HESITANT FUZZY IDEALS IN ABEL-GRASSMANN'S GROUPOIDS

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Abstract. In this paper, we introduced the concepts of hesitant fuzzy sets, hesitant fuzzy product, characteristic hesitant fuzzy set, hesitant fuzzy $\mathcal{AG}$-groupoids, hesitant fuzzy left (resp., right, two-sided) ideal, hesitant fuzzy bi-ideal, hesitant fuzzy interior ideal and hesitant fuzzy quasi-ideal in $\mathcal{AG}$-groupoids, their examples and basic properties are given. We also characterized regular, completely regular, weakly regular and quasi-regular $\mathcal{AG}$-groupoid by the properties of their hesitant fuzzy ideals.

Keywords: hesitant fuzzy sets, hesitant fuzzy $\mathcal{AG}$-groupoid, hesitant fuzzy left (resp., right, two-sided, quasi-, bi-, interior) ideal.

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Introduction

A fuzzy subset f of a given set S (or a fuzzy set in S) is, described as an arbitrary function $f : S \to [0,1]$, where $[0,1]$ is the usual closed interval of real numbers. This fundamental concept of fuzzy set, was introduced by Zadeh in 1965 [17], it has been widely studied, developed and successfully applied in various fields, such as multicriteria decision making (see [19], [20]), fuzzy logic and approximate reasoning [18], and pattern recognition [21]. In real cases, due to the fuzziness and uncertainty of decision making problems, the criteria's weights and evaluation values of alternatives can be inaccurate, uncertain, or incomplete. For the problems like those, fuzzy sets, especially fuzzy numbers, can provide good solutions. However, in fuzzy sets the membership degree of the element is represented by a single value between zero and one, and a major drawback of fuzzy sets is that single values cannot convey information precisely.

In practice, the information regarding alternatives, when referring to a fuzzy set concept, may be incomplete, that is, the sum of membership and non-membership degree of an element in the universe can be less then one. The fuzzy set fails when it comes to managing the insufficient understanding of membership degrees. Thus Atanassov's intuitionistic fuzzy sets, interval valued fuzzy sets, and trapezoidal or triangular intuitionistic fuzzy sets, as the extensions of Zadeh's fuzzy sets, were introduced ([4], [5], [6]). Interval valued fuzzy sets, intuitionistic fuzzy sets has been widely applied in solving multicriteria decision making problems ([6], [7]).

Hesitant fuzzy set is a novel and recent extension of fuzzy sets that aims to model the uncertainty originated by the hesitation that might arise in the assignment of membership degrees of the elements to a fuzzy set. Despite the previous extensions overcome in different ways the managing of simultaneous sources of vagueness, Torra [10], introduced a useful generalization of the fuzzy set that is designed for situations in which it is difficult to determine the membership of an element to a set owing to ambiguity between a few different values. The hesitant fuzzy set permits the membership degree of an element to a set to be represented by a set of possible values between 0 and 1 (see [10], [11]). The hesitant fuzzy set therefore provides a more accurate representation of peoples hesitancy in stating their preferences over objects than the fuzzy set or its classical extensions. Hesitant fuzzy sets are very useful to deal with group decision making problems when experts have hesitation among several possible memberships for an element to a set. During the evaluating process in practice, however, these possible memberships may not be only crisp values in $[0, 1]$, but also interval values. Hesitant fuzzy sets have attracted the attention of many researchers in a short period of time because hesitant situations are very common in different real world problems and this new approach facilitates the management of uncertainty provoked by hesitation. A deep revision of the specialized literature shows the quick growth and applicability of hesitant fuzzy sets which have been extended from different points of view, (see [9], [11], [12], [13], [14], [15], [16]). Jun and Song applied the notion of hesitant fuzzy sets to MTL-algebras and EQ-algebras (see [1], [2]). Additionally, many operators for hesitant fuzzy sets and their extensions have been introduced to deal with such type of information in different applications where decision making has been the most remarkable one. Here the purpose of this paper is to deal with the algebraic structure of $\mathcal{AG}$-groupoids by applying the concept of hesitant fuzzy sets.

In this study, we applied the notion of hesitant fuzzy sets to $\mathcal{AG}$-groupoids. We introduced the notions of hesitant fuzzy product, characteristic hesitant fuzzy set, hesitant fuzzy $\mathcal{AG}$-groupoids, hesitant fuzzy left (resp., right, two-sided) ideal, hesitant fuzzy bi-ideal, hesitant fuzzy interior ideal and hesitant fuzzy quasi-ideal on $\mathcal{AG}$-groupoids and investigate several properties. Moreover, we consider characterizations of regular, completely regular, weakly regular and quasi-regular $\mathcal{AG}$-groupoids in terms of hesitant fuzzy ideals.

Preliminaries notes

The concept of an $\mathcal{AG}$-groupoid (Abel-Grassmann's groupoid) [3] was first given by M. A. Kazim and M. Naseeruddin in 1972.

A groupoid $(\mathcal{S}, .)$ is called an $\mathcal{AG}$-groupoid ($\mathcal{LA}$-semigroup in some articles [8]), if its elements satisfy left invertive law: $(ab)c = (cb)a$ for all $a, b, c \in \mathcal{S}$. In an $\mathcal{AG}$-groupoid medial law [3], $(ab)(cd) = (ac)(bd)$, holds for all a, b, c, $d \in \mathcal{S}$. It is also known that in an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the paramedial law: $(ab)(cd) = (db)(ca)$, [8] holds for all a, b, c, $d \in \mathcal{S}$. If an $\mathcal{AG}$-groupoid contains left identity, then the following law [8] holds,

$$a(bc) = b(ac), \text{for all } a, b, c \in \mathcal{S}. \tag{1}$$

Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and X, Y be subsets of $\mathcal{S}$. Then, the multiplication of X and Y is defined as follows:

$$XY = \{xy \in \mathcal{S} \setminus x \in X \text{ and } y \in Y\}.$$

Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. By $\mathcal{AG}$-subgroupoid of $\mathcal{S}$ we means a non-empty subset A of $\mathcal{S}$ such that $A^2 \subseteq A$, and by a left (resp., right) ideal of $\mathcal{S}$ we mean a non-empty subset I of $\mathcal{S}$ such that $\mathcal{S}I \subseteq I$ ($I\mathcal{S} \subseteq I$). By two-sided ideals or simply ideal, we mean a non-empty subset of $\mathcal{S}$ which is both a left and a right ideal of $\mathcal{S}$. An $\mathcal{AG}$-subgroupoid B of $\mathcal{S}$ is called a bi-ideal of $\mathcal{S}$ if $(B\mathcal{S})B \subseteq B$. A non-empty subset B of $\mathcal{S}$ is called a generalized bi-ideal of $\mathcal{S}$ if $(B\mathcal{S})B \subseteq B$. A non-empty subset Q of $\mathcal{S}$ is called quasi-ideal of $\mathcal{S}$ if $Q\mathcal{S} \cap \mathcal{S}Q \subseteq Q$.

Obviously, every one sided ideal of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal.

Sometimes, it is difficult to determine the membership of an element into a fixed set and which may be caused by a doubt among the set of different values. For the sake of better description of this situation, Torra [10], introduced the concept of hesitant fuzzy sets as a generalization of fuzzy sets. Torra defined hesitant fuzzy sets in terms of a function that returns a set of membership values for each element in the domain. This is formally defined in the next definition.

Definition 1 [10] Let $\mathcal{S}$ be a reference set. Then, we define hesitant fuzzy set on $\mathcal{S}$ in terms of a function h that when applied to $\mathcal{S}$ returns a finite subset of $[0, 1]$. A hesitant fuzzy set $\mathcal{H}$ can also be viewed as the following mathematical representation:

$$\mathcal{H} := \{(x, h(x)) \setminus \forall\ x \in \mathcal{S}\},$$

where $h(x)$ is the set of some different values in $[0, 1]$, denoting the possible membership degrees of the elements of $x \in \mathcal{S}$ to the set $\mathcal{H}$. The set of all hesitant fuzzy sets on $\mathcal{S}$ is denoted by

$$H(\mathcal{S}) = \left\{ \begin{array}{c} \{(x, h(x))\setminus\forall x \in \mathcal{S}\}\setminus \text{ for any } x \in \mathcal{S} \text{ and any} \\ \text{function } h \text{ is the set of some different valuesin } [0, 1] \end{array} \right\}$$

Note that, throughout this paper, the set of all subsets of $[0, 1]$ will be denoted by $\mathcal{P}([0, 1]$.

