SOME GENERAL NUMERICAL RADIUS INEQUALITIES FOR THE OFF-DIAGONAL PARTS OF 2×2 OPERATOR MATRICES

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Abstract. We give some sharp inequalities involving powers of the numerical radii for the off-diagonal parts of 2×2 operator matrices. These inequalities, which are based on some classical convexity inequalities for the nonnegative real numbers, generalize earlier numerical radius inequalities.

Keywords: numerical radius, operator matrix, off-diagonal part, Cartesian decomposition, Jensen's inequality, mixed Schwarz inequality.

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1. Introduction

Let $B(H)$ denote the C^*- algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. For $A \in B(H)$, let $\omega(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of A, respectively. It is well known that $\omega(\cdot)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm $\| \cdot \|$. In fact, for every $A \in B(H)$,

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{1.1}$$

The inequalities in (1.1) are sharp. The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A is normal. For basic properties of the numerical radius, we refer to [4] and [6]. The inequalities in (1.1) have been improved considerably by Kittaneh. It has been shown in [10] and [11], respectively, that if $A \in B(H)$, then

$$\omega(A) \leq \frac{1}{2}\||A| + |A^*|\| \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{\frac{1}{2}}\right), \tag{1.2}$$

where $|A| = (A^*A)^{\frac{1}{2}}$ is the absolute value of A, and

$$\frac{1}{4}\|A^*A + AA^*\| \leq \omega^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|. \tag{1.3}$$

The inequalities in (1.2), which refine the second inequality in (1.1), have been utilized in [10] to derive an estimate for the numerical radius of the Frobenius companion matrix. Such an estimate can be employed to give new bounds for the zeros of polynomials (see, e.g., [9],[10], and references therein).

If $A = B + iC$ is the Cartesian decomposition of A, then B and C are self-adjoint, and $A^*A + AA^* = 2(B^2 + C^2)$. Thus, the inequalities in (1.3) can be written as

$$\frac{1}{2} \| B^2 + C^2 \| \leq \omega^2(A) \leq \| B^2 + C^2 \| . \tag{1.4}$$

The purpose of this paper is to establish a general inequalities involving powers of the numerical radii for the off-diagonal parts of 2×2 operator matrices that are based on the classical convexity inequalities for nonnegative real numbers and some operator inequalities.

Other recent numerical radius inequalities have been obtained by Dragomir [3], El-Haddad [5], and Yamazaki [12]. The inequalities in [3] are related to the Euclidean radius of two Hilbert space operators, the inequalities in [5] involving powers of the numerical radii for Hilbert space operators, and those in [12] involve the Aluthge transform.

2. Main results

To prove our generalized numerical radius inequalities for the off-diagonal parts of 2×2 operator matrices, we need several well known lemmas. The first lemma is a simple consequence of the classical Jensen's inequality concerning the convexity or the concavity of certain power functions. It is a special case of Schlömilch's inequality for weighted means of nonnegative real numbers (see, e.g., [7, p. 26]).

Lemma 2.1 *For $a, b \geq 0$, $0 < \alpha < 1$, and $r \neq 0$, let $M_r(a, b, \alpha) = (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}}$ and let $M_0(a, b, \alpha) = a^\alpha b^{1-\alpha}$. Then*

$$M_r(a, b, \alpha) \leq M_s(a, b, \alpha) \quad for \quad r \leq s.$$

The second lemma is another application of Jensen's inequality (see. e.g., [7, p. 28]).

Lemma 2.2 *For $a, b \geq 0$, and $r > 0$, let $N_r(a, b) = (a^r + b^r)^{\frac{1}{r}}$. Then*

$$N_s(a, b) \leq N_r(a, b) \quad for \quad s \geq r > 0.$$

The third lemma follows from the spectral theorem for positive operators and Jensen's inequality (see, e.g., [8]).

