A CLASS OF HIGH ACCURACY EXPLICIT DIFFERENCE SCHEMES FOR SOLVING THREE-DIMENSIONAL PARABOLIC EQUATIONS

Yongqiang Zhan[1]

School of Computer Engineering
Guangzhou College of South China University of Technology
Guangzhou 510800
P.R. China
e-mail: zhanyongq@126.com

Jinlan Guan

Basic Courses Department
Guangdong Agricultural Industry Business College
Guangzhou 510507
P.R. China

Abstract. A class of explicit difference schemes with high accuracy for solving three-dimensional parabolic equations are given. First, a difference approximation expression of $\frac{\partial u}{\partial t}$ is deduced at a special node (x_j, y_k, z_l, t_{n+1}); a class of explicit difference schemes are constructed by the method of undetermined coefficients, and appropriate parameters are chosen to endow the truncation error of schemes is $O\left(\Delta t^3 + \Delta x^4\right)$. In turn, the new difference schemes are proved to be stable if $r < \frac{1}{4}$ with the Fourier analysis method. Finally, the numerical experiment shows the numerical solutions of difference schemes and the exact solutions are matched and the difference schemes are effective.

Keywords: three-dimensional parabolic equations, explicit difference scheme, truncation error.

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1. Introduction

Partial differential equations (PDE) can be seen in many fields, such as physics, chemistry, biology, economics and financial projections, so PDE is an indispensable mathematical tool in the application of production and life. As we known, it is difficult to solve the PDE to gain its exact analytical solutions, so when solving

[1]Corresponding author.

the PDE problems, it is especially important to get its numerical solutions. By doing this, we can understand the mathematical meaning of these PDE problems.

This paper consider the following three-dimensional parabolic differential equation with initial and boundary conditions:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2},\ 0<x,y,z<1,t>0, \tag{1}$$

$$u(x,y,z,0)=\varphi(x,y,z),\ 0\le x,y,z\le 1, \tag{2}$$

$$u(0,y,z,t)=f_1(y,z,t),u(1,y,z,t)=f_2(y,z,t),\ 0\le y,z\le 1,0\le t\le T, \tag{3}$$

$$u(x,0,z,t)=g_1(x,z,t),u(x,1,z,t)=g_2(x,z,t),\ 0\le x,z\le 1,0\le t\le T, \tag{4}$$

$$u(x,y,0,t)=h_1(x,y,t),u(x,y,1,t)=h_2(x,y,t),\ 0\le x,y\le 1,0\le t\le T, \tag{5}$$

where $\varphi, f_1, f_2, g_1, g_2, h_1$ and h_2 are sufficiently smooth functions.

Equation (1) can describe various physical quantities, and can be encountered in many fields of engineering and science, such as seepage, diffusion, heat conduction and so on.

Various numerical finite difference schemes have been proposed to solve parabolic problems approximately. For multidimensional problems, The explicit difference scheme and implicit difference scheme are the common finite difference schemes. The implicit difference scheme has the advantage of good stability to the explicit one, but it is needed to solve different linear equations on each time layer which will cost to big computation. The alternating-direction implicit (ADI) difference scheme can overcome these disadvantages.

As we known, the ADI scheme is unconditional stable and only need to solve a sequence of tridiagonal linear systems [1], [2]. In recent years, there are many new methods which use ADI scheme to solve the high dimensional parabolic equations [3]–[8], some of them have the accuracy of $O\left(\Delta t^2+\Delta x^4\right)$ [3], [6], [7], [8].

The explicit difference scheme has worse stability than the implicit difference scheme, but has the advantage of smaller amount of calculation. The general explicit scheme is the classical explicit scheme with the stability condition of $r\le\frac{1}{6}$, where $r=\frac{\Delta t}{\Delta x^2}$ is the mesh spacing ratio. Its deficiency is that its accuracy is not high, and its truncation error is $O\left(\Delta t+\Delta x^2\right)$ [1]. Recently, there has been a interest in the development and application of explicit difference schemes for the numerical solution of equation (1) [9]–[12]. In reference [11], Zeng constructed a class of explicit schemes with the truncation error of $O\left(\Delta t^2+\Delta x^4\right)$ and the stability condition of $r\le\frac{1}{6}$. In reference [12], Ma also constructed a class of explicit schemes with the truncation error of $O\left(\Delta t^2+\Delta x^4\right)$ and the stability condition of $r\le\frac{1}{4}$. This paper presents a class of explicit schemes for solving equation (1), the stability condition is $r<\frac{1}{4}$, and the truncation error is $O\left(\Delta t^3+\Delta x^4\right)$, the schemes have higher accuracy than the above schemes.