Let g and h be two hesitant fuzzy sets on $\mathcal{S}$. Then, hesitant union $g \sqcup h$ and hesitant intersection $g \sqcap h$ of g and h are defined to be hesitant fuzzy sets on $\mathcal{S}$ as follows:

$$g \sqcup h :\to \mathcal{P}([0,1], x \mapsto g(x) \sqcup h(x) \text{ and } g \sqcap h :\to \mathcal{P}([0,1], x \mapsto g(x) \sqcap h(x),$$

respectively.

Definition 2 The hesitant fuzzy product of two hesitant fuzzy sets f and g on an $\mathcal{AG}$-groupoid $\mathcal{S}$ is defined to be a hesitant fuzzy set $f\widetilde{\circ}g$ on $\mathcal{S}$ which is given by

$$(f\widetilde{\circ}g)(x) = \begin{cases} \bigcup\limits_{x=yz} \{f(y) \cap g(z)\} & \text{if there exist } x,\, z \in \mathcal{S} \text{ such that } x = yz, \\ \phi & \text{otherwise.} \end{cases}$$

We denote by $H(\mathcal{S})$ the set of all hesitant fuzzy sets on $\mathcal{S}$.

Definition 3 Let f and g be the elements of $H(\mathcal{S})$. Then, f is called the subset of g, denoted by $f \sqsubseteq g$, if $f(x) \subseteq g(x)$ for all $x \in \mathcal{S}$.

Theorem 1 *For any hesitant fuzzy sets $f, g, h \in H(\mathcal{S})$, the following conditions holds.*

(i) $(f\widetilde{\circ}g)\widetilde{\circ}h = f\widetilde{\circ}(g\widetilde{\circ}h)$.

(ii) $f\widetilde{\circ}g \neq g\widetilde{\circ}f$, *in general.*

(iii) $f\widetilde{\circ}(g \sqcup h) = (f\widetilde{\circ}g) \sqcup (f\widetilde{\circ}h)$ *and* $(f \sqcup g)\widetilde{\circ}h = (f\widetilde{\circ}h) \sqcup (g\widetilde{\circ}h)$.

(iv) $f\widetilde{\circ}(g \sqcap h) = (f\widetilde{\circ}g) \sqcap (f\widetilde{\circ}h)$ *and* $(f \sqcap g)\widetilde{\circ}h = (f\widetilde{\circ}h) \sqcap (g\widetilde{\circ}h)$.

(v) *If* $f \sqsubseteq g$, *then* $f\widetilde{\circ}h \sqsubseteq g\widetilde{\circ}h$ *and* $h\widetilde{\circ}f \sqsubseteq h\widetilde{\circ}g$.

(vi) *If* $t, l \in H(\mathcal{S})$ *such that* $t \sqsubseteq f$ *and* $l \sqsubseteq g$, *then* $t\widetilde{\circ}l \sqsubseteq f\widetilde{\circ}g$.

Proposition 1 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, the set $(H(\mathcal{S}), \widetilde{\circ})$ forms an $\mathcal{AG}$-groupoid.*

Proof. It is obvious that $H(\mathcal{S})$ is closed. Let $f, g, h \in H(\mathcal{S})$. Let $a \in \mathcal{S}$. If a is not expressible as $a = bc$ for some $b, c \in \mathcal{S}$. Then, $((f\widetilde{\circ}g)\widetilde{\circ}h)(a) = ((h\widetilde{\circ}g)\widetilde{\circ}f)(a) = \phi$. Let for $a \in \mathcal{S}$ there exist $b, c \in \mathcal{S}$ such that $a = bc$. Then,

$$\begin{aligned} ((f\widetilde{\circ}g)\widetilde{\widetilde{\circ}}h)(a) &= \bigcup_{a=bc} \{(f\widetilde{\circ}g)(b) \cap h(c)\} = \bigcup_{a=bc} \left\{ \bigcup_{b=pq} \{f(p) \cap g(q)\} \cap h(c) \right\} \\ &= \bigcup_{a=(pq)c} \{f(p) \cap g(q) \cap h(c)\} = \bigcup_{a=(cq)p} \{h(c) \cap g(q) \cap f(p)\} \\ &= \bigcup_{a=wp} \left\{ \bigcup_{w=cq} \{h(c) \cap g(q)\} \cap f(p) \right\} = \bigcup_{a=wp} \{(h\widetilde{\circ}g)(w) \cap f(p)\} \\ &= ((h\widetilde{\circ}g)\widetilde{\circ}f)(a). \end{aligned}$$

Thus, $H(\mathcal{S})$ satisfies left invertive law. Hence, $(H(\mathcal{S}), \widetilde{\circ})$ is an $\mathcal{AG}$-groupoid. ∎

Proposition 2 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid, then $H(\mathcal{S})$ satisfies the medial law.*

Proof. Let f, g, h and k be any hesitant fuzzy sets of $H(\mathcal{S})$. Then, by using left invertive law,

$$(f\widetilde{\circ}g)\widetilde{\circ}(h\widetilde{\circ}k) = ((h\widetilde{\circ}k)\widetilde{\circ}g)\widetilde{\circ}f = ((g\widetilde{\circ}k)\widetilde{\circ}h)\widetilde{\circ}f = (f\widetilde{\circ}h)\widetilde{\circ}(g\widetilde{\circ}k). \qquad \blacksquare$$

Theorem 2 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity. Then, the following properties holds in $H(\mathcal{S})$.*

(i) $f\widetilde{\circ}(g\widetilde{\circ}h) = g\widetilde{\circ}(f\widetilde{\circ}h)$, *for all hesitant fuzzy sets $f, g, h \in H(\mathcal{S})$.*

(ii) $(f\widetilde{\circ}g)\widetilde{\circ}(h\widetilde{\circ}k) = (k\widetilde{\circ}h)\widetilde{\circ}(g\widetilde{\circ}f)$, *for all hesitant fuzzy sets $f, g, h, k \in H(\mathcal{S})$.*

Proof. (i) Let $a \in \mathcal{S}$. If $a \neq bc$ for some $b, c \in \mathcal{S}$, then $(f\widetilde{\circ}(g\widetilde{\circ}h))(a) = (g\widetilde{\circ}(f\widetilde{\circ}h))(a) = \phi$. If a is expressible as $a = bc$ for some $b, c \in \mathcal{S}$, then

$$\begin{aligned}(f\widetilde{\circ}(g\widetilde{\circ}h))(a) &= \bigcup_{a=bc}\{f(b) \cap (g\widetilde{\circ}h)(c)\} = \bigcup_{a=bc}\left\{f(b) \cap \bigcup_{c=pq}\{g(p) \cap h(q)\}\right\} \\ &= \bigcup_{a=b(pq)}\{f(b) \cap g(p) \cap h(q)\} = \bigcup_{a=p(bq)}\{g(p) \cap f(b) \cap h(q)\} \\ &= \bigcup_{a=pw}\left\{g(p) \cap \bigcup_{w=bq}\{f(b) \cap h(q)\}\right\} = \bigcup_{a=pw}\{g(p) \cap (f\widetilde{\circ}h)(w)\} \\ &= (g\widetilde{\circ}(f\widetilde{\circ}h))(a).\end{aligned}$$

Hence, $f\widetilde{\circ}(g\widetilde{\circ}h) = g\widetilde{\circ}(f\widetilde{\circ}h)$ for all $a \in \mathcal{S}$.

