EXISTENCE AND UNIQUENESS OF Ψ-BOUNDED SOLUTIONS FOR NONLINEAR MATRIX DIFFERENCE EQUATIONS

T. Srinivasa Rao[1]

G. Suresh Kumar

Ch. Vasavi

Koneru Lakshmaiah University
Department of mathematics
Vaddeswaram, Guntur dt., A.P.
India
e-mails: tsrao011@gmail.com
drgsk006@kluniversity.in
vasavi.klu@gmail.com

M.S.N. Murty

Professor (RETD), Sainivas
D.No:21-47, Opp. State Bank Of India
Bank Street, NUZVID
Krishna dt., A.P.
India
e-mail: drmsn2002@gmail.com

Abstract. Sufficient conditions are established for the existence and uniqueness of Ψ-bounded solutions for nonlinear vector difference equation on $\mathbb{Z}$, using Banach contraction principle. Further, we obtain sufficient conditions for the existence and uniqueness of Ψ-bounded solutions for nonlinear matrix difference equation on $\mathbb{Z}$, using Kronecker product of matrices.

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1. Introduction

The aim of this paper is to give sufficient conditions for the nonlinear matrix difference equation

$$X(n+1) = A(n)X(n)B(n) + F(n, X(n)) \tag{1.1}$$

[1]Corresponding author.

has a unique Ψ-bounded solution on $\mathbb{Z}$, where $A \in \mathbb{R}^{m\times m}$, $F : \mathbb{R} \times \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ and $F(n, O) = O$. Here Ψ is an invertible matrix function on $\mathbb{Z}$. The basic problem under consideration is to determine sufficient conditions for the existence of a solution with some specified boundedness conditions. Classical results of this type, for linear and nonlinear differential equations were given by Coppel [2] and for linear and nonlinear difference equations were given by Agarwal [1]. The problem of Ψ-bounded solutions for the system of linear difference equations have been studied by many authors [3], [4], [6]. Recently, Suresh kumar et al. [7], [8], [9] studied Ψ-bounded solutions for linear matrix difference equations. In [3], Diamandescu proved a necessary and sufficient condition for the existence of Ψ-bounded solutions for the nonhomogeneous linear difference equation on $\mathbb{Z}$. Suresh kumar et al. [7] extended these results to matrix difference equations, using technique of Kronecker product of matrices.

In this paper, we present sufficient conditions for the existence and uniqueness of Ψ-bounded solutions for the nonlinear difference equation

$$x(n+1) = A(n)x(n) + f(n, x(n)) \tag{1.2}$$

on $\mathbb{Z}$. Further, we established sufficient conditions for the existence and uniqueness of Ψ-bounded solutions for nonlinear matrix difference equation (1.1) on $\mathbb{Z}$ with the help of Kronecker product of matrices.

2. Preliminaries

Denote $\mathbb{R}^n$ the Euclidean n-space. $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of all real, integers and nonnegative integers respectively, denote $N(n_0) = \{n_0, n_0+1, n_0+2, \ldots\}$. I_m and O_m denote the unit matrix and zero matrix of order m respectively. For $x = \{x_1, x_2, \ldots, x_m\} \in \mathbb{R}^m$ and $A = [a_{ij}] \in \mathbb{R}^{m\times m}$, we use the following vector and matrix norms

$$\|x\| = \max\{|x_1|, |x_2|, \ldots |x_m|\} \quad \text{and} \quad |A| = \sup_{\|x\|\le 1} \|Ax\|.$$

Let $\Psi_i : \mathbb{Z} \to (0, \infty)$, $i = 1, 2, \ldots, m$ be functions, and define a matrix function

$$\Psi = \text{diag}[\Psi_1, \Psi_2, \ldots \Psi_m].$$

Then $\Psi(n)$ is an invertible matrix function on $\mathbb{Z}$.

Definition 2.1 [5] Let $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{r\times s}$ then the Kronecker product of A and B written $A \otimes B$ is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1q}B \\ a_{21}B & a_{22}B & \ldots & a_{2q}B \\ \ldots & \ldots & \ldots & \ldots \\ a_{p1}B & a_{p2}B & \ldots & a_{pq}B \end{bmatrix}$$

is an $pr \times qs$ matrix and is in $\mathbb{R}^{pr\times qs}$.