The remainder of this paper is organized as follows. In Section 2, we establish a approximate difference scheme of one-order partial derivative $\frac{\partial u}{\partial t}$ at the node of (x_j, y_k, z_l, t_{n+1}). In Section 3, we construct a class of a three-layer explicit difference schemes with the accuracy of $O\left(\Delta t^3 + \Delta x^4\right)$. In Section 4, by using the Fourier analysis method, it is proved that the difference schemes are stable when $r < \dfrac{1}{4}$. In Section 5, we compare the difference of exact solution and the scheme constructed in this paper with that in the reference [12], and compare the computational efficiency of the two difference schemes and the classical explicit scheme. The results shows that the difference schemes in this paper are effective.

2. Difference approximation

Suppose the solution $u\left(x, y, z, t\right)$ of equation (1) is sufficiently smooth, firstly establish the approximate expression of the first order partial derivative of the $u\left(x, y, z, t\right)$ at node (x_j, y_k, z_l, t_{n+1}) at time t, where $x_j = j\Delta x$, $y_k = k\Delta y$, $z_l = l\Delta z, t_n = n\Delta t$, let $u^n_{jkl} = u(x_j, y_k, z_l, t_n)$, by the Taylor expansion, we have

$$u^n_{jkl} = u^{n+1}_{jkl} - \frac{\Delta t}{1!}\left(\frac{\partial u}{\partial t}\right)^{n+1}_{jkl} + \frac{\Delta t^2}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)^{n+1}_{jkl} - \frac{\Delta t^3}{3!}\left(\frac{\partial^3 u}{\partial t^3}\right)^{n+1}_{jkl} + \cdots$$

$$u^{n-1}_{jkl} = u^{n+1}_{jkl} - \frac{(2\Delta t)}{1!}\left(\frac{\partial u}{\partial t}\right)^{n+1}_{jkl} + \frac{(2\Delta t)^2}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)^{n+1}_{jkl} - \frac{(2\Delta t)^3}{3!}\left(\frac{\partial^3 u}{\partial t^3}\right)^{n+1}_{jkl} + \cdots$$

we have

$$\begin{aligned}\left(\frac{\partial u}{\partial t}\right)^{n+1}_{jkl} &= \frac{\frac{3}{2}u^{n+1}_{jkl} - 2u^n_{jkl} + \frac{1}{2}u^{n-1}_{jkl}}{\Delta t} + \frac{1}{3}\Delta t^2\left(\frac{\partial^3 u}{\partial t^3}\right)^{n+1}_{jkl} + \cdots \\ &= \frac{\frac{3}{2}u^{n+1}_{jkl} - 2u^n_{jkl} + \frac{1}{2}u^{n-1}_{jkl}}{\Delta t} + O\left(\Delta t^2\right)\end{aligned}$$

For convenience, denote the difference quotient to be

$$\eta_t u^{n+1}_{jkl} = \frac{\frac{3}{2}u^{n+1}_{jkl} - 2u^n_{jkl} + \frac{1}{2}u^{n-1}_{jkl}}{\Delta t}.$$

It is a finite difference approximation of $\left(\dfrac{\partial u}{\partial t}\right)^{n+1}_{jkl}$.