(ii) Let $a \in \mathcal{S}$ and $a \neq bc$ for some $b, c \in \mathcal{S}$, then $(f\widetilde{\circ}g)\widetilde{\circ}(h\widetilde{\circ}k) = (k\widetilde{\circ}h)\widetilde{\circ}(g\widetilde{\circ}f) = \phi$. Assume that $a \in \mathcal{S}$ is expressible as $a = bc$ for some $b, c \in \mathcal{S}$. Then,

$$\begin{aligned}((f\widetilde{\circ}g)\widetilde{\circ}(h\widetilde{\circ}k))(a) &= \bigcup_{a=bc}\{(f\widetilde{\circ}g)(b) \cap (h\widetilde{\circ}k)(c)\} \\ &= \bigcup_{a=bc}\left\{\bigcup_{b=pq}\{f(p) \cap g(q)\} \cap \bigcup_{c=uv}\{h(u) \cap k(v)\}\right\} \\ &= \bigcup_{a=(pq)(uv)}\{f(p) \cap g(q) \cap h(u) \cap k(v)\} \\ &= \bigcup_{a=(vu)(qp)}\{k(p) \cap h(q) \cap g(u) \cap f(v)\} \\ &= \bigcup_{a=mn}\left\{\bigcup_{m=vu}\{k(p) \cap h(u)\} \cap \bigcup_{n=qp}\{g(q) \cap f(p)\}\right\} \\ &= \bigcup_{a=mn}\{(k\widetilde{\circ}h)(m) \cap (g\widetilde{\circ}f)(n)\} \\ &= ((k\widetilde{\circ}h)\widetilde{\circ}(g\widetilde{\circ}f))(a)\end{aligned}$$

Hence, $(f\widetilde{\circ}g)\widetilde{\circ}(h\widetilde{\circ}k) = (k\widetilde{\circ}h)\widetilde{\circ}(g\widetilde{\circ}f)$ for all $a \in \mathcal{S}$. $\blacksquare$

Definition 4 For a non-empty subset A of an $\mathcal{AG}$-groupoid $\mathcal{S}$, define a map

$$\mathcal{H}_A : \mathcal{S} \to \mathcal{P}([0,1]),\ x \mapsto \begin{cases} [0,1] & \text{if } x \in A, \\ \phi & \text{otherwise.} \end{cases}$$

Then, $\mathcal{H}_A$ is a hesitant fuzzy set on $\mathcal{S}$ and is called characteristic hesitant fuzzy set on $\mathcal{S}$. The hesitant fuzzy set $\mathcal{H}_{\mathcal{S}}$ is called identity hesitant fuzzy set on $\mathcal{S}$.

Theorem 3 *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be hesitant fuzzy sets on $\mathcal{AG}$-groupoid $\mathcal{S}$, where A and B are non-empty subsets of $\mathcal{S}$. Then, the following properties hold.*

(i) *If $A \subseteq B$, then $\mathcal{H}_A \sqsubseteq \mathcal{H}_B$.*

(ii) $\mathcal{H}_A \sqcap \mathcal{H}_B = \mathcal{H}_{A\cap B}$.

(iii) $\mathcal{H}_A \widetilde{\circ} \mathcal{H}_B = \mathcal{H}_{AB}$.

Proof. (i) It is obvious.

(ii) Let $x \in \mathcal{S}$. If $x \in A \cap B$, then $x \in A$ and $x \in B$. Thus, we have

$$(\mathcal{H}_A \sqcap \mathcal{H}_B)(x) = \mathcal{H}_A(x) \cap \mathcal{H}_B(x) = [0,1] = \mathcal{H}_{A\cap B}(x).$$

If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$. Hence, we have

$$(\mathcal{H}_A \sqcap \mathcal{H}_B)(x) = \mathcal{H}_A(x) \cap \mathcal{H}_B(x) = \phi = \mathcal{H}_{A\cap B}(x).$$

Therefore, $\mathcal{H}_A \sqcap \mathcal{H}_B = \mathcal{H}_{A\cap B}$.

(iii) For any $x \in \mathcal{S}$. Let $x \in AB$. Then, there exist $y \in A$ and $z \in B$ such that $x = yz$. Thus, we have

$$(\mathcal{H}_A \widetilde{\circ} \mathcal{H}_B)(x) = \bigcup_{x=yz} \{\mathcal{H}_A(y) \cap \mathcal{H}_B(z)\} \supseteq \mathcal{H}_A(y) \cap \mathcal{H}_B(z) = [0,1],$$

and so $(\mathcal{H}_A \widetilde{\circ} \mathcal{H}_B)(x) = [0,1]$. Since $x \in AB$, we get $\mathcal{H}_{AB}(x) = [0,1]$. Suppose $x \notin AB$. Then $x \neq yz$ for any $y \in A$ and $z \in B$. If $x = yz$ for some $y, z \in \mathcal{S}$, then $y \notin A$ or $z \notin B$. Hence,

$$(\mathcal{H}_A \widetilde{\circ} \mathcal{H}_B)(x) = \bigcup_{x=yz} \{\mathcal{H}_A(y) \cap \mathcal{H}_B(z)\} = \phi = \mathcal{H}_{AB}(x).$$

If $x \neq yz$ for all $x, y \in \mathcal{S}$, then

$$(\mathcal{H}_A \widetilde{\circ} \mathcal{H}_B)(x) = \phi = \mathcal{H}_{AB}(x).$$

In any case, we have $\mathcal{H}_A \widetilde{\circ} \mathcal{H}_B = \mathcal{H}_{AB}$. ∎

Definition 5 A hesitant fuzzy set h on $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$ if it satisfies:

$$(\forall x, y \in \mathcal{S})\ (h(xy) \supseteq h(x) \cap h(y)).$$

Example 1 Let $\mathcal{S} = \{1, 2, 3\}$ be an $\mathcal{AG}$-groupoid with the following Cayley multiplication table.

$\cdot$	1	2	3
1	1	1	1
2	1	1	3
3	1	1	1

Let h be a hesitant fuzzy set on $\mathcal{S}$ defined as follows:

$$h : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} (0.2, 0.8] & \text{if } x = 1, \\ [0.3, 0.7] & \text{if } x = 2, \\ [0.5, 0.7) & \text{if } x = 3. \end{cases}$$

It is easy to observe that h is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$.

If $h(a) = [0,1]\ \forall\ a \in \mathcal{S}$, then it is easy to see that h is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. We denote such type of hesitant fuzzy $\mathcal{AG}$-groupoids by $\widetilde{\mathcal{H}}$. It is obvious that $\widetilde{\mathcal{H}} = \mathcal{H}_{\mathcal{S}}$ that is $\widetilde{\mathcal{H}}(a) = [0,1]\ \forall\ a \in \mathcal{S}$.

Lemma 1 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and f be a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. Then, $\widetilde{\mathcal{H}}\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq \widetilde{\mathcal{H}}, f\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq \widetilde{\mathcal{H}}, \widetilde{\mathcal{H}}\widetilde{\circ}f \sqsubseteq \widetilde{\mathcal{H}}, f \sqcup \widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}$ and $f \sqcap \widetilde{\mathcal{H}} \sqsubseteq f$.*

Proposition 3 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity. Then, $\widetilde{\mathcal{H}}\widetilde{\circ}\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}$.*

Proof. Let $a \in \mathcal{S}$. Since every element $a \in \mathcal{S}$ can be expressed as $a = ea$, where e is the left identity in $\mathcal{S}$, so we have

$$(\widetilde{\mathcal{H}}\widetilde{\circ}\widetilde{\mathcal{H}})(a) = \bigcup_{a=bc} \left\{\widetilde{\mathcal{H}}(b) \cap \widetilde{\mathcal{H}}(c)\right\} \supseteq \left\{\widetilde{\mathcal{H}}(e) \cap \widetilde{\mathcal{H}}(a)\right\} = [0,1].$$

Hence, $(\widetilde{\mathcal{H}}\widetilde{\circ}\widetilde{\mathcal{H}})(a) = [0,1] = \widetilde{\mathcal{H}}(a)\ \forall\ a \in \mathcal{S}$. ■

Theorem 4 *Let f be a hesitant fuzzy set on $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$ if and only if*

$$f\widetilde{\circ}f \sqsubseteq f.$$

Proof. Let f be a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. Let $a \in \mathcal{S}$.

If $a \neq bc$ for some $b, c \in \mathcal{S}$. Then, $(f\widetilde{\circ}f)(a) = \phi \sqsubseteq f(a)$, implies that $(f\widetilde{\circ}f)(a) \sqsubseteq f(a)$. Hence, $f\widetilde{\circ}f \sqsubseteq f$.

If $a = bc$ for some $b, c \in \mathcal{S}$. Then, since f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$, so we have $(f\widetilde{\circ}f)(a) = \bigcup\limits_{a=bc} \{f(b) \cap f(c)\} \subseteq \bigcup\limits_{a=bc} f(bc) = \bigcup\limits_{a=bc} f(a) = f(a)$. Hence, $f\widetilde{\circ}f \sqsubseteq f$.