Definition 2.2 [5] Let $A = [a_{ij}] \in \mathbb{R}^{p\times q}$, then the vectorization operator $Vec : \mathbb{R}^{p\times q} \to \mathbb{R}^{pq}$, defined and denote by

$$\hat{A} = VecA = \begin{bmatrix} A_{.1} \\ A_{.2} \\ \vdots \\ A_{.q} \end{bmatrix}, \text{ where } A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{pj} \end{bmatrix} (1 \le \ j \le \ q) \,.$$

Lemma 2.1 [7] *The vectorization operator $Vec : \mathbb{R}^{m\times m} \to \mathbb{R}^{m^2}$, is a linear and one-to-one operator. In addition, Vec and Vec^{-1} are continuous operators.*

Regarding properties and rules for Kronecker product of matrices we refer to [5].

Now, by applying the Vec operator to the nonlinear matrix difference equation (1.1) and using Kronecker product properties, we have

$$\hat{X}(n+1) = G(n)\hat{X}(n) + \hat{F}(n, \hat{X}(n)), \tag{2.1}$$

where $G(n) = B^T(n) \otimes A(n)$ is a $m^2 \times m^2$ matrix and $\hat{F}(n, \hat{X}(n)) = VecF(n, X(n))$ is a column matrix of order m^2. The equation (2.1) is called the Kronecker product difference equation associated with (1.1). It is clear that, if $X(n)$ is a solution of (1.1) if and only if $\hat{X}(n) = \text{Vec } X(n)$ is a solution of (2.1).

The corresponding homogeneous difference equation of (2.1) is

$$\hat{X}(n+1) = G(n)\hat{X}(n). \tag{2.2}$$

Definition 2.3 A function $\varphi : \mathbb{Z} \to \mathbb{R}^m$ is said to be Ψ-bounded on $\mathbb{Z}$ if $\Psi\varphi$ is bounded on $\mathbb{Z}$ (i.e., there exists $M > 0$ such that $\|\Psi(n)\varphi(n)\| < M$, for all $n \in \mathbb{Z}$).

Extend this definition to matrix functions.

Definition 2.4 A matrix sequence $F : \mathbb{Z} \to \mathbb{R}^{m\times m}$ is said to be Ψ-bounded on $\mathbb{Z}$ if ΨF is bounded on $\mathbb{Z}$ (i.e., there exists $L > 0$ such that $|\Psi(n)F(n)| \le L$, for all $n \in \mathbb{Z}$).

The following lemmas play a vital role in the proof of main results.

Lemma 2.2 *The matrix function $F(n)$ is Ψ-bounded on $\mathbb{Z}$ if and only if the vector function $VecF(n)$ is $I_m \otimes \Psi$-bounded on $\mathbb{Z}$.*

Proof. From the proof of Lemma 2.1, it follows that

$$\frac{1}{m}|A| \le \|VecA\|_{\mathbb{R}^{m^2}} \le |A|,$$

for every $A \in \mathbb{R}^{m\times m}$.

Put $A = \Psi(n)F(n)$ in the above inequality, we have

$$\frac{1}{m}|\Psi(n)F(n)| \le \|(I_m \otimes \Psi(n)).VecF(n)\|_{\mathbb{R}^{m^2}} \le |\Psi(n)F(n)|, \tag{2.3}$$

for all matrix functions $F(n)$, $n \in \mathbb{Z}$. The proof easily follows from inequality (2.3). ∎

Consider the linear difference equation

$$x(n+1) = A(n)x(n), \tag{2.4}$$

where $A(n)$ is an invertible square matrix of order m on $\mathbb{Z}$. The following lemma is well-known and is given in [3].

Lemma 2.3 *Let $Y(n)$ be the fundamental matrix and satisfies $Y(0) = I_m$, then*

(i) $Y(n) = \begin{cases} A(n-1)A(n-2)\dots A(1)A(0), & n > 0 \\ I_m, & n = 0 \\ [A(-1)A(-2)\dots A(n-1)A(n)]^{-1}, & n < 0 \end{cases}$

(ii) $Y(n+1) = A(n)Y(n)$, *for all* $n \in \mathbb{Z}$.

