If f is an epimorphism, then $f(f^{-1}(\nu)) = \nu$,*

(2) *If μ is constant on $ker f$, then $f^{-1}(f(\mu)) = \mu$.*

Lemma 4.3 ([10]) *Let $f : G \to K$ be a homomorphism of groups and ν a fuzzy normal subgroup of K, then $f^{-1}(\nu)$ is a fuzzy normal subgroup of G.*

Theorem 4.4 (First fuzzy isomorphism theorem) *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is soft homomorphism from (F, A) to (H, B) and μ is a fuzzy normal subgroup of G with $G_\mu \supset ker f$, then*

(1) *$(F/\mu, A) \simeq (f(F)/f(\mu), A)$,*

(2) *If g is a bijective mapping, then $(F/\mu, A) \simeq (H/f(\mu), B)$.*

Proof. (1) Since (F, A) is a normalistic soft group over G and μ is a fuzzy normal subgroup of G, $(F/\mu, A)$ is a normalistic soft group over G/μ. (f, g) is a soft homomorphism from (F, A) to (H, B), then for any $t \in A$, $f(F(t)) = H(g(t))$ is a normal subgroup of K, and then we can deduce that $(f(F)/f(\mu), A)$ is a normalistic soft group over $K/f(\mu)$. We define $\varphi : G/\mu \to K/f(\mu)$ by $\varphi(\mu_x) = (f(\mu))_{f(x)}$.

(a) If $\mu_x = \mu_y$, then $\mu(xy^{-1}) = \mu(e)$. Since $G_\mu \supset kerf$, then μ is a constant on $kerf$, by Lemma 4.2, $f^{-1}(f(\mu)) = \mu$. Thus we have $(f^{-1}(f(\mu)))(xy^{-1}) = (f^{-1}(f(\mu)))(e)$, i.e., $f(\mu)(f(xy^{-1})) = f(\mu)(f(e))$, then $f(\mu)(f(x)(f(y))^{-1}) = f(\mu)(e')$, and so $(f(\mu))_{f(x)} = (f(\mu))_{f(y)}$. Hence φ is well-defined.

(b) $\varphi(\mu_x\mu_y) = \varphi(\mu_{xy}) = (f(\mu))_{f(xy)} = (f(\mu))_{f(x)f(y)} = (f(\mu))_{f(x)}(f(\mu))_{f(y)} = \varphi(\mu_x)\varphi(\mu_y)$. Hence φ is a homomorphism.

(c) φ is an epimorphism: For any $(f(\mu))_y \in K/f(\mu)$, since f is epimorphic, there exists $x \in G$ such that $f(x) = y$. So $\varphi(\mu_x) = (f(\mu))_{f(x)} = (f(\mu))_y$.

(d) φ is an monomorphism: $(f(\mu))_{f(x)} = (f(\mu))_{f(y)}$, thus $f(\mu)(f(x)(f(y))^{-1}) = f(\mu)(e')$, so $f(\mu)(f(xy^{-1})) = f(\mu)(f(e))$, i.e., $(f^{-1}(f(\mu)))(xy^{-1}) = (f^{-1}(f(\mu)))(e)$, then $\mu(xy^{-1}) = \mu(e)$, thus $\mu_x = \mu_y$.

By (a),(b),(c) and (d), we deduce that φ is an isomorphism from G/μ to $K/f(\mu)$.

Let $\phi : A \to A$ defined by $\phi(t) = t$. Then ϕ is bijective.

Moreover, we have $\varphi((F/\mu)(t)) = \{(f(\mu))_{f(x)} | x \in F(t)\} = f(F(t))/f(\mu) = f(F(\phi(t)))/f(\mu) = (f(F)/f(\mu))(\phi(t))$. Hence, (φ, ϕ) is an isomorphism and $(F/\mu, A) \simeq (f(F)/f(\mu), A)$.