Lemma 2.3 *Let $A \in B(H)$ be positive, and let $x \in H$ be any unit vector. Then*

(a) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \quad for \quad r \geq 1.$

(b) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r \quad for \quad 0 < r \leq 1.$

The fourth lemma is an immediate consequence of the spectral theorem for self-adjoint operators. For generalizations of this lemma, we refer to [8].

Lemma 2.4 *Let $A \in B(H)$ be self-adjoint, and let $x \in H$ be any vector. Then*

$$| \langle Ax, x\rangle | \leq \langle | A | x, x\rangle.$$

The fifth lemma is a generalized for the mixed Schwarz inequality which has been proved by Kittaneh [8].

Lemma 2.5 *Let T be an operator in $B(H)$ and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$| \langle Tx, y\rangle | \leq \| f(| T |)x \| \| g(| T^* |)x \| \qquad for\, all \quad x, y \in H.$$

The sixth lemma contains two parts. Part (a) is well known and can be found in [2, p. 10]. Part (b) is also known and can be found in [1].

Lemma 2.6 *Let $X, Y \in B(H)$. Then*

(a) $\omega\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\{\omega(X), \omega(Y)\}.$

(b) $\omega\left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\right) = \max\{\omega(X+Y), \omega(X-Y)\}.$

In particular,

$$\omega\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = \omega(Y).$$

Our first result is a generalization of the first inequality in (1.2).

Theorem 2.7 *Let $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ be a 2×2 operator matrix in $B(H_1 \bigoplus H_2)$, and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$, and $r \geq 1$. Then*

$$\omega^r(S) \leq \frac{1}{2} \max\{\|f^{2r}(|C|) + g^{2r}(|B^*|)\|, \|f^{2r}(|B|) + g^{2r}(|C^*|)\|\}. \tag{2.1}$$

Proof. For every unit vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (H_1 \bigoplus H_2)$, by using Lemma 2.5, Lemma 2.1, and Lemma 2.3(a) we have

$$\begin{aligned}
&\mid \langle SX, X\rangle \mid \leq \parallel f\left(\mid S \mid\right) X \parallel \parallel g\left(\mid S^* \mid\right) X \parallel \\
&= \langle f^2(\mid S \mid)X, X\rangle^{\frac{1}{2}} \langle g^2(\mid S^* \mid)X, X\rangle^{\frac{1}{2}} \\
&= \left\langle f^2\left(\begin{bmatrix} \mid C \mid & 0 \\ 0 & \mid B \mid \end{bmatrix}\right) X, X\right\rangle^{\frac{1}{2}} \left\langle g^2\left(\begin{bmatrix} \mid B^* \mid & 0 \\ 0 & \mid C^* \mid \end{bmatrix}\right) X, X\right\rangle^{\frac{1}{2}} \\
&= \left\langle \begin{bmatrix} f^2(\mid C \mid) & 0 \\ 0 & f^2(\mid B \mid) \end{bmatrix} X, X\right\rangle^{\frac{1}{2}} \left\langle \begin{bmatrix} g^2(\mid B^* \mid) & 0 \\ 0 & g^2(\mid C^* \mid) \end{bmatrix} X, X\right\rangle^{\frac{1}{2}} \\
&\leq \frac{1}{2}\left(\left\langle \begin{bmatrix} f^2(\mid C \mid) & 0 \\ 0 & f^2(\mid B \mid) \end{bmatrix} X, X\right\rangle + \left\langle \begin{bmatrix} g^2(\mid B^* \mid) & 0 \\ 0 & g^2(\mid C^* \mid) \end{bmatrix} X, X\right\rangle\right) \\
&\leq \left(\frac{1}{2}\left(\left\langle \begin{bmatrix} f^2(\mid C \mid) & 0 \\ 0 & f^2(\mid B \mid) \end{bmatrix} X, X\right\rangle^r + \left\langle \begin{bmatrix} g^2(\mid B^* \mid) & 0 \\ 0 & g^2(\mid C^* \mid) \end{bmatrix} X, X\right\rangle^r\right)\right)^{\frac{1}{r}} \\
&\leq \left(\frac{1}{2}\left(\left\langle \begin{bmatrix} f^{2r}(\mid C \mid) & 0 \\ 0 & f^{2r}(\mid B \mid) \end{bmatrix} X, X\right\rangle + \left\langle \begin{bmatrix} g^{2r}(\mid B^* \mid) & 0 \\ 0 & g^{2r}(\mid C^* \mid) \end{bmatrix} X, X\right\rangle\right)\right)^{\frac{1}{r}} \\
&\leq \left(\frac{1}{2}\left\langle \begin{bmatrix} f^{2r}(\mid C \mid) + g^{2r}(\mid B^* \mid) & 0 \\ 0 & f^{2r}(\mid B \mid) + g^{2r}(\mid C^* \mid) \end{bmatrix} X, X\right\rangle\right)^{\frac{1}{r}}.
\end{aligned}$$