3. Construction of the difference scheme

Let Δt denote the step length of time and $\Delta x = \Delta y = \Delta z$ be the step length of space in the direction of x, y, z, respectively. We approximate equation (1) with the following parametered difference equation

$$\begin{aligned}&\eta_t u^{n+1}_{jkl} + \theta_1 \Delta_t u^{n-1}_{jkl} + \theta_2 \Delta_t\left(\frac{1}{2}\Diamond\right)u^{n-1}_{jkl} \\ &\quad = \frac{1}{\Delta x^2}\left[\left(\frac{\theta_3}{4}\Box + \frac{\theta_4}{2}\Diamond\right)u^n_{jkl} + \left(\frac{\theta_5}{4}\Box + \frac{\theta_6}{2}\Diamond\right)u^{n-1}_{jkl}\right],\end{aligned} \tag{6}$$

where u_{jkl}^n denotes the value of u at node $(j\Delta x, k\Delta y, l\Delta z, n\Delta t)$, $\Delta_t u_{jkl}^n = \dfrac{u_{jkl}^{n+1} - u_{jkl}^n}{\Delta t}$, respectively, and

$$\begin{aligned}
\Box u_{jkl}^n &= \left({}_x\Box + {}_y\Box + {}_z\Box\right) u_{jkl}^n,\\
\Diamond u_{jkl}^n &= \left({}_x\Diamond + {}_y\Diamond + {}_z\Diamond\right) u_{jkl}^n,\\
{}_x\Box u_{jkl}^n &= u_{j,k+1,l+1}^n + u_{j,k-1,l+1}^n + u_{j,k+1,l-1}^n + u_{j,k-1,l-1}^n - 4u_{jkl}^n\\
{}_x\Diamond u_{jkl}^n &= u_{j,k+1,l}^n + u_{j,k-1,l}^n + u_{j,k,l+1}^n + u_{j,k,l-1}^n - 4u_{jkl}^n
\end{aligned}$$

the rest can be inferred by analogy. $\theta_1 - \theta_6$ are parameters to be determined. A proper choice of undetermined parameters $\theta_1 - \theta_6$ can make difference equation (6) approach equation (1), and not only has truncation error with order as high as possible, but also has higher stability.

When the solution of equation (1) is smooth enough, we can get the following relation:

$$\frac{\partial^n}{\partial t^n}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^m u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^{m+2n} u \tag{7}$$

Substituting the Taylor series of u at node $(j\Delta x, k\Delta y, l\Delta z, n\Delta t)$ into (6) and using relation (7), we can obtain

$$\begin{aligned}
&(1+\theta_1)\frac{\partial u}{\partial t} + \Delta t\left(1 - \frac{\theta_1}{2}\right)\frac{\partial^2 u}{\partial t^2} + \theta_2 \Delta x^2 \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{6}(1+\theta_1)\frac{\partial^3 u}{\partial t^3} - \frac{\theta_2 \Delta x^2 \Delta t}{2}\frac{\partial^3 u}{\partial t^3}\\
&\quad = (\theta_3+\theta_4+\theta_5+\theta_6)\frac{\partial u}{\partial t} - \Delta t(\theta_5+\theta_6)\frac{\partial^2 u}{\partial t^2}\\
&\quad + \frac{\Delta t^2}{2}(\theta_5+\theta_6)\frac{\partial^3 u}{\partial t^3} + \frac{\Delta x^2}{12}(\theta_3+\theta_4+\theta_5+\theta_6)\frac{\partial^2 u}{\partial t^2}\\
&\quad + \frac{\Delta x^2}{12}\left[(\theta_3+\theta_5) - 2(\theta_4+\theta_6)\right]\left(\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^2 \partial z^2} + \frac{\partial^4 u}{\partial x^2 \partial z^2}\right)\\
&\quad - \frac{\Delta x^2 \Delta t}{12}(\theta_5+\theta_6)\frac{\partial^3 u}{\partial t^3}\\
&\quad + \frac{\Delta x^2 \Delta t}{12}(2\theta_6 - \theta_5)\left(\frac{\partial^5 u}{\partial x^2 \partial y^2 \partial t} + \frac{\partial^5 u}{\partial y^2 \partial z^2 \partial t} + \frac{\partial^5 u}{\partial x^2 \partial z^2 \partial t}\right)\\
&\quad + O\left(\Delta t^3 + \Delta x^4\right)
\end{aligned}$$

In order to make the truncation error of scheme (6) getting to $O\left(\Delta t^3 + \Delta x^4\right)$, the following equation system should be available.

