ON ROUGH APPROXIMATIONS OF SUBGROUPS VIA CONJUGACY

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Abstract. Conjugacy is a very significant equivalence relation in the theory of groups and it has several important applications as well. In this present paper, we use this equivalence relation to generate Rough sets. A few results are presented in this context by assigning a group structure to the universe set. Some interesting properties of Lower and Upper approximations of subgroups are investigated.

Keywords: group, subgroup, conjugacy, rough set, upper approximation, lower approximation, information system, equivalence relation.

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1. Introduction

The problem of imperfect knowledge has been tackled for a long time by philosophers, logicians and mathematicians. Recently it became a crucial issue for computer scientists, particularly in the area of Artificial Intelligence. There are many approaches to the problem of how to understand and manipulate imperfect knowledge. The most successful approaches to tackle this problem are the Fuzzy set theory and the Rough set theory. Theories of Fuzzy sets and Rough sets are powerful mathematical tools for modeling various types of uncertainties. Fuzzy set theory was introduced by *L. A. Zadeh* in his classical paper [6] of 1965.

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A polish applied mathematician and computer scientist *Zdzislaw Pawlak* introduced Rough set theory in his classical paper [3] of 1982. Rough set theory is a new mathematical approach to imperfect knowledge. This theory presents still another attempt to deal with uncertainty or vagueness. The Rough set theory has attracted the attention of many researchers and practitioners who contributed essentially to its development and application. Rough sets have been proposed for a very wide variety of applications. In particular, the Rough set approach seems to be important for Artificial Intelligence and cognitive sciences, especially for machine learning, knowledge discovery, data mining, pattern recognition and approximate reasoning.

In this present work, we construct Rough sets by considering the equivalence relation Conjugacy on a group. We investigate a few results on lower and upper approximations of subgroups.

2. Preliminaries

Definition 2.1 A relation R on a non-empty set S is said to be an *equivalence relation* on S if

(a) xRx for all $x \in S$ (*reflexivity*)

(b) $xRy \Leftrightarrow yRx$ (*symmetry*)

(c) xRy and $yRz \Rightarrow xRz$ (*transitivity*)

We denote the equivalence class of an element $x \in S$ with respect to the equivalence relation R by the symbol $R[x]$ and $R[x] = \{y \in S : yRx\}$.

Definition 2.2 Let U be a universe set and $X \subseteq U$. Let R be an equivalence relation on U . Then we define the following.

(a) The *lower approximation* of X with respect to R is the set

$$R_*(X) = \{x : R[x] \subseteq X\}$$

(b) The *upper approximation* of X with respect to R is the set

$$R^*(X) = \{x : R[x] \cap X \neq \phi\}$$

(c) The *boundary region* of X with respect to R is the set

$$B_R(X) = R^*(X) - R_*(X)$$

It is clear that $R_*(X) \subseteq X \subseteq R^*(X)$.

Definition 2.3 A set $X \subseteq U$ is said to be a *Rough set* with respect to an equivalence relation R on U, if the boundary region $\mathcal{B}_{\mathcal{R}}(X) = R^*(X) - R_*(X)$ is non-empty.

Definition 2.4 A non-empty set G of elements is said to form a *group* if there is defined a binary operation, called the product and denoted by $*$, such that

(a) $a, b \in G \quad \Rightarrow a * b \in G$ (*Closure*)

(b) $a * (b * c) = (a * b) * c$ for all a, b, c in G (*Associative law*)

(c) there exists an element $e \in G$ such that $a * e = e * a = a \; \forall \; a \in G$. The element e is called the identity element in G. (*Existence of identity*)

(d) for every $a \in G$ there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$. The element a^{-1} is called the inverse element of a in G. (*Existence of inverse*)

Definition 2.5 A group G with respect to a binary operation $*$ is said to be an *abelian group* if $a * b = b * a$ for all a, b in G.

Definition 2.6 A non-empty subset H of a group G with respect to a binary operation $*$ is said to be a *subgroup* of G if H itself forms a group with respect to $*$.