Thus,

$$\mid \langle SX, X\rangle \mid^r \leq \frac{1}{2}\left\langle \begin{bmatrix} f^{2r}(\mid C \mid) + g^{2r}(\mid B^* \mid) & 0 \\ 0 & f^{2r}(\mid B \mid) + g^{2r}(\mid C^* \mid) \end{bmatrix} X, X\right\rangle,$$

and so

$$\begin{aligned}
\omega^r\left(S\right) &= \sup\{\mid \langle SX, X\rangle \mid^r : X \in (H_1 \oplus H_2), \parallel X \parallel = 1\} \\
&\leq \frac{1}{2}\sup\left\{\left\langle \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} X, X\right\rangle : X \in (H_1 \oplus H_2), \parallel X \parallel = 1\right\} \\
&= \frac{1}{2}\max\{\|\lambda\|, \|\mu\|\},
\end{aligned}$$

where

$$\lambda = f^{2r}(\mid C \mid) + g^{2r}(\mid B^* \mid)$$

and

$$\mu = f^{2r}(\mid B \mid) + g^{2r}(\mid C^* \mid),$$

as required. ∎

Inequality (2.1) includes several numerical radius inequalities for operator matrices. Samples of inequalities are demonstrated in the following remarks.

Remark 2.8 For $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$, $\alpha \in (0,1)$, in inequality (2.1), we get the following inequality

$$\omega^r(S) \le \frac{1}{2}\max\{\| \mid C \mid^{2r\alpha} + \mid B^* \mid^{2r(1-\alpha)} \|, \| \mid B \mid^{2r\alpha} + \mid C^* \mid^{2r(1-\alpha)} \|\}.$$

Remark 2.9 If $B = C$ in the above Remark, and by using Lemma 2.6(b), then

$$\omega^r(B) = \omega^r(S) \le \frac{1}{2}\| \mid B \mid^{2r\alpha} + \mid B^* \mid^{2r(1-\alpha)} \|,$$

and this inequality is given in Theorem 1 in [5].

Now, the second result is a generalization of the second inequality in (1.3).

Theorem 2.10 *Let* $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ *be a* 2×2 *operator matrix in* $B(H_1 \bigoplus H_2)$*, and let* f *and* g *be nonnegative functions on* $[0,\infty)$ *which are continuous and satisfying the relation* $f(t)g(t) = t$ *for all* $t \in [0,\infty)$*, and* $r \ge 1$ *and* $0 < k < 1$*. Then*

$$\omega^{2r}(S) \le \max\{\|kf^{\frac{2r}{k}}(|C|)+(1-k)g^{\frac{2r}{1-k}}(|B^*|)\|, \|kf^{\frac{2r}{k}}(|B|)+(1-k)g^{\frac{2r}{1-k}}(|C^*|)\|\}.$$