$$\begin{cases}
1+\theta_1 = \theta_3+\theta_4+\theta_5+\theta_6\\
1 - \dfrac{\theta_1}{2} + \dfrac{\theta_2}{r} = -\theta_5 - \theta_6 + \dfrac{1}{12r}(\theta_3+\theta_4+\theta_5+\theta_6)\\
(\theta_3+\theta_5) - 2(\theta_4+\theta_6) = 0\\
\dfrac{1}{6}(1+\theta_1) - \dfrac{\theta_2}{2r} = \left(\dfrac{1}{2} - \dfrac{1}{12r}\right)(\theta_5+\theta_6)\\
2\theta_6 - \theta_5 = 0
\end{cases} \tag{8}$$

where $r = \dfrac{\Delta t}{\Delta x^2}$. Let $\theta_6 = \theta$, the solution of the above equation system is:

$$\begin{aligned} \theta_1 &= \frac{16r + 6\theta - 1}{2r + 1}; \qquad \theta_2 = \frac{\theta - 12r^2\theta + 12r^2}{2(2r+1)}; \\ \theta_3 &= \frac{2(6r + \theta - 2r\theta)}{2r + 1}; \quad \theta_4 = \frac{6r + \theta - 2r\theta}{2r + 1}; \theta_5 = 2\theta. \end{aligned}$$

Substituting the above values into (6), we obtain the following single parameter three level explicit difference schemes with its truncation error getting to $O(\Delta t^3 + \Delta x^4)$.

$$\begin{aligned} & 6(2r+1)\, u_{jkl}^{n+1} \\ (9) \qquad & = [12(1 - 2\theta - 4r) + 2r(6r + \theta - 2r\theta)\Box + (8r^2\theta - \theta + 2r\theta)\Diamond]\, u_{jkl}^{n} \\ & + [(60r + 24\theta - 6) + 2r(2r+1)\theta\Box + (-8r^2\theta + 12r^2 + 2r\theta + \theta)\Diamond]\, u_{jkl}^{n-1}. \end{aligned}$$

4. Analysis of stability

According to the Fourier method for analyzing stability, the two-level equation system equivalent to (9) is

$$(10) \quad \begin{cases} u_{jkl}^{n+1} = \dfrac{12(1 - 2\theta - 4r) + 2r(6r + \theta - 2r\theta)\Box + (8r^2\theta - \theta + 2r\theta)\Diamond}{6(2r+1)} u_{jkl}^{n} \\ \qquad + \dfrac{60r + 24\theta - 6 + 2r(2r+1)\theta\Box + (-8r^2\theta + 12r^2 + 2r\theta + \theta)\Diamond}{6(2r+1)} v_{jkl}^{n} \\ v_{jkl}^{n+1} = u_{jkl}^{n} \end{cases}$$

Let

$$(11) \qquad u_{jkl}^{n} = U^n e^{i(j\theta + k\varphi + l\psi)}, v_{jkl}^{n} = V^n e^{i(j\theta + k\varphi + l\psi)}$$

where $i = \sqrt{-1}$. And through a simple calculation, we know

$$(12) \qquad \Box u_{jkl}^{n} = -4s_2 u_{jkl}^{n}, \Diamond u_{jkl}^{n} = -8s_1 u_{jkl}^{n}$$

where

$$\begin{aligned} s_1 &= \sin^2\frac{\theta}{2} + \sin^2\frac{\varphi}{2} + \sin^2\frac{\psi}{2} \in [0, 3] \\ s_2 &= \sin^2\frac{\theta + \varphi}{2} + \sin^2\frac{\theta - \varphi}{2} + \sin^2\frac{\varphi + \psi}{2} + \sin^2\frac{\varphi - \psi}{2} + \sin^2\frac{\theta + \psi}{2} \\ &\quad + \sin^2\frac{\theta - \psi}{2} \in [0, 6] \end{aligned}$$