Conversely, suppose that $f\widetilde{\circ}f \sqsubseteq f$. Let $a, b, c \in \mathcal{S}$ and $a = bc$, then we have $f(bc) = f(a) \supseteq (f\widetilde{\circ}f)(a) = \bigcup\limits_{a=bc} \{f(b) \cap f(c)\} \supseteq f(b) \cap f(c)$. Hence, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. ■

Theorem 5 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and A be a non-empty subset of $\mathcal{S}$. Then, A is an $\mathcal{AG}$-subgroupoid of $\mathcal{S}$ if and only if $\mathcal{H}_A$ is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$.*

Proof. Suppose that A is an $\mathcal{AG}$-subgroupoid of $\mathcal{S}$, then $AA \subseteq A$. Thus, we have $\mathcal{H}_A \tilde{\circ} \mathcal{H}_A = \mathcal{H}_{AA} \sqsubseteq \mathcal{H}_A$. By Theorem 4, $\mathcal{H}_A$ is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$.

Conversely, assume that $\mathcal{H}_A$ is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$ and let $a \in AA$. Then, by using Theorem 4, we have $\mathcal{H}_A(a) \supseteq (\mathcal{H}_A \tilde{\circ} \mathcal{H}_A)(a) = \mathcal{H}_{AA}(a) = [0,1] \Rightarrow \mathcal{H}_A(a) = [0,1] \Rightarrow a \in A$. Thus, $AA \subseteq A$. Hence, A is an $\mathcal{AG}$-subgroupoid of $\mathcal{S}$. ■

Definition 6 A hesitant fuzzy set h on an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a hesitant fuzzy left (resp., right) ideal on $\mathcal{S}$ if it satisfies:

$$(\forall x, y \in \mathcal{S})\ (h(xy) \supseteq h(y)\ (\text{resp. }, h(xy) \supseteq h(x)).$$

A hesitant fuzzy set h on $\mathcal{S}$ is called a hesitant fuzzy two-side ideal on $\mathcal{S}$ if it is both hesitant fuzzy left ideal and hesitant fuzzy right ideal on $\mathcal{S}$.

Example 2 Let $\mathcal{S} = \{a, b, c, d\}$ be an $\mathcal{AG}$-groupoid with the following Cayley multiplication table.

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Let h be a hesitant fuzzy set on $\mathcal{S}$ defined as follows:

$$h : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} (0.2, 0.8] & \text{if } x = a, \\ [0.3, 0.7] & \text{if } x = b, \\ [0.5, 0.7] & \text{if } x = c, \\ (0.5, 0.6) \cup (0.6, 0.7) & \text{if } x = d. \end{cases}$$

Then, it is easy to verify that h is a hesitant fuzzy two-sided ideal on $\mathcal{S}$.

Obviously, every hesitant fuzzy left (resp., right) ideal on $\mathcal{S}$ is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. But the converse is not true as seen in the following example.

Example 3 Let $\mathcal{S} = \{0, 1, 2, 3, 4, 5\}$ be an $\mathcal{AG}$-groupoid with the following Cayley multiplication table.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	1	2	2	3	1	1
3	0	1	1	1	3	3
4	0	3	4	5	1	1
5	0	1	1	1	4	5

Let h be a hesitant fuzzy set on $\mathcal{S}$ defined as follows:

$$h : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} [0,1) & \text{if } x = 0, \\ \{0\} \cup [0.4, 1) & \text{if } x = 1, \\ [0.5, 0.8] \cup \{0.9\} & \text{if } x \in \{2, 4\}, \\ [0.5, 0.9] & \text{if } x = 3, \\ \{0, 0.2\} \cup [0.5, 1) & \text{if } x = 5. \end{cases}$$

Then, h is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. But it is not a hesitant fuzzy left ideal on $\mathcal{S}$, since $h(3.5) = h(3) = [0.5, 0.9] \not\supseteq \{0, 0.2\} \cup [0.5, 1) = h(5)$.

Example 4 Let $\mathcal{S} = \{0, 1, 2, 3, 4, 5\}$ be an $\mathcal{AG}$-groupoid with the Cayley multiplication table which is appeared in Example 3. Let g be a hesitant fuzzy set on $\mathcal{S}$ defined as follows:

$$g : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} [0.3, 1] & \text{if } x \in \{0, 1\}, \\ [0.4, 0.9] & \text{if } x = 2, \\ (0.3, 1) & \text{if } x = 3, \\ (0.4, 0.9) & \text{if } x \in \{4, 5\}. \end{cases}$$

Then, g is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. But it is not a hesitant fuzzy right ideal on $\mathcal{S}$, since $g(3.4) = g(2) = [0.4, 0.9] \not\supseteq (0.3, 1) = g(3)$.

Theorem 6 *Let f be a hesitant fuzzy set on $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, f is a hesitant fuzzy left (resp., right) ideal on $\mathcal{S}$ if and only if $(\widetilde{\mathcal{H}}\tilde{\circ}f \sqsubseteq f)$ (resp., $f\tilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f$).*

Proof. Assume that f be a hesitant fuzzy left ideal on $\mathcal{S}$. Let $a \in \mathcal{S}$. If $a \neq bc$ for some $b, c \in \mathcal{S}$, then $(\widetilde{\mathcal{H}}\tilde{\circ}f)(a) = \phi \sqsubseteq f(a)$, that is $\widetilde{\mathcal{H}}\tilde{\circ}f \sqsubseteq f(a)$. If a is expressible as $a = bc$ for some $b, c \in \mathcal{S}$. Then, since f is a hesitant fuzzy left ideal on $\mathcal{S}$, we have $(\widetilde{\mathcal{H}}\tilde{\circ}f)(a) = \bigcup_{a=bc} \left\{\widetilde{\mathcal{H}}(b) \cap f(c)\right\} \subseteq \{[0,1] \cup f(bc)\} = \bigcup_{a=bc} \{[0,1] \cap f(a)\} = f(a)$. Hence, $\widetilde{\mathcal{H}}\tilde{\circ}f \sqsubseteq f$.

Conversely, let $\widetilde{\mathcal{H}}\tilde{\circ}f \sqsubseteq f$. Let there exist $b, c \in \mathcal{S}$ such that $a = bc$. Then, we have $f(bc) = f(a) \supseteq (\widetilde{\mathcal{H}}\tilde{\circ}f)(a) = \bigcup_{a=bc} \left\{\widetilde{\mathcal{H}}(b) \cap f(c)\right\} \supseteq \widetilde{\mathcal{H}}(b) \cap f(c) = [0,1] \cap f(c) = f(c)$. Hence, f is a hesitant fuzzy left ideal on $\mathcal{S}$.

The other case can be seen in a similar way. ∎

Theorem 7 *For any non-empty subset A of an $\mathcal{AG}$-groupoid $\mathcal{S}$, the following are equivalent.*

(i) *A is left (resp., right, two-sided) ideal of $\mathcal{S}$.*

(ii) *The characteristic hesitant fuzzy set $\mathcal{H}_A$ on $\mathcal{S}$ is a left (resp., right, two-sided) ideal on $\mathcal{S}$.*

Proof. (i) $\Rightarrow$ (ii) : Let A be left ideal of an $\mathcal{AG}$-groupoid $\mathcal{S}$. For any $x, y \in \mathcal{S}$, if $y \notin A$ then $\mathcal{H}_A(y) = \phi \subseteq \mathcal{H}_A(xy)$. If $y \in A$, then $x, y \in A$, since A is left ideal of $\mathcal{S}$. Hence, $\mathcal{H}_A(xy) = [0,1] = \mathcal{H}_A(y)$. Therefore, $\mathcal{H}_A$ is a hesitant fuzzy left ideal on $\mathcal{S}$. Similarly, $\mathcal{H}_A$ is a hesitant fuzzy right and hesitant fuzzy two-sided ideal on $\mathcal{S}$ when A is a right ideal and two-sided ideal of $\mathcal{S}$.

(ii) $\Rightarrow$ (i) : Let $\mathcal{H}_A$ be a hesitant fuzzy left ideal on $\mathcal{S}$. Let $x \in \mathcal{S}$ and $y \in A$. Then $\mathcal{H}_A(y) = [0,1]$, and so $\mathcal{H}_A(xy) \supseteq \mathcal{H}_A(y) = [0,1]$, that is, $\mathcal{H}_A(xy) = [0,1]$. Thus $x, y \in A$ and therefore A is a left ideal of $\mathcal{S}$. Similarly, we can show that A is a right ideal and two-sided ideal of $\mathcal{S}$, when $\mathcal{H}_A$ is a hesitant fuzzy right ideal and hesitant fuzzy two-sided ideal on $\mathcal{S}$. ∎

Theorem 8 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and f be a hesitant fuzzy set on $\mathcal{S}$. If f is a hesitant fuzzy left (resp., right, two-sided) ideal on $\mathcal{S}$, then f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$.*

Proof. Let f be a hesitant fuzzy left ideal on $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, $f(ab) \supseteq f(b)$ $\forall\ a, b \in \mathcal{S}$ implies that $f(ab) \supseteq f(b) \supseteq f(a) \cap f(b)$. Hence, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$.