(2) If g is a bijective, we define $\varphi : G/\mu \to K/f(\mu)$ as same as (1), and we have $\varphi((F/\mu)(t)) = \{(f(\mu))_{f(x)} | x \in F(t)\} = f(F(t))/f(\mu) = H(g(t))/f(\mu)$. Hence, (φ, g) is an isomorphism so that $(F/\mu, A) \simeq (H/f(\mu), B)$. ∎

Theorem 4.5 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is soft homomorphism from (F, A) to (H, B) and ν is a fuzzy normal subgroup of K, then*

(1) $(F/f^{-1}(\nu), A) \simeq (f(F)/\nu, A)$,

(2) *If g is a bijective mapping, then* $(F/f^{-1}(\nu), A) \simeq (H/\nu, B)$.

Proof. It follows from Lemma 4.3, that $(F/f^{-1}(\nu), A)$ and $(f(F)/\nu, A)$ are normalistic soft groups over groups $G/f^{-1}(\nu)$ and K/ν, respectively. Since f is epimorphic, by Lemma 4.2, we have $\nu = f(f^{-1}(\nu))$. Now, we show that $G_{f^{-1}(\nu)} \supset kerf$. For any $x \in kerf$, then $f(x) = e' = f(e)$, and so $\nu(f(x)) = \nu(f(e))$, i.e., $f^{-1}(\nu)(x) = f^{-1}(\nu)(e)$. Hence $x \in G_f^{-1}(\nu)$ and $G_f^{-1}(\nu) \supset kerf$. By Theorem 4.4, we have (1) $(F/f^{-1}(\nu), A) \simeq (f(F)/\nu, A)$ and (2) if g is a bijective mapping, $(F/f^{-1}(\nu), A) \simeq (H/\nu)\, B)$. ∎

Lemma 4.6 *Let χ_S be a characteristic function of a subset S of a group G. Then χ_S is a fuzzy normal subgroup of G if and only if S is a normal subgroup of G.*

Theorem 4.7 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is soft homomorphism from (F, A) to (H, B) and N a is normal subgroup of G with $N \supset kerf$, then*

(1) $(F/\chi_N, A) \simeq (f(F)/\chi_{f(N)}, A)$,

(2) *If g is a bijective mapping, then* $(F/\chi_N, A) \simeq (H/\chi_{f(N)}, B)$.

Proof. By Lemma 4.6, χ_N and $\chi_{f(N)}$ are both fuzzy normal subgroups of G and K, respectively. So they are fuzzy normal subgroups of (F, A) and (H, B), respectively. Putting $\mu = \chi_N$ in Theorem 4.4, then $G_\mu = G_{\chi_N} = N \supset kerf$, and we have $f(\mu) = f(\chi_N) = \chi_{f(N)}$. Then (1) $(F/\chi_N, A) \simeq (f(F)/\chi_{f(N)}, A)$ and (2) if g is a bijective mapping, $(F/\chi_N, A) \simeq (H/\chi_{f(N)}, B)$. ■

Theorem 4.8 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is soft homomorphism from (F, A) to (H, B) and J is a normal subgroup of K, then*

(1) $(F/\chi_{f^{-1}(J)}, A) \simeq (f(F)/\chi_J, A)$,

(2) *If g is a bijective mapping, then* $(F/\chi_{f^{-1}(J)}, A) \simeq (H/\chi_J, B)$.