Remark 2.7 A non-empty subset H of a group G is a subgroup of G if and only if (i) $a, b \in H \quad \Rightarrow a * b \in H$ and (ii) $a \in H \quad \Rightarrow a^{-1} \in H$

Definition 2.8 A subgroup H of a group G is said to be a *normal subgroup* of G if $x \in G$ and $h \in H$ then $xhx^{-1} \in H$.

3. Construction of rough sets

In the Literature of Rough set theory, information systems are considered. An *information system* is a pair $(U, \mathcal{A})$ where $\mathcal{A}$ is a set of attributes. Each attribute $a \in \mathcal{A}$ is a mapping $a : U \to V_a$ where V_a is the range set of the attribute $a \in \mathcal{A}$. Corresponding to each attribute $a \in \mathcal{A}$, an equivalence relation R_a is defined on U such that $xR_ay \Leftrightarrow a(x) = a(y)$. Rough sets are constructed through this relation as usual.

In this section, we slightly deviate from the above traditional setting to construct Rough sets. We consider the relation *conjugacy* on a group to construct Rough sets and present a few results in this context. In what follows G stands for a group and we take the universe set U to be G.

Definition 3.1 If $x, y \in G$ then y is said to be a conjugate of x in G if there exists an element $c \in G$ such that $y = c^{-1}xc$. We write, for this, $x \sim y$ and refer to this relation as *conjugacy*.

Theorem 3.2 Conjugacy *is an equivalence relation on* G.

Remark 3.3 Since conjugacy is an equivalence relation on G, the group G is partitioned into disjoint equivalence classes. If $x \in G$ then the equivalence class of x under the relation conjugacy is the set $C[x] = \{c^{-1}xc : c \in G\}$ and it is called the *conjugate class* of x in G.

Definition 3.4 The lower approximation, upper approximation and boundary region of a subset X of G under the equivalence relation conjugacy on G are as follows.

(a) $C_*(X) = \{x \in G : C[x] \subseteq X\}$,

(b) $C^*(X) = \{x \in G : C[x] \cap X \neq \phi\}$,

(c) $\mathcal{B}_\sim(X) = C^*(X) - C_*(X)$.

4. Rough approximations of subgroups

In this section, we present a few results on lower and upper approximations of subgroups of G under the relation conjugacy.

Proposition 4.1 *A group G is abelian $\Leftrightarrow C_*(X) = X = C^*(X)$ for any subset X of G.*

Proof. Let G be a group. Suppose that G is abelian. Then the conjugate class of an element $x \in G$ is $C[x] = \{x\}$. Let X be any subset of G. If $X = \phi$ or $X = G$, then $C_*(X) = X = C^*(X)$. Now suppose that $X \neq \phi$ and $X \neq G$. Let $x \in X$. Since $C[x] = \{x\} \subset X, \quad x \in C_*(X) \Rightarrow X \subseteq C_*(X) \Rightarrow X = C_*(X)$.

Let $y \in C^*(X)$. Then $\{y\} \cap X \neq \phi \Rightarrow y \in X$. This shows that $C^*(X) \subseteq X$. Hence $C_*(X) = X = C^*(X)$ for any subset X of G.

Conversely, suppose that $C_*(X) = X = C^*(X)$ for any subset X of G. Let $x, y \in G$. Since $C_*(\{y\}) = \{y\} = C^*(\{y\})$, we have $C[y] \subseteq \{y\}$ and $C[y] \cap \{y\} \neq \phi \Rightarrow C[y] = \{y\}$. Since $x^{-1}yx \in C[y]$, $x^{-1}yx = y \Rightarrow xy = yx$. Hence G is abelian. ∎

Remark 4.2 By the above Proposition 4.1, it follows that there does not exist Rough sets in an abelian group under the equivalence relation conjugacy.