(iii) *the solution of* (2.4) *with the initial condition* $x(0) = x_0$ *is*

$$x(n) = Y(n)x_0, \ \ n \in \mathbb{Z}.$$

(iv) $Y(n)$ *is invertible for each* $n \in \mathbb{Z}$ *and*

$$Y^{-1}(n) = \begin{cases} A^{-1}(0)A^{-1}(1)\dots A^{-1}(n-2)A^{-1}(n-1), & n > 0 \\ I_m, & n = 0 \\ A(-1)A(-2)\dots A(n-1)A(n), & n < 0. \end{cases}$$

Lemma 2.4 [7] *Let $Y(n)$ and $Z(n)$ be the fundamental matrices for the matrix difference equations*

$$X(n+1) = A(n)X(n) \tag{2.5}$$

and

$$X(n+1) = B^T(n)X(n), \tag{2.6}$$

$n \in \mathbb{Z}$ *respectively. Then the matrix* $Z(n) \otimes Y(n)$ *is fundamental matrix of* (2.2).

Lemma 2.5 *Let $Y(n)$ be an invertible matrix function on $\mathbb{N}$ and let P be a projection. If there exists a positive constant $L > 1$ such that*

$$\sum_{k=n_0}^{n-1} |\Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)| \le L, \ \ \text{for all } n \in \mathbb{N}(n_0), \tag{2.7}$$

then there exists a constant $L_1 > 0$ such that

$$|\Psi(n)Y(n)P| \le L_1 \left(\frac{L-1}{L}\right)^{n-n_0}, \ \ \text{for all } n \in \mathbb{N}(n_0). \tag{2.8}$$

Proof. Let $a(n) = |\Psi(n+1)Y(n+1)P|^{-1}$. From the identity

$$\Psi(n)Y(n)P\left(\sum_{k=n_0}^{n-1} a(k)\right) = \sum_{k=n_0}^{n-1} \Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)\Psi(k+1)Y(k+1)Pa(k),$$

it follows that

$$\tag{2.9} \begin{aligned} &|\Psi(n)Y(n)P|\left(\sum_{k=n_0}^{n-1} a(k)\right)\\ &\le \sum_{k=n_0}^{n-1} |\Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)|\ |\Psi(k+1)Y(k+1)P|a(k) \le L. \end{aligned}$$

Setting $b(n) = \sum_{k=n_0}^{n-1} a(k)$, we obtain $b(n) - b(n-1) = |\Psi(n)Y(n)P|^{-1}$, for $n \in \mathbb{N}(n_0+1)$. After substituting in (2.9), we have

$$b(n) - b(n-1) \ge \frac{b(n)}{L} \quad \text{and} \quad b(n) \ge \frac{L}{L-1} b(n-1),$$

which implies that

$$b(n) \ge \left(\frac{L}{L-1}\right)^{n-1-n_0} b(n_0+1), \text{ for all } n \in \mathbb{N}(n_0+1).$$

From (2.9), we get

$$|\Psi(n)Y(n)P|b(n) \le L,$$

which implies that

$$\begin{aligned} |\Psi(n)Y(n)P| &\le L(b(n))^{-1} \le L\left(\frac{L-1}{L}\right)^{n-n_0-1} b^{-1}(n_0+1)\\ &= L\left(\frac{L-1}{L}\right)^{n-n_0-1} |\Psi(n_0+1)Y(n_0+1)P|. \end{aligned}$$

If we choose $L_1 = \max\left\{|\Psi(n_0)Y(n_0)P|, \frac{L^2}{L-1}|\Psi(n_0+1)Y(n_0+1)P|\right\}$, then (2.8) follows. ∎

Lemma 2.6 *Let $Y(n)$ be an invertible matrix which is defined on $\mathbb{N}$ and let P be a projection. If there exists a constant $L > 0$ such that*

$$\tag{2.10} \sum_{k=n}^{\infty} |\Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)\| \le L, \text{ for all } n \in \mathbb{N},$$

then for any vector $\xi \in \mathbb{R}^m$ such that $P\xi \ne 0$,

$$\tag{2.11} \limsup_{n\to\infty} \|\Psi(n)Y(n)P\xi\| = \infty.$$

Proof. For any $n \in \mathbb{N}(n_0)$, we have $\|\Psi(n+1)Y(n+1)P\xi\| > 0$. Then, from

$$\sum_{k=n}^{n_1} \|\Psi(k+1)Y(k+1)P\xi\|^{-1}\Psi(n)Y(n)P\xi$$
$$=\sum_{k=n}^{n_1} \|\Psi(k+1)Y(k+1)P\xi\|^{-1}\Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)\Psi(k+1)Y(k+1)P\xi$$

and (2.10), we get

$$\|\Psi(k+1)Y(k+1)P\xi\| \sum_{k=n}^{n_1} \|\Psi(k+1)Y(k+1)P\xi\|^{-1} \le L, \ \text{for } n_1 \ge n, n, n_1 \in \mathbb{N}(n).$$