Proof. We know that $\chi_{f^{-1}(J)}$ and χ_J are both fuzzy normal groups of (F, A) and (H, B), respectively. Setting $\nu = \chi_J$ in Theorem 4.5, we have $f^{-1}(\nu) = f^{-1}(\chi_J) = \chi_{f^{-1}(J)}$. In fact, for any $x \in G$, if $x \in f^{-1}(J)$, then $f(x) \in J$, $f^{-1}(\chi_J)(x) = \chi_J(f(x)) = 1 = \chi_{f^{-1}(J)}(x)$, if $x \notin f^{-1}(J)$, then $f(x) \notin J$, $f^{-1}(\chi_J)(x) = \chi_J(f(x)) = 0 = \chi_{f^{-1}(J)}(x)$. Hence,

(1) $(F/\chi_{f^{-1}(J)}, A) \simeq (f(F)/\chi_J, A)$ and

(2) if g is a bijective mapping, then $(F/\chi_{f^{-1}(J)}, A) \simeq (H/\chi_J, B)$. ■

We can easy verify that $\chi_{\{e\}}f = \chi_{kerf}$ and $G/\chi_{\{e\}} \cong G$ for any group G. Putting $N = kerf$ in Theorem 4.7, we have the following.

Theorem 4.9 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is a soft homomorphism from (F, A) to (H, B), then*

(1) $(F/\chi_{kerf}, A) \simeq (f(F), A)$,

(2) *If g is a bijective mapping, then* $(F/\chi_{kerf}, A) \simeq (H, B)$.

Let N be a normal subgroup of a group G. Recall that a quotient group G/N induced by a normal subgroup N is determined by an equivalent relation $\sim$, where $x \sim y$ is defined by $xy^{-1} \in N$. For no confusion, we write $x \sim y(N)$ to show that x is equivalent to y with respect to the normal subgroup N, and $x \sim y(N)$ to mean that x is equivalent to y with respect to the fuzzy normal subgroup χ_N.

Lemma 4.10 *Suppose that N is a normal subgroup of a group G. Then $x \sim y(N)$ if and only if $x \sim y(\chi_N)$.*

Proof. $x \sim y(N) \iff xy^{-1} \in N \iff \chi_N(xy^{-1}) = 1 \iff \chi_N(xy^{-1}) = \chi_N(e)$ $\iff x \sim y(\chi_N)$. The proof is complete. ∎

By Lemma 4.10, we have $G/N \cong G/\chi_N$, where N is the normal subgroup of group G. Applying Theorem 4.7, Theorem 4.8, Theorem 4.9, we have the following corollaries.

Corollary 4.11 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is a soft homomorphism from (F, A) to (H, B) and N is a normal subgroup of G with $F(t) \supset N \supset kerf$ for all $t \in A$, then*

(1) $(F/N, A) \simeq (f(F)/f(N), A)$,

(2) *If g is a bijective mapping, then $(F/N, A) \simeq (H/f(N), B)$.*

Corollary 4.12 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is a soft homomorphism from (F, A) to (H, B) and J is a normal subgroup of K with $F(t) \supset f^{-1}(J)$ for all $t \in A$, then*

(1) $(F/f^{-1}(J), A) \simeq (f(F)/J, A)$,

(2) *If g is a bijective mapping, then $(F/f^{-1}(J), A) \simeq (H/J, B)$.*

Corollary 4.13 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is a soft homomorphism from (F, A) to (H, B) with $F(t) \supset kerf$ for all $t \in A$, then*

(1) $(F/kerf, A) \simeq (f(F), A)$,

(2) *If g is a bijective mapping, then $(F/kerf, A) \simeq (H, B)$.*

Theorem 4.14 (Second fuzzy isomorphism theorem) *Let (F, A) be a normalistic soft group over group G and μ, ν two fuzzy normal subgroups of G with $\mu(e) = \nu(e)$, then*

$$(F_\mu F_\nu/\nu, A) \simeq (F_\mu/(\mu \cap \nu), A).$$

Proof. We know that $(F_\mu F_\nu/\nu, A)$ and $(F_\mu/(\mu\cap\nu), A)$ are normalistic soft groups over $G_\mu G_\nu/\nu$ and $G_\mu/(\mu \cap \nu)$, respectively. For any $x \in G_\mu G_\nu$, $x = ab$, where $a \in G_\mu$ and $b \in G_\nu$. Define $\varphi : G_\mu G_\nu/\nu \to G_\mu/(\mu \cap \nu)$ by