Example 4.3 Consider the symmetric group $S_3 = \{e, x, x^2, y, xy, x^2y\}$ of order $|S_3| = 6$, where e is the identity permutation, x is the cyclic permutation $(1\ 2\ 3)$ and y is the transposition $(1\ 2)$. It can be observed that $x^3 = e$ and $y^2 = e$. This group S_3 is a finite non-abelian group under the product of permutations. In this group S_3, the conjugate classes are as follows.

(a) $C[e] = \{e\}$

(b) $C[x] = C[x^2] = \{x, x^2\}$

(c) $C[y] = C[xy] = C[x^2y] = \{y, xy, x^2y\}$.

Let $A = \{e, x\}$. Then $C_*(A) = \{e\}$ and $C^*(A) = \{e, x, x^2\}$. Then $\mathcal{B}_\sim(A) = C^*(A) - C_*(A) = \{x, x^2\}$. Hence A is a Rough set in S_3 under the relation conjugacy. From this example it is clear that Rough sets do exist in non-abelian groups.

Proposition 4.4 *If H is a subgroup of G then the lower approximation $C_*(H)$ of H is a subgroup of G.*

Proof. Let H be a subgroup of G. Let $x, y \in C_*(H)$. Then $C[x] \subseteq H$ and $C[y] \subseteq H$. Let $z \in C[xy]$.

$\Rightarrow \quad z = c^{-1}(xy)c$ for some $c \in G$

$\Rightarrow \quad z = (c^{-1}xc)\,(c^{-1}yc)$.

Since $c^{-1}xc \in C[x]$ and $c^{-1}yc \in C[y]$, we have $c^{-1}xc \in H$ and $c^{-1}yc \in H$

$\Rightarrow \quad (c^{-1}xc)\,(c^{-1}yc) \in H$

$\Rightarrow \quad z \in H$

Hence $C[xy] \subseteq H$. This shows that $xy \in C_*(H)$.

Let $w \in C[y^{-1}]$. Then $w = d^{-1}y^{-1}d$ for some $d \in G$.

Since $d^{-1}y^{-1}d \in C[y]$, $d^{-1}y^{-1}d \in H$

$\Rightarrow \quad d^{-1}y^{-1}d \in H$

$\Rightarrow \quad w \in H$

Hence $C[y^{-1}] \subseteq H$. This proves that $y^{-1} \in C_*(H)$. Hence $C_*(H)$ is a subgroup of G. ∎

Remark 4.5 The converse of the above Proposition 4.4 is not true. That is, a subset H of G need not be a subgroup of G even though $C_*(H)$ of H is a subgroup of G. It can be seen from the following example.

Example 4.6 In S_3, take $H = \{e, x^2\}$. Then $C_*(H) = \{e\}$. Now it can be observed that H is not a subgroup of G but $C_*(H)$ is a subgroup of G.

Proposition 4.7 *If H is a cyclic subgroup of G then the lower approximation $C_*(H)$ of H is a cyclic subgroup of G.*

Proposition 4.8 *A subgroup H of a group G is a normal subgroup of G if and only if $C_*(H) = H$.*

Proof. Suppose that H is a subgroup of G. Let H be a normal subgroup of G.

Let $x \in H$ and let $z \in C[x]$.

$\Rightarrow \quad z = c^{-1}xc$ for some $c \in G$.

Since $c \in G$, $c^{-1} \in G$.

Since $c^{-1} \in G$ and $x \in H$, $c^{-1}xc \in H \Rightarrow z \in H$.

This shows that $C[x] \subseteq H \Rightarrow x \in C_*(H)$. Hence $C_*(H) = H$.

Conversely, suppose that $C_*(H) = H$. Let $c \in G$ and $x \in H$

$\Rightarrow \quad c^{-1} \in G$ and $x \in C_*(H)$

$\Rightarrow \quad c^{-1} \in G$ and $C[x] \subseteq H$

$\Rightarrow \quad cxc^{-1} \in C[x]$

$\Rightarrow \quad cxc^{-1} \in H$.