Substituting (11) into (10) and using (12), we obtain

$$\begin{bmatrix} U^{k+1} \\ V^{k+1} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} U^k \\ V^k \end{bmatrix} = \mathbf{G}(\mathbf{s_1}, \mathbf{s_2}) \begin{bmatrix} U^k \\ V^k \end{bmatrix}$$

where

$$g_{11} = \frac{12\left(1-2\theta-4r\right)-8r\left(6r+\theta-2r\theta\right)s_2-8\left(8r^2\theta-\theta+2r\theta\right)s_1}{6\left(2r+1\right)}$$
$$g_{12} = \frac{60r+24\theta-6-8r\left(2r+1\right)\theta s_2-8\left(-8r^2\theta+12r^2+2r\theta+\theta\right)s_1}{6\left(2r+1\right)},$$
$$g_{21} = 1,$$
$$g_{22} = 0$$

The characteristic equation of propagation matrix $\mathbf{G}\left(\mathbf{s_1},\mathbf{s_2}\right)$ is

$$\lambda^2 - g_{11}\lambda - g_{12} = 0. \tag{13}$$

Lemma 1. [13] *The two roots of real coefficient quadratic equation* (13) *are less than or equal to* 1 *in norm if and only if*

$$|g_{11}| \le 1 - g_{12} \le 2. \tag{14}$$

Lemma 2. [13] *The difference scheme* (9) *is stable, i.e., the family of matrices* $\mathbf{G^n}\left(\mathbf{s_1},\mathbf{s_2}\right), \left(\left(s_1,s_2\right)\in\left[0,3\right]\times\left[0,6\right], n=1,2,...\right)$ *is uniformly bounded if and only if*

(1) $|\lambda_{1,2}| \le 1$ ($\lambda_{1,2}$ *are roots of* (13));

(2) (s_1,s_2) *which assures* $1-g_{11}^2/4 = g_{11}^2+4g_{12} = 0$ *is not existent or not in the region of* $[0,3]\times[0,6]$.

Theorem 1. *A sufficient condition for scheme* (9) *being stable is*

$$r < \frac{1}{4}, \quad -3r < \theta \le \frac{12r^2}{8r^2-2r-1}.$$

Proof. If $g_{12} \ne -1, 1-g_{11}^2/4 = g_{11}^2+4g_{12} = 0$ do not hold for any (s_1,s_2). By Lemmas 1 and 2, the stability conditions of scheme (9) become

$$-1+g_{12} \le g_{11} \le 1-g_{12} < 2.$$

From $g_{11} \le 1-g_{12}$, we have

$$\frac{12r+6-16r\left(3r+\theta\right)s_2-32r\left(3r+\theta\right)s_1}{6\left(2r+1\right)} \le 1. \tag{15}$$

Hence

$$3r+\theta \ge 0. \tag{16}$$

Because $1-g_{12} < 2$, we have

$$24\left(3r+\theta\right)-8r\left(2r+1\right)\theta s_2-8\left(-8r^2\theta+12r^2+2r\theta+\theta\right)s_1 > 0. \tag{17}$$

A sufficient condition which assures the above inequality hold is

$$\begin{cases} 3r+\theta>0 & (18)\\ \theta\le 0 & (19)\\ \left(8r^2-2r-1\right)\theta\ge 12r^2. & (20)\end{cases}$$

It is equivalent to

$$\begin{cases} 8r^2-2r-1<0\\ -3r<\theta\le \dfrac{12r^2}{8r^2-2r-1}\end{cases}$$

i.e.,

$$\begin{cases} r<\dfrac{1}{4} & (21)\\ \theta>-3r & (22)\\ \theta\le \dfrac{12r^2}{8r^2-2r-1} & (23)\end{cases}$$

Using $-1+g_{12}\le g_{11}$, we obtain

$$12r+6\theta-3-2r^2\left(2\theta-3\right)s_2-2\left(-8r^2\theta+6r^2+\theta\right)s_1\le 0. \tag{24}$$

A sufficient condition which assures the above inequality hold is

$$\begin{cases} 2\theta-3\le 0 & (25)\\ -8r^2\theta+6r^2+\theta\le 0 & (26)\\ 12r+6\theta-3-12r^2\left(2\theta-3\right)-6\left(-8r^2\theta+6r^2+\theta\right)\le 0 & (27)\end{cases}$$

Considering the conditions (19) and (21), conditions (25) and (27) hold. From (26) we have

$$\theta\le\frac{6r^2}{8r^2-1}. \tag{28}$$

When $r<\frac{1}{4}$, condition (23) is superior to (28).