Therefore, $\sum\limits_{k=n}^{\infty} \|\Psi(k+1)Y(k+1)P\xi\|^{-1}$ exists and so

$$\limsup_{n\to\infty} \|\Psi(n+1)Y(n+1)P\xi\|^{-1} = 0, \ \text{or } \limsup_{n\to\infty} \|\Psi(n+1)Y(n+1)P\xi\| = \infty. \ ■$$

3. Main result

In this section, first we obtain sufficient conditions for the existence and uniqueness of Ψ-bounded solution of the nonlinear difference equation (1.2), using Banach contraction principle.

Let the vector space $\mathbb{R}^m$ be represented as a direct sum of three subspaces X_-, X_0, X_+ such that a solution $y(n)$ of (2.4) is Ψ-bounded on $\mathbb{Z}$ if and only if $y(0) \in X_0$ and Ψ-bounded on $\mathbb{Z}$ if and only if $y(0) \in X_- \oplus X_0$. Also let P_{-1}, P_0, P_1 denote the corresponding projections of $\mathbb{R}^m$ onto X_-, X_0, X_+ respectively.

In the general case where ($P_0 \neq 0$), the solution for (1.2) is as follows

$$x(n) = \sum_{k=-\infty}^{n-1} Y(n)P_{-1}Y^{-1}(k+1)f(k, x(k)) + \sum_{k=0}^{n-1} Y(n)P_0Y^{-1}(k+1)f(k, x(k)) - \sum_{k=n}^{\infty} Y(n)P_1Y^{-1}(k+1)f(k, x(k)). \tag{3.1}$$

For simplicity, assume that the linear equation (2.4) has no nontrivial Ψ-bounded solution ($P_0 = 0$).

Theorem 3.1 (Existence and Uniqueness) *Suppose that there exist supplementary projections P_{-1},P_1 and a positive constant K such that*

$$\sum_{k=-\infty}^{n-1} |\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)| + \sum_{k=n}^{\infty} |\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)| \le K. \tag{3.2}$$

Let $f(n,x)$ be a vector function such that

$$||\Psi(n)[f(n,x)-f(n,y)]\| \leq \alpha||\Psi(n)(x-y)\|, \tag{3.3}$$

for $n\in\mathbb{Z}$, $\|\Psi x\|\leq\rho$, $\|\Psi y\|\leq\rho$, where $\alpha K<1$, then the equation (1.2) has a unique Ψ-bounded solution $x(n)$ for which $\|\Psi x\|\leq\rho$.

Proof. From Lemmas 2.5 and 2.6, the condition (3.2) implies that $|\Psi(n)Y(n)P_{-1}\xi|$ is unbounded for $n\leq 0$ if $P_{-1}\xi\neq 0$ and bounded for $n\geq 0$, and that $|\Psi(n)Y(n)P_1\xi|$ is unbounded for $n\geq 0$ if $P_1\xi\neq 0$ and bounded for $n\leq 0$. Hence the linear equation (2.4) has no nontrivial Ψ-bounded solution.

Let $x(n)$ be the solution of (1.2), then from (3.2) and (3.3) the function

$$\begin{aligned} y(n)=x(n)&-\sum_{k=-\infty}^{n-1}Y(n)P_{-1}Y^{-1}(k+1)f(k,x(k))\\ &+\sum_{k=n}^{\infty}Y(n)P_1Y^{-1}(k+1)f(k,x(k)), \end{aligned} \tag{3.4}$$