$$\varphi(\nu_x) = (\mu \cap \nu)_a.$$

(1) If $\nu_x = \nu_y$, where $y = a_1b_1$, $a_1 \in G_\mu$ and $b_1 \in G_\nu$, then $\nu(ab(a_1b_1)^{-1}) = \nu(abb_1^{-1}a_1^{-1}) = \nu(a_1^{-1}abb_1^{-1}) = \nu(a_1^{-1}a(b_1b^{-1})^{-1}) = \nu(e)$. Hence $\nu(a_1^{-1}a) = \nu(b_1b^{-1})$ $= \nu(e)$. Thus $(\mu \cap \nu)(aa_1^{-1}) = \mu(aa_1^{-1}) \wedge \nu(aa_1^{-1}) = \mu(e) \wedge \nu((a_1^{-1}a)^{-1}) = \mu(e) \wedge \nu(e) = (\mu \cap \nu)(e)$. That is $(\mu \cap \nu)_a = (\mu \cap \nu)_{a_1}$. Hence φ is well-defined.

(2) For any $\nu_x, \nu_y \in G_\mu G_\nu/\nu$, where $x = ab$, $y = a_1b_1$, $a, a_1 \in G_\mu$, $b, b_1 \in G_\nu$. then $xy = aba_1b_1$. Since G_μ is normal, $ba_1b_1 \in G_\mu$. Hence $\varphi(\nu_x\nu_y) = \varphi(\nu_{xy}) = (\mu\cap\nu)_{a(ba_1b_1)} = (\mu\cap\nu)_a(\mu\cap\nu)_{ba_1b_1}$. Now we show $(\mu\cap\nu)_{ba_1b_1} = (\mu\cap\nu)_{a_1}$. In fact, $(\mu\cap\nu)((ba_1b_1)a_1^{-1}) = \mu(ba_1b_1a_1^{-1}) \wedge \nu(ba_1b_1a_1^{-1}) = \mu((ba_1b_1)a_1^{-1}) \wedge \nu(b(a_1b_1a_1^{-1})) =$

$\mu(e) \wedge \nu(e) = (\mu \cap \nu)(e)$. Hence $\varphi(\nu_x \nu_y) = (\mu \cap \nu)_a (\mu \cap \nu)_{a_1} = \varphi(\nu_x)\varphi(\nu_y)$ and φ is a homomorphism.

(3) For any $(\mu \cap \nu)_a \in G_\mu/(\mu \cap \nu)$, taking $b \in G_\nu$, then $x = ab \in G_\mu G_\nu$. Hence $f(\nu_x) = (\mu \cap \nu)_a$ and φ is an epimorphism.

(4) For any $x, y \in G_\mu G_\nu$, where $x = ab$, $y = a_1 b_1$, $a, a_1 \in G_\mu$, $b, b_1 \in G_\nu$, if $(\mu \cap \nu)_a$=$(\mu \cap \nu)_{a_1}$, then $(\mu \cap \nu)(aa_1^{-1})$=$(\mu \cap \nu)(e)$, i.e., $\mu(aa_1^{-1}) \wedge \nu(aa_1^{-1}) = \mu(e) \wedge \nu(e)$. Since $\mu(e) = \nu(e)$ and $\mu(aa_1^{-1}) = \mu(e)$, then $\nu(aa_1^{-1}) = \nu(e)$. Hence $\nu(xy^{-1}) = \nu(ab(a_1b_1)^{-1}) = \nu(abb_1^{-1}a_1^{-1}) = \nu(a_1^{-1}abb_1^{-1}) \geq \nu(a_1^{-1}a) \wedge \nu(bb_1^{-1}) = \nu((a_1^{-1}a)^{-1}) \wedge \nu(bb_1^{-1}) = \nu(e) \wedge \nu(e) = \nu(e)$. Thus $\nu_x = \nu_y$. Hence φ is a monomorphism.