Similarly, we can show that if f is a hesitant fuzzy (right, two-sided) ideal on $\mathcal{S}$, then f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. ∎

Definition 7 Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. A hesitant fuzzy $\mathcal{AG}$-groupoid f on $\mathcal{S}$ is called a hesitant fuzzy bi-ideal on $\mathcal{S}$ if it satisfies:

$$(\forall\ a, b, c \in \mathcal{S})\ (h((ab)c) \supseteq h(a) \cap h(c)).$$

Example 5 Let $\mathcal{S} = \{1, 2, 3\}$ be an $\mathcal{AG}$-groupoid with the following Cayley multiplication table.

$\cdot$	1	2	3
1	1	1	1
2	1	1	3
3	1	1	1

Let f be a hesitant fuzzy set on $\mathcal{S}$ defined as follows:

$$f : \mathcal{S} \to \mathcal{P}([0,1]),\ x \mapsto \begin{cases} (0.3, 0.9] & \text{if } x = 1, \\ [0.4, 0.8] & \text{if } x = 2, \\ [0.6, 0.8] & \text{if } x = 3. \end{cases}$$

Then, it is easy to verify that f is a hesitant fuzzy bi-ideal on $\mathcal{S}$.

Theorem 9 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and f be a hesitant fuzzy set on $\mathcal{S}$. Then, f is a hesitant fuzzy bi-ideal on $\mathcal{S}$ if and only if*

$$f \widetilde{\circ} f \sqsubseteq f \text{ and } (f \widetilde{\circ} \widetilde{\mathcal{H}}) \widetilde{\circ} f \sqsubseteq f.$$

Proof. Let f be a hesitant fuzzy bi-ideal on $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$ and by Theorem 4, we have $f \widetilde{\circ} f \sqsubseteq f$. Next let $a \in \mathcal{S}$. If a is not expressible as $a = bc$ for some $b, c \in \mathcal{S}$, then $((f \widetilde{\circ} \widetilde{\mathcal{H}}) \widetilde{\circ} f)(a) = \phi \sqsubseteq f$. If a is expressible as $a = bc$ for some $b, c \in \mathcal{S}$, then we have

$$\begin{aligned}((f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f)(a) &= \bigcup_{a=bc}\left\{(f\widetilde{\circ}\widetilde{\mathcal{H}})(b)\cap f(c)\right\} = \bigcup_{a=bc}\left\{\bigcup_{b=mn}\left\{f(m)\cap\widetilde{\mathcal{H}}(n)\right\}\cap f(c)\right\}\\ &= \bigcup_{a=bc}\left\{\bigcup_{b=mn}\{f(m)\cap[0,1]\}\cap f(c)\right\} = \bigcup_{a=(mn)c}\{f(m)\cap f(c)\}\\ &\subseteq \bigcup_{a=(mn)c} f((mn)c) = f_a.\end{aligned}$$

Hence, $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq f$.

Conversely, let $f\widetilde{\circ}f \sqsubseteq f$. Then, by Theorem 4, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. Now, since $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq f$, let $a,b,c\in\mathcal{S}$, then we have

$$\begin{aligned}f((ab)c) &\supseteq ((f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f)((ab)c) = \bigcup_{((ab)c)=pq}\left\{(f\widetilde{\circ}\widetilde{\mathcal{H}})(p)\cap f(q)\right\}\\ &= \bigcup_{((ab)c)=pq}\left\{\bigcup_{p=mn}\left\{f(m)\cap\widetilde{\mathcal{H}}(n)\right\}\cap f(q)\right\}\\ &= \bigcup_{((ab)c)=(mn)q}\{f(m)\cap f(q)\}\supseteq f(a)\cap f(c).\end{aligned}$$

This shows that, $f((ab)c)\supseteq f(a)\cap f(c)\ \forall\ a,b,c\in\mathcal{S}$. Hence, f is a hesitant fuzzy bi-ideal on $\mathcal{S}$. ■

Theorem 10 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, every hesitant fuzzy left (resp., right, two-sided) ideal on $\mathcal{S}$ is a hesitant fuzzy bi-ideal on $\mathcal{S}$.*

Proof. Let f be a hesitant fuzzy left (resp., right, two-sided) ideal on $\mathcal{S}$. Then, by Theorem 8, f is a hesitant fuzzy left (resp., right, two-sided) ideal on $\mathcal{S}$. Also $f((ab)c)\supseteq f(c)\supseteq f(a)\cap f(c)\ \forall\ a,b,c\in\mathcal{S}$. Hence, f is a hesitant fuzzy bi-ideal on $\mathcal{S}$. ■

Theorem 11 *Let A be a non-empty subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, A is a bi-ideal of $\mathcal{S}$ if and only if the characteristic hesitant fuzzy set $\mathcal{H}_A$ is a hesitant fuzzy bi-ideal on $\mathcal{S}$.*

Definition 8 Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and f be a hesitant fuzzy set on $\mathcal{S}$. Then, f is called a hesitant fuzzy interior ideal on $\mathcal{S}$, if

$$(\forall\ a,b,c\in\mathcal{S})\ (f((ab)c)\supseteq f(b)).$$

Example 6 Let $\mathcal{S}=\{p,q,r\}$ be an $\mathcal{AG}$-groupoid with the following Cayley multiplication table.

$\cdot$	p	q	r
p	r	r	q
q	q	q	q
r	q	q	q

Let f be a hesitant fuzzy set on $\mathcal{S}$ defined as follows:

$$f : \mathcal{S} \to \mathcal{P}([0,1]),\ x \mapsto \begin{cases} [0.5, 0.7) & \text{if } x = p, \\ [0.3, 0.7] & \text{if } x = q, \\ (0.3, 0.6] & \text{if } x = r. \end{cases}$$

Then, it is easy to verify that f is a hesitant fuzzy interior on $\mathcal{S}$.

Theorem 12 *A heitant fuzzy set f on $\mathcal{AG}$-groupoid $\mathcal{S}$ is a hesitant fuzzy interior ideal on $\mathcal{S}$ if and only if*

$$(\widetilde{\mathcal{H}}\tilde{\circ}f)\tilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f.$$

Proof. Let $a \in \mathcal{S}$ and let f be a hesitant fuzzy interior ideal on $\mathcal{S}$. If a is not expressible as $a = bc$ for some $b, c \in \mathcal{S}$, then $((\widetilde{\mathcal{H}}\tilde{\circ}f)\tilde{\circ}\widetilde{\mathcal{H}})(a) = \phi \subseteq f(a)$. Thus, $(\widetilde{\mathcal{H}}\tilde{\circ}f)\tilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f$. If a is expressible as $a = bc$ for some $b, c \in \mathcal{S}$, then we have

$$\begin{aligned} ((\widetilde{\mathcal{H}}\tilde{\circ}f)\tilde{\circ}\widetilde{\mathcal{H}})(a) &= \bigcup_{a=bc} \left\{(\widetilde{\mathcal{H}}\tilde{\circ}f)(b) \cap \widetilde{\mathcal{H}}(c)\right\} = \bigcup_{a=bc} \left\{ \bigcup_{b=pq} \left\{\widetilde{\mathcal{H}}(p) \cap f(q)\right\} \cap \widetilde{\mathcal{H}}(c) \right\} \\ &= \bigcup_{a=bc} \left\{ \bigcup_{b=pq} \{[0,1], f(q)\} \cap [0,1] \right\} \subseteq \bigcup_{a=bc} \left\{ \bigcup_{b=pq} \{[0,1], f((pq)c)\} \cap [0,1] \right\} \\ &= \bigcup_{a=(pq)c} \{[0,1], f((pq)c)\} \cap [0,1] = f(a). \end{aligned}$$

Also if $b \neq pq$, then $(\widetilde{\mathcal{H}}\tilde{\circ}f)(b) = \phi$, and so $((\widetilde{\mathcal{H}}\tilde{\circ}f)\tilde{\circ}\widetilde{\mathcal{H}})(a) = \phi \subseteq f(a)$.

In any case, we have $(\widetilde{\mathcal{H}}\tilde{\circ}f)\tilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f$.