exists and is Ψ-bounded for all $n\in\mathbb{Z}$. Moreover, it follows that

$$\begin{aligned} y(n+1)=x(n+1)&-\sum_{k=-\infty}^{n}Y(n+1)P_{-1}Y^{-1}(k+1)f(k,x(k))\\ &+\sum_{k=n+1}^{\infty}Y(n+1)P_1Y^{-1}(k+1)f(k,x(k))\\ =Ax(n)&+f(n,x(n))-Y(n+1)P_{-1}Y^{-1}(n+1)f(n,x(n))\\ &-\sum_{k=-\infty}^{n-1}A(n)Y(n)P_{-1}Y^{-1}(k+1)f(k,x(k))\\ &-Y(n+1)P_1Y^{-1}(n+1)f(n,x(n))\\ &+\sum_{k=n}^{\infty}A(n)Y(n)P_1Y^{-1}(k+1)f(k,x(k))\\ =A(n)&\Bigg[x(n)-\sum_{k=-\infty}^{n-1}Y(n)P_{-1}Y^{-1}(k+1)f(k,x(k))\\ &-\sum_{k=n}^{\infty}Y(n)P_1Y^{-1}(k+1)f(k,x(k))\Bigg]+f(n,x(n))-f(n,x(n))\\ =A(n)&y(n). \end{aligned}$$

Therefore, $y(n)$ is a Ψ-bounded solution of (1.2). Thus $y(n)=0$ that is

$$x(n)=\sum_{k=-\infty}^{n-1}Y(n)P_{-1}Y^{-1}(k+1)f(k,x(k))-\sum_{k=n}^{\infty}Y(n)P_1Y^{-1}(k+1)f(k,x(k)). \tag{3.5}$$

Define C_Ψ=$\{x : \mathbb{R} \to \mathbb{R}^m : x$ is Ψ-bounded functions on $\mathbb{Z}$ such that $\|\Psi x\| \leq \rho\}$ and $\|x\|_\Psi = \sup_{n\in\mathbb{Z}} \|\Psi(n)x(n)\|$. Clearly, this defines a norm on C_Ψ and $(C_\Psi, ||\cdot\|_\Psi)$ is a Banach space. Let T be a mapping defined by

$$(3.6)\quad Tx(n)=\sum_{k=-\infty}^{n-1} Y(n)P_{-1}Y^{-1}(k+1)f(k,x(k))-\sum_{k=n}^{\infty} Y(n)P_1Y^{-1}(k+1)f(k,x(k)),$$

for all $x \in C_\Psi$. Consider

$$\begin{aligned}\|\Psi(n)Tx(n)\| = |&\sum_{k=-\infty}^{n-1} \Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k,x(k))\\ &-\sum_{k=n}^{\infty}\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k,x(k))|\\ \leq \Big[&\sum_{k=-\infty}^{n-1} |\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)|\\ &+\sum_{k=n}^{\infty}|\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)|\Big]|\|\Psi(k)f(k,x(k))\|\\ \leq\, &K\alpha\|\Psi(k)x(k)\|\\ \leq\, &K\alpha\rho < \rho.\end{aligned}$$

which implies $Tx(n) \in C_\Psi$ and hence $T : C_\Psi \to C_\Psi$.

Now, we show that T is a contraction mapping on C_Ψ. Consider

$$\begin{aligned}\|\Psi(n)(Tx-Ty)(n)\| = \Big\|&\sum_{k=-\infty}^{n-1} \Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k,x(k))\\ &-\sum_{k=n}^{\infty}\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k,x(k))\\ &-\Big[\sum_{k=-\infty}^{n}\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k,y(k))\\ &-\sum_{k=n}^{\infty}\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k,y(k))\Big]\Big\|\\ \leq \Big[&\sum_{k=-\infty}^{n}|\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)|\\ &+\sum_{k=n}^{\infty}|\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)|\Big]\\ &\qquad\|\Psi(k)[f(k,x(k))-f(k,y(k))]\|\\ \leq\, &K\alpha\|\Psi(k)(x(k)-y(k))\|.\end{aligned}$$

Thus $\|Tx-Ty\|_\Psi \leq K\alpha\|x-y\|_\Psi$.

Therefore T is a contraction mapping on C_Ψ. Hence by Banach contraction principle, T has a unique fixed point $x(n)$ on C_Ψ. Thus, the nonlinear difference equation (1.2) has a unique fixed point for which $\|\Psi x\| \le \rho$.