Hence H is a normal subgroup of G. ∎

Proposition 4.9 *For any subset A of G, the following conditions are equivalent to one another.*

(a) $C_*(A) = A$,

(b) $C^*(A) = (A)$,

(c) $C_*(A) = C^*(A)$.

Proof. For any subset A of G, we always have $C_*(A) \subseteq A \subseteq C^*(A)$. Suppose $C_*(A) = A$ and let $x \in C^*(A)$. Then $C[x] \cap A \neq \phi$
$\Rightarrow$ $C[x] \cap C_*(A) \neq \phi$
$\Rightarrow$ there exists a point y in G such that $y \in C[x] \cap C_*(A)$
$\Rightarrow$ $y \sim x$ and $y \in C_*(A)$
$\Rightarrow$ $C[x] = C[y]$ and $C[y] \subseteq A$
$\Rightarrow$ $C[x] \subseteq A$
$\Rightarrow$ $x \in A$.
This shows that $C^*(A) \subseteq A$. Hence $C^*(A) = A$.
This completes the proof of (a) $\Rightarrow$ (b).

Now, consider $C^*(A) = A$. Let $x \in A$. Then $x \in C^*(A) \Rightarrow C[x] \cap A \neq \phi$.
Let $z \in C[x]$. Then $x \sim z \Rightarrow z \in A$.
This proves that $C[x] \subseteq A$ and hence $x \in C_*(A)$.
Hence $C_*(A) = A$. This completes the proof of (b) $\Rightarrow$ (a).
Obviously, (c) is equivalent to both (a) and (b). ∎

Proposition 4.10 *A subgroup H of G is a normal subgroup of G if and only if $C^*(H) = H$.*

Remark 4.11 The lower and upper approximations of a subset A of G can be equal even though A is not a subgroup of G. It can be observed from the following example.

Example 4.12 In S_3, take $B = \{e, y, xy, x^2y\}$. Then B is not a subgroup of G, but $C^*(B) = B = C_*(B)$.

Example 4.13 *Let $K = \{e, y\}$. Then K is a cyclic subgroup of S_3 generated by y but K is not a normal subgroup of S_3. It can be observed that $C^*(K) = \{e, y, xy, x^2y\}$ is not a cyclic subgroup of S_3. From this example, it follows that the upper approximation of a cyclic subgroup of G is not necessarily a cyclic subgroup of G.*

Example 4.14 In S_3, $C^*(W) = \{e, y, xy, x^2\}$ for $W = \{e, x\}$. Here $C^*(W)$ is a normal subgroup of S_3 eventhough W is not a subgroup of S_3. Also $C^*(W)$ is a cyclic subgroup of S_3, generated by x and $C^*(W) = A_3$, the Alternating group in S_3. From this example, it can be observed that the upper approximation of a subset can be a normal subgroup of G.

5. Homomorphisms and rough approximations

In this section, we establish some interesting results on the Rough approximations of homomorphic images of subsets of G. In what follows G and G' stand for two groups.

Definition 5.1 A mapping $f : G \to G'$ is said to be a *homomorphism* of G into G' if $f(xy) = f(x)\ f(y)$ for all x and y in G.

Definition 5.2 A homomorphism $f : G \to G'$ is said to be an *isomorphism* of G into G' if f is one-one.

Definition 5.3 A homomorphism $f : G \to G$ is said to be an *automorphism* of G onto itself if f is a bijection.

Proposition 5.4 *If $f : G \to G'$ is an isomorphism of G onto G', then*

$$C_*(f(A)) = f(C_*(A)) \text{ for any subset } A \text{ of } G.$$

Proof. Let $f : G \to G'$ be an isomorphism of G onto G' and let A be any subset of G. If $A = \phi$ or $A = G$ then $C_*(f(A)) = f(C_*(A))$. Now suppose that $A \neq \phi$ and $A \neq G$. Let $y \in f(C_*(A))$

$\Rightarrow \quad y = f(x)$ for some $x \in C_*(A)$.