By (1), (2), (3) and (4), we deduce that φ is an isomorphism from $G_\mu G_\nu/\nu$ to $G_\mu/(\mu \cap \nu)$.

Let $\phi : A \to A$ defined by $\phi(t) = t$. Then ϕ is bijective.

Also, we have $\varphi(((F/\nu)/(F_\mu/\nu))(t)) = F(t)/\mu = F(\varphi(t))/\mu = (F/\mu)(\phi(t))$. Hence $(\varphi, \phi) : (F_\mu F_\nu/\nu, A) \simeq (F_\mu/(\mu \cap \nu), A)$. ■

Corollary 4.15 *Let (F, A) be a normalistic soft group over group G and N, J two normal subgroups of G, then*

$$(F_{\chi_N} F_{\chi_J}/\chi_J, A) \simeq (F_{\chi_N}/\chi_{N \cap J}, A).$$

Corollary 4.16 *Let (F, A) be a normalistic soft group over group G and N, J two normal subgroups of G with $F_{\chi_N}(t) \supset J$ for any $t \in A$, then*

$$(F_{\chi_N} F_{\chi_J}/J, A) \simeq (F_{\chi_N}/(N \cap J), A).$$

Theorem 4.17 (Third fuzzy isomorphism theorem) *Let (F, A) be a normalistic soft group over group G and μ, ν two fuzzy normal subgroups of G with $F_\mu(t) = G_\mu$ for all $t \in A$, then*

$$((F/\nu)/(F_\mu/\nu), A) \simeq (F/\mu, A).$$

Proof. $((F/\nu)/(F_\mu/\nu), A)$ is a normalistic soft group over $(G/\nu)/(G_\mu/\nu)$ and $(F/\mu, A)$ is a normalistic soft group over G/μ. Define $\varphi : (G/\nu)/(G_\mu/\nu) \to G/\mu$ by $\varphi(\nu_x(G_\mu/\nu)) = \mu_x$.

(1) If $\nu_x(G_\mu/\nu) = \nu_y(G_\mu/\nu)$, then $\nu_x(\nu_y)^{-1} = \nu_{xy^{-1}} \in G_\mu/\nu$, then $xy^{-1} \in G_\mu$, i.e., $\mu(xy^{-1}) = \mu(e)$ and $\mu(x) = \mu(y)$. Hence, $\mu_x = \mu_y$ and φ is well-defined.

(2) For any $\nu_x(G_\mu/\nu), \nu_y(G_\mu/\nu) \in (G/\nu)/(G_\mu/\nu)$, $\varphi[(\nu_x(G_\mu/\nu))(\nu_y(G_\mu/\nu))] = \varphi[(\nu_x\nu_y)(G_\mu/\nu)] = \varphi[\nu_{xy}(G_\mu/\nu)] = \mu_{xy} = \mu_x\mu_y = \varphi(\nu_x(G_\mu/\nu)\varphi(\nu_y(G_\mu/\nu)$. Thus, φ is a homomorphism.

(3) If $\mu_x = \mu_y$, then $\mu(xy^{-1}) = \mu(e)$ and $xy^{-1} \in G/\mu$, $\nu_{xy^{-1}} = \nu_x(\nu_y)^{-1} \in G_\mu/\nu$, we have $\nu_x(G_\mu/\nu) = \nu_y(G_\mu/\nu)$. Thus, φ is a monomorphism.

(4) For any $\mu_x \in G/\mu$, there exists $\nu_x(G_\mu/\nu) \in (G/\nu)/(G_\mu/\nu)$ such that $\varphi(\nu_x(G_\mu/\nu)) = \mu_x$, so φ is an epimorphism.

It follows that φ is an isomorphism from (1), (2), (3) and (4).