Conversely, if $x(n)$ is a solution of (1.2) such that $\|\Psi x\| \le \rho$, then $y = x - Tx$ is a Ψ-bounded solution of the linear equation (2.4), therefore $y = 0$. ■

Example 3.1 Consider the nonlinear difference equation (1.2) with

$$A(n) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \text{ and } f(n, x(n)) = \frac{1}{10} \begin{bmatrix} 3^{-|n|} \tan^{-1}(x_1(n)) \\ 5^{-|n|} \frac{x_2(n)}{1+n^2} \end{bmatrix}.$$

The fundamental matrix of (2.4) is

$$Y(n) = \begin{bmatrix} 2^n & 0 \\ 0 & 3^{-n} \end{bmatrix}.$$

Consider

$$\Psi(n) = \begin{bmatrix} 3^{-n} & 0 \\ 0 & 5^n \end{bmatrix}, \text{ for all } n \in \mathbb{Z}.$$

Clearly, the linear equation (2.4) has no nontrivial Ψ-bounded solution on $\mathbb{Z}$. There exist supplementary projections

$$P_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

such that

$$\sum_{k=-\infty}^{n-1} |\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)| = \left(\frac{1}{3}\right) \sum_{k=-\infty}^{n-1} \left(\frac{2}{3}\right)^{k-(n+1)} = 1,$$

$$\sum_{k=n}^{\infty} |\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)| = (3) \sum_{k=n}^{\infty} \left(\frac{5}{3}\right)^{n-k} = 7.5,$$

which implies

$$\sum_{k=-\infty}^{n-1} |\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)| + \sum_{k=n}^{\infty} |\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)| = 8.5.$$

And also

$$\|\Psi(n)[f(n,x) - f(n,y)]\| \le \frac{1}{10}\|\Psi(n)[x-y]\|.$$

Therefore, all the conditions of Theorem 3.1 are satisfied with $\alpha = 1/10$ and $K = 8.5$. Hence the nonlinear difference equation (1.2) has a unique Ψ-bounded solution

$$x(n) = \frac{1}{10} \begin{bmatrix} \sum_{k=-\infty}^{n-1} (2)^{n-(k+1)} 3^{-|k|} \tan^{-1}(x_1(k)) \\ \sum_{k=n}^{\infty} 3^{(k+1)-n} 5^{-|k|} \frac{x_2(k)}{1+k^2} \end{bmatrix}$$

on $\mathbb{Z}$.

Now, we obtain sufficient conditions for the existence and uniqueness of the nonlinear matrix difference equation (1.1), using Theorem 3.1 and the technique of Kronecker product of matrices.

Let the matrix space $\mathbb{R}^{m\times m}$ be represented as a direct sum of three subspaces Y_-, Y_0, Y_+ such that a solution $V(n)$ of

$$X(n+1) = A(n)X(n)B(n) \tag{3.7}$$

is a Ψ-bounded solution on $\mathbb{Z}$ if and only if $V(0) \in Y_0$ and Ψ-bounded on $\mathbb{N}$ if and only if $V(0) \in Y_- \oplus Y_0$. Also, let R_{-1}, R_0, R_1 denote the corresponding projections of $\mathbb{R}^{m\times m}$ onto Y_-, Y_0, Y_+ respectively.

Then, the vector space $\mathbb{R}^{m^2}$ represents direct sum of three subspaces S_-, S_0, S_+ such that a solution $\hat{V}(n) = VecV(n)$ of (2.2) is $(I_m \otimes \Psi)$-bounded on $\mathbb{Z}$ if and only if $\hat{V}(0) \in S_0$ and $(I_m \otimes \Psi)$-bounded on $\mathbb{N}$ if and only if $\hat{V}(0) \in S_- \oplus S_0$. Also, let Q_{-1}, Q_0, Q_1 denote the corresponding projections of $\mathbb{R}^{m^2}$ onto S_-, S_0, S_+ respectively.

In the general case where $(Q_0 \neq 0)$, the solution for (2.1) is as follows

$$\begin{aligned} \hat{X}(n) = &\sum_{k=-\infty}^{n-1} (Z(n)\otimes Y(n))Q_{-1}(Z^{-1}(k+1)\otimes Y^{-1}(k+1))\hat{F}(k,\hat{X}(k)) \\ &+\sum_{k=0}^{n-1}(Z(n)\otimes Y(n))Q_0(Z^{-1}(k+1)\otimes Y^{-1}(k+1))\hat{F}(k,\hat{X}(k)) \\ &-\sum_{k=n}^{\infty}(Z(n)\otimes Y(n))Q_1(Z^{-1}(k+1)\otimes Y^{-1}(k+1))\hat{F}(k,\hat{X}(k)). \end{aligned} \tag{3.8}$$

For simplicity, assume that the linear equation (2.2) has no nontrivial $(I_m \otimes \Psi)$-bounded solution $(Q_0 = 0)$.