By combining inequalities (21)–(23), we complete the proof. ■

In particular, taking $\theta=\dfrac{12r^2}{8r^2-2r-1}$, we obtain a three-level explicit scheme as follows:

$$\begin{aligned} &6\left(2r+1\right)u_{jkl}^{n+1}\\ &=\left[-\frac{288r^2}{8r^2-2r-1}-48r+12+\left(12r^2+\frac{24r^3-48r^4}{8r^2-2r-1}\right)\Box+\frac{12r^2\left(8r^2+2r-1\right)}{8r^2-2r-1}\Diamond\right]u_{jkl}^{n}\\ &+\left[60r-6+\frac{288r^2}{8r^2-2r-1}+\frac{24r^3\left(2r+1\right)}{8r^2-2r-1}\Box\right]u_{jkl}^{n-1}\end{aligned} \tag{29}$$

5. Numerical experiment

Consider initial and boundary value problem as follows:

$$(30)\quad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}, \qquad (0 < x, y, z < 1, t > 0) \\ u(x, y, z, 0) = \sin(x + y + z), \quad (0 \le x, y, z \le 1) \\ u(0, y, z, t) = e^{-3t} \sin(y + z), \ u(1, y, z, t) = e^{-3t} \sin(1 + y + z), \\ \qquad (0 \le y, z \le 1, t \ge 0) \\ u(x, 0, z, t) = e^{-3t} \sin(x + z), \ u(x, 1, z, t) = e^{-3t} \sin(x + 1 + z), \\ \qquad (0 \le x, z \le 1, t \ge 0) \\ u(x, y, 0, t) = e^{-3t} \sin(x + y), \ u(x, y, 1, t) = e^{-3t} \sin(x + y + 1), \\ \qquad (0 \le x, y \le 1, t \ge 0). \end{cases}$$

Taking $\Delta x = \Delta y = \Delta z = 0.1, \Delta t = r\Delta x^2 = r/100, r = 1/10, 1/8, 1/6, 1/5$. For convenience, we use the exact solution of (30) $u(x, y, z, t) = e^{-3t} \sin(x + y + z)$ to calculate the value of the first level u^1_{jkl}.

Table 1 shows the comparison of the exact solutions and the scheme (29) constructed in this paper and that in the reference [12] at the time strata n=200. From Table 1, one can easily see that numerical results of schemes (29) are completely identical with theoretical analysis, it has higher accuracy than the scheme in reference [12]. Table 2 shows the comparison of the efficiency of the scheme (29) constructed in this paper and the classical scheme one and that in the reference [12]. The result is tested by the unit of second. From Table 2, we can see that the three schemes' computational efficiency are similar. Among these three schemes, the classical explicit scheme has the highest computational efficiency, the second one is the scheme in this paper, and the lowest is the one constructed in the reference [12]. From the result of these two tables, we can see that the schemes constructed in this paper is a class of high accuracy and efficiency difference schemes.

Table 1 Comparison of calculating results of difference schemes with exact solution

r	result	(x,y,z)				
		$(0.1, 0.1, 0.1)$	$(0.3, 0.3, 0.3)$	$(0.5, 0.5, 0.5)$	$(0.7, 0.7, 0.7)$	$(0.9, 0.9, 0.9)$
$\frac{1}{10}$	exact solution	0.162 184 928	0.429 898 922	0.547 436 855	0.473 739 344	0.234 551 051
	scheme (29)	0.162 184 912	0.429 898 757	0.547 436 580	0.473 739 170	0.234 551 032
	reference [12]	0.162 184 903	0.429 898 666	0.547 436 427	0.473 739 073	0.234 551 021
$\frac{1}{8}$	exact solution	0.139 593 861	0.370 017 431	0.471 183 268	0.407 751 232	0.201 879 960
	scheme (29)	0.139 593 848	0.370 017 301	0.471 183 051	0.407 751 095	0.201 879 945
	reference [12]	0.139 593 844	0.370 017 253	0.471 182 970	0.407 751 043	0.201 879 940
$\frac{1}{6}$	exact solution	0.108 715 808	0.288 169 865	0.366 957 898	0.317 556 979	0.157 224 271
	scheme (29)	0.108 715 802	0.288 169 798	0.366 957 786	0.317 556 908	0.157 224 263
	reference [12]	0.108 715 792	0.288 169 789	0.366 957 777	0.317 556 899	0.157 224 253
$\frac{1}{5}$	exact solution	0.089 008 975	0.235 933 531	0.300 439 716	0.259 993 664	0.128 724 346
	scheme (29)	0.089 008 976	0.235 933 539	0.300 439 730	0.259 993 673	0.128 724 347
	reference [12]	0.089 008 960	0.235 933 366	0.300 439 441	0.259 993 490	0.128 724 327