Proof. For every unit vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (H_1 \bigoplus H_2)$, by using Lemma 2.5, Lemma 2.3(b), Lemma 2.1, and Lemma 2.3(a), we have

$$\begin{aligned}
&\mid \langle SX, X\rangle \mid^2 \le \langle f^2(\mid S \mid) X, X\rangle \langle g^2(\mid S^* \mid) X, X\rangle \\
&= \left\langle \begin{bmatrix} f^2(\mid C \mid) & 0 \\ 0 & f^2(\mid B \mid) \end{bmatrix} X, X \right\rangle \left\langle \begin{bmatrix} g^2(\mid B^* \mid) & 0 \\ 0 & g^2(\mid C^* \mid) \end{bmatrix} X, X \right\rangle \\
&\le \left\langle \begin{bmatrix} f^{\frac{2}{k}}(\mid C \mid) & 0 \\ 0 & f^{\frac{2}{k}}(\mid B \mid) \end{bmatrix} X, X \right\rangle^k \left\langle \begin{bmatrix} g^{\frac{2}{1-k}}(\mid B^* \mid) & 0 \\ 0 & g^{\frac{2}{1-k}}(\mid C^* \mid) \end{bmatrix} X, X \right\rangle^{1-k} \\
&\le \left(k \left\langle \begin{bmatrix} f^{\frac{2}{k}}(\mid C \mid) & 0 \\ 0 & f^{\frac{2}{k}}(\mid B \mid) \end{bmatrix} X, X \right\rangle^r \right. \\
&\qquad \left. +(1-k) \left\langle \begin{bmatrix} g^{\frac{2}{1-k}}(\mid B^* \mid) & 0 \\ 0 & g^{\frac{2}{1-k}}(\mid C^* \mid) \end{bmatrix} X, X \right\rangle^r \right)^{\frac{1}{r}} \\
&\le \left(k \left\langle \begin{bmatrix} f^{\frac{2r}{k}}(\mid C \mid) & 0 \\ 0 & f^{\frac{2r}{k}}(\mid B \mid) \end{bmatrix} X, X \right\rangle \right. \\
&\qquad \left. +(1-k) \left\langle \begin{bmatrix} g^{\frac{2r}{1-k}}(\mid B^* \mid) & 0 \\ 0 & g^{\frac{2r}{1-k}}(\mid C^* \mid) \end{bmatrix} X, X \right\rangle \right)^{\frac{1}{r}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mid \langle SX, X\rangle \mid^{2r}\\
&\leq \left\langle \left(k \begin{bmatrix} f^{\frac{2r}{k}}(\mid C \mid) & 0 \\ 0 & f^{\frac{2r}{k}}(\mid B \mid) \end{bmatrix} + (1-k) \begin{bmatrix} g^{\frac{2r}{1-k}}(\mid B^* \mid) & 0 \\ 0 & g^{\frac{2r}{1-k}}(\mid C^* \mid) \end{bmatrix} \right) X, X \right\rangle \\
&= \left\langle \begin{bmatrix} kf^{\frac{2r}{k}}(\mid C \mid) + (1-k) g^{\frac{2r}{1-k}}(\mid B^* \mid) & 0 \\ 0 & kf^{\frac{2r}{k}}(\mid B \mid) + (1-k) g^{\frac{2r}{1-k}}(\mid C^* \mid) \end{bmatrix} X, X \right\rangle,
\end{aligned}$$

and so

$$\begin{aligned}
\omega^{2r}(S) &= \sup\{\mid \langle SX, X\rangle \mid^{2r} : X \in (H_1 \oplus H_2), \parallel X \parallel = 1\} \\
&\leq \sup\left\{ \left\langle \begin{bmatrix} \beta & 0 \\ 0 & \gamma \end{bmatrix} X, X \right\rangle : X \in (H_1 \oplus H_2), \parallel X \parallel = 1 \right\} \\
&= \max\{\|\beta\|, \|\gamma\|\},
\end{aligned}$$

where

$$\beta = kf^{\frac{2r}{k}}(\mid C \mid) + (1-k) g^{\frac{2r}{1-k}}(\mid B^* \mid)$$

and

$$\gamma = kf^{\frac{2r}{k}}(\mid B \mid) + (1-k) g^{\frac{2r}{1-k}}(\mid C^* \mid),$$

as required. ∎

Now, Theorem 2.10 includes several numerical radius inequalities for operator matrices, and so we give some inequalities in the following remarks.