Since $x \in C_*(A),\ C[x] \subseteq A$. Let $z \in C[y]$

$\Rightarrow \quad z = c^{-1}yc$ for some $c \in G'$.

Since $c \in G'$, there exists $d \in G$ such that $c = f(d)$.

Hence $z = c^{-1}yc = f(d^{-1})f(x)f(d) = f(d^{-1}xd)$.

Since $d^{-1}xd \in C[x], d^{-1}xd \in A$

$\Rightarrow \quad f(d^{-1}xd) \in f(A)$

$\Rightarrow \quad z \in f(A)$.

Hence $C[y] \subseteq f(A)$.

This proves that

$$f(C_*(A)) \subseteq C_*(f(A)) \tag{1}$$

Let $y \in C_*(f(A))$. Then $C[y] \subset f(A)$.

$\Rightarrow \quad y = f(x)$ for some $x \in A$.

Let $z \in C[x]$. Then $z = c^{-1}xc$ for some $c \in G$

$\Rightarrow \quad f(z) = f(c^{-1}xc) = f(c^{-1})y\ f(c)$

$\Rightarrow \quad f(z) \in C[y]$

$\Rightarrow \quad f(z) \in f(A)$

$\Rightarrow \quad f(z) \in f(a)$ for some $a \in A$

$\Rightarrow \quad z = a$ and hence $z \in A$.

Hence $C[x] \subseteq A$

$\Rightarrow x \in C_*(A)$

$\Rightarrow y \in f(C_*(A))$.

This shows that

$$C_*(f(A)) \subseteq f(C_*(f(A)) \tag{2}$$

From (1) and (2), we have

$$C_*(f(A)) = f(C_*(A)).$$ ∎

Proposition 5.5 *If $f : G \to G'$ is a homomorphism of G onto G' then*

$$C^*(f(A)) = f(C^*(A)) \text{ for any subset } A \text{ of } G.$$

Proof. Let $f : G \to G'$ be a homomorphism of G onto G' and let A be any subset of G. If $A = \phi$ or $A = G$, then $C^*(f(A)) = f(C^*(A))$.

Now, suppose that $A \neq \phi$ and $A \neq G$. Let $y \in f(C^*(A))$.
$\Rightarrow$ $y = f(x)$ for some $x \in C^*(A)$.
Since $x \in C^*(A)$, $C[x] \cap A \neq \phi$
$\Rightarrow$ there exists z such that $z \in C[x] \cap A$
$\Rightarrow$ $z = c^{-1}xc$ for some $c \in G$ and $z \in A$
$\Rightarrow$ $f(z) = f(c^{-1})y\, f(c)$
$\Rightarrow$ $y = f(czc^{-1}) = f(d^{-1}zd)$ where $d = c^{-1}$
$\Rightarrow$ $y = f(d^{-1})f(z)f(d)$
$\Rightarrow$ $f(z) \in C[y]$ and $f(z) \in f(A)$.
$\Rightarrow$ $f(z) \in C[y] \cap f(A)$.
Hence $C[y] \cap f(A) \neq \phi$ and $y \in C^*(f(A))$. This proves that

$$f(C^*(A)) \subseteq C^*(f(A)) \tag{1}$$

Let $y \in C^*(f(A))$. Then $C[y] \cap f(A) \neq \phi$
$\Rightarrow$ there exists $z \in G$ such that $f(z) \in C[y] \cap f(A)$
$\Rightarrow$ $f(z) = c^{-1}yc$ for some $c \in G'$ and $z \in A$.
Since $c \in G'$, there exists $d \in G$ such that $c = f(d)$. Hence $f(z) = f(d^{-1})y\, f(d)$
$\Rightarrow$ $y = f(d)f(z)f(d^{-1}) = f(dzd^{-1})$.
Since $z = d^{-1}(dzd^{-1})d$, $z \in C[dzd^{-1}] \cap A$
$\Rightarrow$ $C[dzd^{-1}] \cap A \neq \phi$
$\Rightarrow dzd^{-1} \in C^*(A)$
$\Rightarrow f(dzd^{-1}) \in f(C^*(A))$
$\Rightarrow$ $y \in f(C^*(A))$.
This shows that