Conversely, let $(\widetilde{\mathcal{H}}\tilde{\circ}f)\tilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f$. Let $a, b, c \in \mathcal{S}$, then

$$\begin{aligned} f((ab)c) &\supseteq ((\widetilde{\mathcal{H}}\tilde{\circ}f)\tilde{\circ}\widetilde{\mathcal{H}})((ab)c) = \bigcup_{(ab)c=mn} \left\{(\widetilde{\mathcal{H}}\tilde{\circ}f)(m) \cap \widetilde{\mathcal{H}}(n)\right\} \\ &\supseteq (\widetilde{\mathcal{H}}\tilde{\circ}f)(ab) \cap \widetilde{\mathcal{H}}(c) = (\widetilde{\mathcal{H}}\tilde{\circ}f)(ab) \cap [0,1] \\ &= \bigcup_{ab=pq} \left\{\widetilde{\mathcal{H}}(p) \cap f(q)\right\} \supseteq \widetilde{\mathcal{H}}(a) \cap f(b) = [0,1] \cap f(b) = f(b). \end{aligned}$$

This shows that, f is a hesitant fuzzy interior ideal on $\mathcal{S}$. ∎

Theorem 13 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, a hesitant fuzzy left (resp., right, two-sided) ideal f on $\mathcal{S}$ is a hesitant fuzzy interior ideal on $\mathcal{S}$ but the converse is not true in general.*

Proof. Assume that f be a hesitant fuzzy ideal on $\mathcal{S}$. Let $a, b, c \in \mathcal{S}$, then $f((ab)c) \supseteq f(ab) \supseteq f(b)$. This implies that f is a hesitant fuzzy interior ideal on $\mathcal{S}$. Conversely, since in Example 6, f is a hesitant fuzzy interior ideal on $\mathcal{S}$ and it is not a hesitant fuzzy ideal on $\mathcal{S}$ because $f(pq) = f(r) = (0.3, 0.6] \not\supseteq [0.3, 0.7] = f(q)$. ∎

Theorem 14 *Let A be a non-empty subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, A is an interior ideal of $\mathcal{S}$ if and only if the characteristic hesitant fuzzy set $\mathcal{H}_A$ is a hesitant fuzzy interior ideal on $\mathcal{S}$.*

If $h(a) = [0,1]\ \forall\ a \in \mathcal{S}$, then it is easy to see that h is a hesitant fuzzy $\mathcal{AG}$-groupoid, left (resp., right, bi-ideal, interior ideal, generalized bi-ideal) on $\mathcal{S}$. We denote such type of hesitant fuzzy $\mathcal{AG}$-groupoid, hesitant fuzzy left (resp., right, bi-ideal, interior ideal, generalized bi-ideal) by $\widetilde{\mathcal{H}}$. It is obvious that $\widetilde{\mathcal{H}} = \mathcal{H}_{\mathcal{S}}$ that is $\widetilde{\mathcal{H}}(a) = [0,1]\ \forall\ a \in \mathcal{S}$.

Definition 9 A hesitant fuzzy set f on $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a hesitant fuzzy quasi-ideal on $\mathcal{S}$ if the following condition is valid.

$$(f\widetilde{\circ}\widetilde{\mathcal{H}}) \sqcap (\widetilde{\mathcal{H}}\widetilde{\circ}f) \sqsubseteq f.$$

Example 7 Let $\mathcal{S} = \{0,a,b,c\}$ be an $\mathcal{AG}$-groupoid with the following Caley multiplication table.

$\cdot$	0	a	b	c
0	0	0	0	0
a	c	a	b	0
b	0	0	0	0
c	0	c	b	0

Let f be a hesitant fuzzy set defined on $\mathcal{S}$ as follows:

$$f : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} \epsilon & \text{if } x \in \{0,a\} \\ \phi & \text{if } x \in \{b,c\} \end{cases}$$

where ϵ is a non-empty subset of $[0,1]$. Then, f is a hesitant fuzzy quasi-ideal on $\mathcal{S}$.

Proposition 4 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, every hesitant fuzzy quasi-ideal on $\mathcal{S}$ is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$.*

Proof. Assume that f be a hesitant fuzzy quasi-ideal on $\mathcal{S}$. Since $f \sqsubseteq \widetilde{\mathcal{H}}$, $f\widetilde{\circ}f \sqsubseteq \widetilde{\mathcal{H}}\widetilde{\circ}f$ and $f\widetilde{\circ}f \sqsubseteq f\widetilde{\circ}\widetilde{\mathcal{H}}$. So we have, $f\widetilde{\circ}f \sqsubseteq (\widetilde{\mathcal{H}}\widetilde{\circ}f) \sqcap (f\widetilde{\circ}\widetilde{\mathcal{H}}) \sqsubseteq f$ implies that $f\widetilde{\circ}f \sqsubseteq f$. Hence, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. ■

Proposition 5 *Every hesitant fuzzy left (resp., right) ideal on $\mathcal{AG}$-groupoid $\mathcal{S}$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$.*

Proof. Assume that f is a hesitant fuzzy quasi-ideal on $\mathcal{S}$. Now, since $\widetilde{\mathcal{H}}\widetilde{\circ}f \sqsubseteq f$, so we have $(\widetilde{\mathcal{H}}\widetilde{\circ}f)\sqcap(f\widetilde{\circ}\widetilde{\mathcal{H}}) \sqsubseteq \widetilde{\mathcal{H}}\widetilde{\circ}f \sqsubseteq f$. Hence, f is a hesitant fuzzy left ideal on $\mathcal{S}$.■

Proposition 6 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, every idempotent hesitant fuzzy quasi-ideal on $\mathcal{S}$ is a hesitant fuzzy bi-ideal on $\mathcal{S}$.*

Proof. Let f be a hesitant fuzzy quasi-ideal on $\mathcal{AG}$-groupoid$\mathcal{S}$. Then, by Proposition 4, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. Next, since f is idempotent hesitant fuzzy quasi-ideal on $\mathcal{S}$, so we have

$$(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq (\widetilde{\mathcal{H}}\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq \widetilde{\mathcal{H}}\widetilde{\circ}f \text{ and } (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f = (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}(f\widetilde{\circ}f) = (f\widetilde{\circ}f)\widetilde{\circ}(\widetilde{\mathcal{H}}\widetilde{\circ}f) \sqsubseteq f\widetilde{\circ}\widetilde{\mathcal{H}}$$

This implies that, $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq (\widetilde{\mathcal{H}}\widetilde{\circ}f) \sqcap (f\widetilde{\circ}\widetilde{\mathcal{H}}) \sqsubseteq f$ implies that $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq f$. Hence, by using Theorem 9, f is a hesitant fuzzy bi-ideal on $\mathcal{S}$. ■

Theorem 15 *Let Q be a non-empty subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, Q is a quasi-ideal of $\mathcal{S}$ if and only if the characteristic hesitant fuzzy set $\mathcal{H}_Q$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$.*

Proof. We first assume that Q is a quasi-ideal of $\mathcal{S}$. Let a be any element of $\mathcal{S}$. If $a \in Q$, then

$$((\mathcal{H}_Q \tilde{\circ} \widetilde{\mathcal{H}}) \sqcap (\widetilde{\mathcal{H}} \tilde{\circ} \mathcal{H}_Q))(a) \sqsubseteq [0,1] = \mathcal{H}_Q(a).$$

If $a \notin Q$, then $[\mathcal{H}_Q](a) = \phi$. On the other hand assume that

$$((\mathcal{H}_Q \tilde{\circ} \widetilde{\mathcal{H}}) \sqcap (\widetilde{\mathcal{H}} \tilde{\circ} \mathcal{H}_Q))(a) = [0,1].$$

Then

$$\bigcup_{a=xy} \left\{ \mathcal{H}_Q(x) \cap \widetilde{\mathcal{H}}(y) \right\} = (\mathcal{H}_Q \tilde{\circ} \widetilde{\mathcal{H}})(a) = [0,1]$$

and

$$\bigcup_{a=xy} \left\{ \widetilde{\mathcal{H}}(x) \cap \mathcal{H}_Q(y) \right\} = (\widetilde{\mathcal{H}} \tilde{\circ} \mathcal{H}_Q)(a) = [0,1].$$

This implies that, there exist b, c, d and e of $\mathcal{S}$ with $a = bc = de$ such that $\mathcal{H}_Q(b) = [0,1]$ and $\mathcal{H}_Q(e) = [0,1]$. Hence $a = bc = de \in Q\mathcal{S} \cap \mathcal{S}Q \subseteq Q$, which contradicts that $a \notin Q$. Thus, we have $(\mathcal{H}_Q \tilde{\circ} \widetilde{\mathcal{H}}) \sqcap (\widetilde{\mathcal{H}} \tilde{\circ} \mathcal{H}_Q) \sqsubseteq \mathcal{H}_Q$ and so $\mathcal{H}_Q$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$.