Let $\phi : A \to A$ defined by $\phi(t)$=t. Then ϕ is bijective. That is, $\varphi((G_\mu G_\nu)(t)/\nu)$ =$F_\mu(t)/(\mu \cap \nu)$=$F_\mu(\phi(t))/(\mu \cap \nu)$. Hence $(\varphi, \phi) : ((F/\nu)/(F_\mu/\nu), A) \simeq (F/\mu, A)$. ■

Corollary 4.18

(1) *Let (F, A) be a normalistic soft group over group G and N, J two normal subgroups of G with $F_{\chi_J}(t) = J$ for all $t \in A$, then*

$$((F/\chi_N)/(J/\chi_N), A) \simeq (F/\chi_J, A).$$

(2) *Let (F, A) be a normalistic soft group over group G and N, J two normal subgroups of G with $N \subset J$ and $F_{\chi_J}(t) = J$ for all $t \in A$, then*

$$((F/N)/(J/N), A) \simeq (F/J, A).$$

Acknowledgement. This research is partially supported by a grant of National Natural Science Foundation of China (11561023).

References

[1] Aktaş, H., Çağman, N., *Soft sets and soft groups*, Inform. Sci., 177 (2007), 2726-2735.

[2] Ali, M.I., Mahmood, T., Muti Ur Rehman, M., Fahim Aslam, M., *On lattice ordered soft sets*, Applied Soft Computing, 36 (2015), 499-505.

[3] Das, P.S., *Fuzzy groups and level subgroups*, J. Math. Anal. Appl., 84 (1981), 264-269.

[4] Fang, J.X., *Fuzzy homomorphism and fuzzy isomorphism*, Fuzzy Sets and Systems, 63 (1994), 237-242.

[5] Maji, P.K., Roy, A.R., Biswas, R., *An application of soft sets in a decision making problem*, Comput. Math. Appl., 44 (2002), 1077-1083.

[6] Maji, P.K., Roy, A.R., Biswas, R., *Soft set theory*, Comput. Math. Appl., 45 (2003), 555-562.

[7] Liu, X.P., Xiang, D.J., Zhan, J.M., *Fuzzy isomorphism theorems of soft rings*, Neural. Comput. Applic., 21 (2012), 391-397.

[8] Liu, X.P., Xiang, D.J., Zhan, J.M., *Isomorphism theorems for soft ring*, Algebra Colloquiun, 19 (2012), 649-656.

[9] Liu, Y.L., Meng, J., Xin, X.L., *Quotient rings induced via fuzzy ideals*, Korean J. Comput. Appl. Math., 8 (2001), 631-643.

[10] Liu, Y.L., Liu, S., *Fuzzy isomorphism theorems of groups*, Far East J. Appl. Math., 16 (2004), 77-89.

[11] Molodtsov, D., *Soft set theory-first results*, Comput. Math. Appl., 37 (1999), 19-31.

[12] Mukherjee, T.K., Sen, M.K., *On fuzzy ideals on a ring I*, Fuzzy Sets and Systems, 21 (1987), 99-104.

[13] Rosenfeld, A., *Fuzzy groups*, J. Math. Anal. Appl., 35 (1971), 512-517.

[14] Sezgin, A., Atagün, A.O., *Soft groups and normalistic soft groups*, Comput. Math. Appl., 62 (2011), 685-698.

[15] Zadeh, L.A., *Fuzzy sets*, Inform. Control, 8 (1965), 338-353.

[16] Zhan, J., *The Uncertainties of Ideal Theory on Hemirings*, Science Press, 2015.

[17] Zhan, J., Çağman, N., Singin, A., *Applications of soft union sets to hemirings via SU-h-ideals*, J. Intell. Fuzzy Syst., 26 (2014), 1363-1370.

[18] Zhan, J., Shun, K.P., *Applications of soft union sets in h-hemiregular and h-intrahemiregular hemirings*, Bulletion of the Malaysian Mathmatical Sciences Society, 38 (2015), 805-825.

Accepted: 13.10.2015