$$C^*(f(A)) \subseteq f(C^*(f(A))) \tag{2}$$

From (1) and (2), we have

$$C^*(f(A)) = f(C^*(A)).$$ ∎

Proposition 5.6 *If $f : G \to G'$ is an isomorphism of G into G' then*

$$C_*(f(A)) \subseteq f(C_*(A)) \text{ for any subset } A \text{ of } G.$$

Proposition 5.7 *If* $f : G \to G'$ *is a homomorphism of* G *into* G' *then* $C^*(f(A)) \subseteq f(C^*(A))$ *for any subset* A *of* G.

Proposition 5.8 *If* $f : G \to G'$ *is an isomorphism of* G *into* G' *then* $f(\mathcal{B}_\sim(A)) = \mathcal{B}_\sim(f(A))$ *for any subset* A *of* G.

6. Topological aspects

In this section, we assign a topological structure to G and investigate some topological aspects in this context.

Definition 6.1 A *topological group* is a group G together with a topology on G that satisfies the following two properties.

(a) The map $\varphi : G \times G \to G$ defined by $\varphi(g, h) = gh$ is continuous when $G \times G$ is endowed with the product topology.

(b) The map $G \to G$ defined by $\psi(g) = g^{-1}$ is continuous.

Proposition 6.2 *A collection* $\mathcal{T} = \{A : C_*(A) = A\}$ *of subsets of* G *forms a topology on* G *and hence* $(G, \mathcal{T})$ *is a topological space.*

Proposition 6.3 *The map* $\psi : (G, \mathcal{T}) \to (G, \mathcal{T})$ *defined by* $\psi(g) = g^{-1}$ *for every* $g \in G$ *is continuous.*

Proof. Let U be an open set in $(G, \mathcal{T})$ and let $U^{-1} = \{a^{-1} : a \in U\}$.
If U is empty then so is U^{-1}. Now, suppose that U is non-empty.
Since U is open, $C_*(U) = U$.
If $y \in U^{-1}$, then $y = x^{-1}$ for some $x \in U$.
Since $x \in U$, $\quad x \in C_*(U) \quad \Rightarrow \quad C[x] \subseteq U$.
Let $z \in C[y]$. Then $z = c^{-1}yc$ for some $c \in G$
$\Rightarrow \quad z^{-1} = c^{-1}xc$
$\Rightarrow \quad z^{-1} \in C[x]$
$\Rightarrow \quad z^{-1} \in U$
$\Rightarrow \quad z \in U^{-1}$.
Hence $C[y] \subseteq U^{-1}$
$\Rightarrow \quad y \in C_*(U^{-1})$.
This shows that $C_*(U^{-1}) = U^{-1}$. Hence U^{-1} is open in $(G, \mathcal{T})$. ∎

Example 6.4 The topology $\mathcal{T} = \{A : C_*(A) = A\}$ on S_3 is given by

$$\mathcal{T} = \{\phi, S_3, \{e\}, \{x, x^2\}, \{e, x, x^2\}, \{y, xy, x^2y\}, \{e, y, xy, x^2y\}, \{x, x^2, y, xy, x^2y\}\}$$

The map $\varphi : S_3 \times S_3 \to S_3$ defined by $\varphi(g, h) = gh$ is not continuous when $S_3 \times S_3$ is endowed with the product topology. Because $x.x^2 = x^3 = e \in \{e\}$ and $\{e\} \in \mathcal{T}$, but there does not exist two open neighborhoods V_x and V_x^2 of x and x^2 respectively in $(S_3, \mathcal{T})$ such that $V_x \, V_x^2 \subseteq \{e\}$.

Now, it is so natural to ask the following Question-5.5. The investigation of the following problem is under progress. The results that are under investigation in this context may appear in later Research Articles.

Question 6.5 *Under what conditions $(G, \mathcal{T})$ forms a topological group ?*

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