Conversely, suppose that $\mathcal{H}_Q$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$. Let a be any element of $Q\mathcal{S} \cap \mathcal{S}Q$. Then, $bx = a = yc$ for some $x, y \in \mathcal{S}$. It follows from Example 2, that

$$\begin{aligned} \mathcal{H}_Q(a) &\supseteq ((\mathcal{H}_Q \tilde{\circ} \widetilde{\mathcal{H}}) \sqcap (\widetilde{\mathcal{H}} \tilde{\circ} \mathcal{H}_Q))(a) = (\mathcal{H}_Q \tilde{\circ} \widetilde{\mathcal{H}})(a) \cap (\widetilde{\mathcal{H}} \tilde{\circ} \mathcal{H}_Q)(a) \\ &= \left(\bigcup_{a=uv} \left\{ \mathcal{H}_Q(u) \cap \widetilde{\mathcal{H}}(v) \right\} \right) \cap \left(\bigcup_{a=uv} \{ \widetilde{\mathcal{H}}(u) \cap \mathcal{H}_Q(v) \} \right) \\ &= \left(\bigcup_{a=uv} \{ \mathcal{H}_Q(u) \} \right) \cap \left(\bigcup_{a=uv} \{ \mathcal{H}_Q(v) \} \right) = [0,1] \end{aligned}$$

and so $a \in Q$. Thus, $Q\mathcal{S} \cap \mathcal{S}Q \subseteq Q$. Hence, Q is a quasi-ideal of $\mathcal{S}$. ∎

Characterizations of regular $\mathcal{AG}$-groupoids

In this section, we characterize a regular $\mathcal{AG}$-groupoid in terms of hesitant fuzzy ideals.

Definition 10 An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called regular $\mathcal{AG}$-groupoid if for every element $a \in \mathcal{S}$ there exist an element $x \in \mathcal{S}$ such that

$$a = (ax)a.$$

Theorem 16 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Then, for every hesitant fuzzy right ideal f and for every hesitant fuzzy left ideal g on $\mathcal{S}$, $f\widetilde{\circ}g = f \sqcap g$.*

Proof. Let f be a hesitant fuzzy right ideal and g be a hesitant fuzzy left ideal on a regular $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, $f\widetilde{\circ}g \sqsubseteq f\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f$ and $f\widetilde{\circ}g \sqsubseteq \widetilde{\mathcal{H}}\widetilde{\circ}g \sqsubseteq g$. This implies that, $f\widetilde{\circ}g \sqsubseteq f \sqcap g$. Now let a be any element of $\mathcal{S}$. Then, since $\mathcal{S}$ is regular $\mathcal{AG}$-groupoid, so there exist an element $x \in \mathcal{S}$ such that $a = (ax)a$. Thus, we have $(f\widetilde{\circ}g)(a) = \bigcup\limits_{a=xy} \{f(x) \cap g(y)\} \supseteq f(ax) \cap g(a) \supseteq f(a) \cap g(a) = (f \cap g)(a)$. This implies that, $f\widetilde{\circ}g \sqsupseteq f \sqcap g$. Hence, $f\widetilde{\circ}g = f \sqcap g$. ∎

Corollary 1 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Then, for every hesitant fuzzy ideal f and for every hesitant fuzzy ideal g on $\mathcal{S}$, $f\widetilde{\circ}g = f \sqcap g$.*

Proposition 7 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Then, every hesitant fuzzy right ideal on $\mathcal{S}$ is idempotent.*

Proof. Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid and g be a hesitant fuzzy right ideal on $\mathcal{S}$. Then, $g\widetilde{\circ}g \sqsubseteq g\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq g$. Next, since $\mathcal{S}$ is regular, so for any $a \in \mathcal{S}$ there exist $x \in \mathcal{S}$ such that $a = (ax)a$. Thus, we have $(g\widetilde{\circ}g)(a) = \bigcup\limits_{a=(ax)a} \{g(ax) \cap g(a)\} \supseteq g(a) \cap g(a) = g(a)$. Hence, $g \sqsubseteq (g\widetilde{\circ}g)$ and so $(g)^2 = g\widetilde{\circ}g = g$. ∎

Corollary 2 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Then, every hesitant fuzzy ideal on $\mathcal{S}$ is idempotent.*

Corollary 3 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Then, the set of all hesitant fuzzy ideals on $\mathcal{S}$ forms a samilattice structure under the hesitant fuzzy product.*

Proposition 8 *Every hesitant fuzzy right ideal on a regular $\mathcal{AG}$-groupoid $\mathcal{S}$ is a hesitant fuzzy left ideal on $\mathcal{S}$.*

Proof. Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid and f be a hesitant fuzzy right ideal on $\mathcal{S}$. Then, since $\mathcal{S}$ is regular, so for any $a \in \mathcal{S}$ there exist $x \in \mathcal{S}$ such that $a = (ax)a$. Thus, we have $f(ab) = f(((ax)a)b) = f((ba)(ax)) \supseteq f(ba) \supseteq f(b)$. Hence, f is a hesitant fuzzy left ideal on $\mathcal{S}$. ∎

Proposition 9 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and let the set of all hesitant fuzzy ideals on $\mathcal{S}$ forms a regular $\mathcal{AG}$-groupoid on $\mathcal{S}$ under the hesitant fuzzy product. Then, every hesitant fuzzy ideal on $\mathcal{S}$ has the form $f = (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f$.*

Proof. Let f be a hesitant fuzzy ideal on $\mathcal{S}$. Then, by assumption there exist a hesitant fuzzy ideal on $\mathcal{S}$ such that $f = (f\widetilde{\circ}g)\widetilde{\circ}f$ Thus,

$$f = (f\widetilde{\circ}g)\widetilde{\circ}f \sqsubseteq (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq (f\widetilde{\circ}\widetilde{\mathcal{H}}) \sqcap (\widetilde{\mathcal{H}}\widetilde{\circ}f) \sqsubseteq f \sqcap f = f.$$

Now since $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f\widetilde{\circ}\widetilde{\mathcal{H}}$ and $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq (\widetilde{\mathcal{H}}\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq \widetilde{\mathcal{H}}\widetilde{\circ}f$, hence, $f = (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f$. ∎

Proposition 10 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid and let f be a hesitant fuzzy interior ideal on $\mathcal{S}$. Then, f is a hesitant fuzzy ideal on $\mathcal{S}$.*

Proof. Let f be a hesitant fuzzy interior ideal on $\mathcal{S}$ and a be any element of $\mathcal{S}$. Then, since $\mathcal{S}$ is regular so there exist $x \in \mathcal{S}$ such that $a = (ax)a$. Thus, we have$f(ab) = f(((ax)a)b) \supseteq f(a)$. Hence, f is a hesitant fuzzy right ideal on $\mathcal{S}$. ■

Theorem 17 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Then, for every hesitant fuzzy bi-ideal f on $\mathcal{S}$, $f = (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f$.*

Proof. Let f be a hesitant fuzzy bi-ideal on $\mathcal{S}$ and let a be an arbitrary element of $\mathcal{S}$. Then, since $\mathcal{S}$ is regular, so there exist $x \in \mathcal{S}$ such that $a = (ax)a$. Thus, we have

$$\begin{aligned}((f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f)(a) &= \bigcup_{a=bc}\left\{(f\widetilde{\circ}\widetilde{\mathcal{H}})(b) \cap f(c)\right\} \supseteq (f\widetilde{\circ}\widetilde{\mathcal{H}})(xa) \cap f(a)\\ &= \bigcup_{ax=pq}\left\{f(p) \cap \widetilde{\mathcal{H}}(q)\right\} \cap f(a) \supseteq f(a) \cap \widetilde{\mathcal{H}}(x) \cap f(a)\\ &= f(a) \cap [0,1] \cap f(a) = f(a)\end{aligned}$$

This shows that, $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsupseteq f$. By assumption, since f is a hesitant fuzzy bi-ideal on $\mathcal{S}$, so $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq f$. Hence, $f = (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f$. ■

Proposition 11 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Let f be a hesitant fuzzy right ideal and g be a hesitant fuzzy left ideal on $\mathcal{S}$. Then, $f\widetilde{\circ}g$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$.*

Proof. Let f be a hesitant fuzzy right ideal and g be a hesitant fuzzy left ideal on $\mathcal{S}$. It follows that $f \sqcap g$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$. Now, since $\mathcal{S}$ is regular, so by Theorem 16, $f\widetilde{\circ}g$=$f \sqcap g$. Hence, $f\widetilde{\circ}g$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$. ■

Characterizations of completely regular $\mathcal{AG}$-groupoids

In this section, we characterize a completely regular $\mathcal{AG}$-groupoid in terms of hesitant fuzzy ideals. An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called left regular if for all $a \in \mathcal{S}$ there exist $x \in \mathcal{S}$ such that

$$a = xa^2 = x(aa).$$

An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called right regular if for all $a \in \mathcal{S}$ there exist $x \in \mathcal{S}$ such that

$$a = a^2x = (aa)x.$$

Definition 11 An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called completely regular $\mathcal{AG}$-groupoid if $\mathcal{S}$ is regular, left regular and right regular.