Theorem 3.2 *Suppose that there exist supplementary projections Q_{-1}, Q_1 and a positive constant M such that*

$$\begin{aligned} &\sum_{k=-\infty}^{n-1} |(Z(n)\otimes \Psi(n)Y(n))Q_{-1}(Z^{-1}(k+1)\otimes (Y^{-1}(k+1)\Psi^{-1}(k)))| \\ &+\sum_{k=n}^{\infty} |(Z(n)\otimes\Psi(n)Y(n))Q_1(Z^{-1}(k+1)\otimes(Y^{-1}(k+1)\Psi^{-1}(k)))| \leq M. \end{aligned} \tag{3.9}$$

Let $F(n,X)$ be a matrix function such that

$$|\Psi(n)(F(n,U) - F(n,V))| \leq \beta|\Psi(n)(U-V)| \tag{3.10}$$

for $n \in \mathbb{Z}$, $|\Psi U| \leq \gamma$, $|\Psi V| \leq \gamma$, where $m\beta M < 1$, then the equation (1.1) has a unique Ψ-bounded solution $X(n)$ from which $|\Psi X| \leq \gamma$.

Proof. Let $F(n,X)$ be a matrix function satisfies (3.10). From inequality (2.3), we have

$$\begin{aligned} |(I_m \otimes \Psi(n+1))(\hat{F}(n,\hat{U}) - \hat{F}(n,\hat{V}))| &\leq |\Psi(n+1)(F(n,U) - F(n,V))| \\ &\leq \beta|\Psi(n)(U-V)| \\ &\leq m\beta|(I_m \otimes \Psi(n))(\hat{U} - \hat{V})|, \end{aligned}$$

for $\hat{U}, \hat{V} \in \mathbb{R}^{m^2}$. Also

$$\begin{aligned} |(I_m \otimes \Psi(n))\hat{U}| &\leq |\Psi(n)U(n)| \leq \gamma, \\ |(I(n) \otimes \Psi(n))\hat{V}| &\leq |\Psi(n)V(n)| \leq \gamma. \end{aligned}$$

From Kronecker product properties, equations (3.9) and (3.10), we have that the fundamental matrix of (2.2) satisfies condition (3.2) of Theorem 3.1, and the vectorization function $\hat{F}(n,\hat{X})$ satisfies condition (3.3). Therefore, from Theorem 3.1, the Kronecker product difference equation (2.1) has a unique $(I_m \otimes \Psi)$-bounded solution on $\mathbb{Z}$. From Lemma 2.2, the matrix difference equation (1.1) has a unique Ψ-bounded solution on $\mathbb{Z}$. ∎

Example 3.2 Consider the nonlinear matrix difference equation (1.1) with

$$A(n) = \begin{bmatrix} 5 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, \; B(n) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}, \; F(n,X(n)) = \frac{1}{30} \begin{bmatrix} \frac{5^{-|n|}}{1+|n|}x_1(n) & 3^{-|n|}\tan^{-1} x_3(n) \\ \frac{5^{-|n|}}{1+n^2}x_2(n) & \frac{3^{-|n|}}{1+|n|}\tan^{-1} x_4(n) \end{bmatrix}$$

Then $Y(n) = \begin{bmatrix} 5^n & 0 \\ 0 & 3^{-n} \end{bmatrix}$ and $Z(n) = \begin{bmatrix} 3^{-n} & 0 \\ 0 & 2^n \end{bmatrix}$ are fundamental matrices for (2.5) and (2.6) respectively.

Let $\Psi(n) = \begin{bmatrix} 5^{-n} & 0 \\ 0 & 3^n \end{bmatrix}$, for all $n \in \mathbb{Z}$. Then, their exist supplementary projections

$$Q_{-1} = \begin{bmatrix} I_2 & O_2 \\ O_2 & O_2 \end{bmatrix} \text{ and } Q_1 = \begin{bmatrix} O_2 & O_2 \\ O_2 & I_2 \end{bmatrix},$$

such that condition (3.9) and (3.10) satisfied with $M = 15/2$ and $\beta = 1/30$. Moreover, $m\beta M = 1/2 < 1$. Therefore all the conditions of Theorem 3.2 are satisfied. Hence nonlinear difference equation (1.1) has a unique Ψ-bounded solution on $\mathbb{Z}$.

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