Table 2 Comparison of the calculation efficiency among three kinds of difference schemes (unit:s)

difference scheme	r=0.1,n=100	r=0.2,n=100	r=0.1,n=200	r=0.2,n=200
classical explicit scheme	21.868 497	23.463 492	46.445 442	46.507 839
scheme (29)	22.362 985	23.629 915	46.671 319	46.826 391
reference [12] scheme	22.398 267	23.784 045	46.732 014	46.874 218

6. Conclusions

In this paper, we proposed a class of explicit difference schemes for solving three-dimensional parabolic problems. The stable character of the schemes are which has been verified by a discrete Fourier analysis. The schemes which proposed in this paper is fourth-order accurate in space and third-order accurate in time and allows a considerable saving in computing time. Numerical examples are given to test its high accuracy and to show its superiority over some other schemes in terms of accuracy and computational costs.

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References

[1] Lu, J.F., Guan, Z., *Numerical solution of partial differential equations*, Tsinghua University Press, Beijing, 2010, 112-120.

[2] Yu, D.H., Tang, H.Z., *Numerical solution of differential equations*, Science Press, Beijing, 2004, 165-172.

[3] Karaa, S., *A high-order compact ADI method for solving three-dimensional unsteady convection-diffusion problems,* Numerical Methods for Partial Differential Equations, 22 (4) (2006), 983-993.

[4] Dai, W.Z., Nassar, R., *A new ADI scheme for solving three-dimensional parabolic equations with first-order derivatives and variable coefficients*, Journal of Computational Analysis and Applications, 2 (4) (2000), 293-308.

[5] Dai, W., Nassar, R., *A second-order ADI scheme for three-dimensional parabolic differential equations*, Numerical Methods for Partial Differential Equations, 14 (2) (1998), 159-168.

[6] Mohanty, R.K., *High accuracy difference schemes for a class of three space dimensional singular parabolic equations with variable coefficients*, Journal of Computational and Applied Mathematics, 89 (1) (1998), 39-51.

[7] Karaa, S., Zhang, J., *High order ADI method for solving unsteady convection-diffusion problems*, Journal of Computational Physics, 198 (1) (2004), 1-9.

[8] Qin, J.G., *The new alternating direction implicit difference methods for solving three-dimensional parabolic equations*, Applied Mathematical Modelling, 34 (4) (2010), 890-897.

[9] Ma, M.S., *A new high-order accuracy explicit difference scheme for solving three-dimensional parabolic equation*, Applied Mathematics and Mechanics, 19 (5) (1998), 498-502.

[10] Ma, M.S., Ma, J.Y., Gu, S.M., *A high-order accuracy explicit difference scheme with branching stability for solving higher-dimensional heat-conduction equation*, Applied Mathematics and Mechanics, 23 (3) (2008), 447-452.

[11] Zeng, W.P., *A class of two-level explicit difference schemes for solving three dimensional heat conduction equation*, Applied Mathematics and Mechanics, 21 (9) (2000), 1071-1078.

[12] Ma, M.S., Wang T.K., *A family of high-order accuracy explicit difference schemes with branching stability for solving 3-D parabolic partial differential equation*, Applied Mathematics and Mechanics, 21 (10) (2000), 1207-1212.

[13] MA, S.L., *The necessary and sufficient condition for the two-order matrix family* $\mathbf{G^n}\left(\mathbf{k}, \mathbf{\Delta t}\right)$ *uniformly bounded and its applications to the stability of difference equations*, Numerical Mathematics A Journal of Chinese Universities, 2 (2) (1980), 41-53.

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