Theorem 18 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, the following coditions are equivalent.*

(i) *$\mathcal{S}$ is left regular.*

(ii) *$f(a) = f(a^2)$ for all $a \in \mathcal{S}$, where f is any hesitant fuzzy left ideal on $\mathcal{S}$.*

Proof. (i) $\Rightarrow$ (ii) : Let $\mathcal{S}$ be a left regular $\mathcal{AG}$-groupoid and f be any hesitant fuzzy left ideal on $\mathcal{S}$. Then, since $\mathcal{S}$ is a left regular so for any $p \in \mathcal{S}$ there exist $q \in \mathcal{S}$ such that $p = q(pp)$. Thus,

$$f(p) = f(q(pp)) \supseteq f(pp) \supseteq f(p).$$

This shows that, $f(p) = f(p^2)$.

(ii) $\Rightarrow$ (i) : Let $f(p) = f(p^2)$ for all $p \in \mathcal{S}$. Since $L(p^2)$ is a left ideal of $\mathcal{S}$, the characteristic hesitant fuzzy set $\mathcal{H}_{L(p^2)}$ is a hesitant fuzzy left ideal on $\mathcal{S}$. Since $p^2 \in L(p^2)$, we have

$$\mathcal{H}_{L(p^2)}(p) = \mathcal{H}_{L(p^2)}(p^2) = [0, 1].$$

This implies that, $p \in L(p^2) = \mathcal{S}(pp)$. This obviously means that $\mathcal{S}$ is a left regular. ∎

Theorem 19 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, the following coditions are equivalent.*

(i) *$\mathcal{S}$ is left regular.*

(ii) *$f(a) = f(a^2)$ for all $a \in \mathcal{S}$, where f is any hesitant fuzzy right ideal on $\mathcal{S}$.*

Characterizations of weakly regular $\mathcal{AG}$-groupoids

In this section, we characterize a weakly regular $\mathcal{AG}$-groupoid in terms of hesitant fuzzy ideals.

Definition 12 An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called weakly regular $\mathcal{AG}$-groupoid if for all $a \in \mathcal{S}$ there exist $b, c \in \mathcal{S}$ such that

$$a = (ab)(ac)$$

Theorem 20 *Let $\mathcal{S}$ be a weakly regular $\mathcal{AG}$-groupoid. Let f be any hesitant fuzzy right ideal and g be any hesitant fuzzy ideal on $\mathcal{S}$. Then, $f \sqcap g = f \widetilde{\circ} g$.*

Proof. Let f be any hesitant fuzzy right ideal and g be any hesitant fuzzy ideal on weakly regular $\mathcal{AG}$-groupoid $\mathcal{S}$. Let a be any element of $\mathcal{S}$. Then, since $\mathcal{S}$ is weakly regular, so there exist $b, c \in \mathcal{S}$ such that $a = (ab)(ac)$. Thus, we have

$$(f \widetilde{\circ} g)(a) = \bigcup_{a=(ab)(ac)} \{f(ab) \cap g(ac)\} \supseteq f(a) \cap f(a) = (f \cap g)(a).$$

This shows that, $f \sqcap g \sqsubseteq f \widetilde{\circ} g$. Now since $f \sqcap g \sqsupseteq f \widetilde{\circ} g$ holds for all hesitant fuzzy right ideal and for all hesitant fuzzy left ideal on $\mathcal{S}$. Hence, $f \sqcap g = f \widetilde{\circ} g$. ∎

Characterizations of quasi-regular $\mathcal{AG}$-groupoids

In this section, we characterize quasi-regular $\mathcal{AG}$-groupoid in terms of hesitant fuzzy ideals. An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called left quasi-regular if for all $a \in \mathcal{S}$ there exist $b, c \in \mathcal{S}$ such that $a = (ba)(ca)$ and $\mathcal{S}$ is called right quasi-regular if for all $a \in \mathcal{S}$ there exist $b, c \in \mathcal{S}$ such that $a = (ab)(ac)$.

Definition 13 An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called quasi-regular, if it is both left quasi-regular and right quasi-regular.

Theorem 21 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, $\mathcal{S}$ is left (resp., right) quasi-regular if and only if every hesitant fuzzy left (resp., right) ideal on $\mathcal{S}$ is idempotent.*

Proof. Let $\mathcal{S}$ be left quasi-regualr and f be a hesitant fuzzy left ideal on $\mathcal{S}$. Then, for any $x \in \mathcal{S}$ there exist $b, c \in \mathcal{S}$ such that $x = (bx)(cx)$. Thus, we have $(f\widetilde{\circ}f)(x) = \bigcup\limits_{x=(bx)(cx)} \{f(bx) \cap f(cx)\} \supseteq f(bx) \cap f(cx) \supseteq f(x) \cap f(x) = f(x)$. This shows that, $f\widetilde{\circ}f \supseteq f$. Also it is obvious that $f\widetilde{\circ}f \subseteq f$. Thus, $f\widetilde{\circ}f = f$. Hence, f is idempotent.

Conversely, let every hesitant fuzzy left ideal on $\mathcal{S}$ is idempotent. Let $x \in \mathcal{S}$. Then, since $L[x]$ is a principal left ideal of $\mathcal{S}$, so the characteristic hesitant fuzzy set is a hesitant fuzzy left ideal on $\mathcal{S}$. Thus, by using assumption we have

$$\mathcal{H}_{L[x]L[x]}(x) = (\mathcal{H}_{L[x]}\widetilde{\circ}\mathcal{H}_{L[x]})(x) = \mathcal{H}_{L[x]}(x) = [0, 1].$$

This implies that, $a \in L[x]L[x] = (\mathcal{S}x)(\mathcal{S}x)$. Hence, $\mathcal{S}$ is left quasi-regular.

The other case can be shown in a similar way. ∎

Proposition 12 *Let $\mathcal{S}$ be left (resp., right) quasi-regular $\mathcal{AG}$-groupoid and f be a hesitant fuzzy right ideal on $\mathcal{S}$. Then, f is a hesitant fuzzy ideal on $\mathcal{S}$.*

Proof. Let $\mathcal{S}$ be left (resp., right) quasi-regular $\mathcal{AG}$-groupoid and f be a hesitant fuzzy right ideal on $\mathcal{S}$. Then, since $\widetilde{\mathcal{H}}$ is a hesitant fuzzy right ideal of itself on $\mathcal{S}$, and by using assumption $\widetilde{\mathcal{H}}$ is idempotent, so we have$\widetilde{\mathcal{H}}\widetilde{\circ}f = (\widetilde{\mathcal{H}}\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f = (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f$. Hence, f is a hesitant fuzzy ideal on $\mathcal{S}$. ∎

Theorem 22 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Let for every hesitant fuzzy ideal f on $\mathcal{S}$, $f = (f\widetilde{\circ}\widetilde{\mathcal{H}})^2 \sqcap (\widetilde{\mathcal{H}}\widetilde{\circ}f)^2$, then $\mathcal{S}$ is quasi-regular.*

Proof. Let f be any hesitant fuzzy right ideal on $\mathcal{S}$. Then, by assumption we have $f = (f\widetilde{\circ}\widetilde{\mathcal{H}})^2 \sqcap (\widetilde{\mathcal{H}}\widetilde{\circ}f)^2 \sqsubseteq (f\widetilde{\circ}\widetilde{\mathcal{H}})^2 \sqsubseteq f\widetilde{\circ}f \sqsubseteq f\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f$. Thus, $f = f\widetilde{\circ}f = (f)^2$ and from Theorem 21, it follows that $\mathcal{S}$ is right quasi-regular. In similar way we can show that $\mathcal{S}$ is left quasi-regular. ∎

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