Remark 2.11 If $f(t) = t^k$ and $g(t) = t^{1-k}$, $k \in (0,1)$, in Theorem 2.10, then we get the following inequality

$$\omega^{2r}(S) \leq \max\{\parallel k \mid C \mid^{2r} + (1-k) \mid B^* \mid^{2r} \parallel, \parallel k \mid B \mid^{2r} + (1-k) \mid C^* \mid^{2r} \parallel\}.$$

Remark 2.12 If $B = C$ in the above Remark, and by using Lemma (2.6)b, then we have

$$\omega^{2r}(B) = \omega^{2r}(S) \leq \parallel k \mid B \mid^{2r} + (1-k) \mid B^* \mid^{2r} \parallel,$$

and this inequality can be found in Theorem 2 in [5].

Remark 2.13 If we take $r = 1$ and $k = \frac{1}{2}$ in the last Remark, we find

$$\omega^2(B) \leq \frac{1}{2} \parallel\mid B \mid^2 + \mid B^* \mid^2 \parallel,$$

which is the second inequality in (1.3).

Our next results are generalizations of the second inequality in (1.4).

Theorem 2.14 *Let* $R = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ *be a* 2×2 *operator matrix in* $B(H_1 \oplus H_2)$, *with the Cartesian decomposition* $R = S + iT$ *and* $1 \le r \le 2$. *Then*

$$\omega^r(R) \le \frac{1}{2^r}\max\{\|\mid C+B^* \mid^r + \mid C-B^* \mid^r\|, \|\mid B+C^* \mid^r + \mid B-C^* \mid^r\|\}.$$

Proof. For every unit vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (H_1 \oplus H_2)$, and for $1 \le r \le 2$, we have

$$\begin{aligned}
\mid \langle RX, X\rangle \mid &= \mid \langle (S+iT)X, X\rangle \mid \\
&= \mid \langle SX, X\rangle + i\langle TX, X\rangle \mid \\
&= \sqrt{\langle SX, X\rangle^2 + \langle TX, X\rangle^2} \\
&\le \sqrt[r]{\mid \langle SX, X\rangle \mid^r + \mid \langle TX, X\rangle \mid^r} \quad \text{(by Lemma 2.2)} \\
&\le \sqrt[r]{\langle \mid S \mid X, X\rangle^r + \langle \mid T \mid X, X\rangle^r} \quad \text{(by Lemma 2.4)} \\
&\le \sqrt[r]{\langle \mid S \mid^r X, X\rangle + \langle \mid T \mid^r X, X\rangle} \quad \text{(by Lemma (2.3)a)} \\
&= \sqrt[r]{\langle (\mid S \mid^r + \mid T \mid^r)X, X\rangle}.
\end{aligned}$$

Thus,

$$\mid \langle RX, X\rangle \mid^r \le \langle (\mid S \mid^r + \mid T \mid^r)X, X\rangle$$

and so,

$$\begin{aligned}
\omega^r(R) &= \sup\{\mid \langle RX, X\rangle \mid^r : X \in (H_1 \oplus H_2), \| X \| = 1\} \\
&\le \sup\{\langle (\mid S \mid^r + \mid T \mid^r)X, X\rangle : X \in (H_1 \oplus H_2), \| X \| = 1\} \\
&= \frac{1}{2^r}\max\{\|\mid C+B^* \mid^r + \mid C-B^* \mid^r\|, \|\mid B+C^* \mid^r + \mid B-C^* \mid^r\|\},
\end{aligned}$$

as required. ∎

Remark 2.15 Let $B = C$ and $r = 2$ in Theorem (2.14). Then we get

$$\begin{aligned}
\omega^2\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) &= \omega^2(B) \qquad \text{(by Lemma (2.6)b)} \\
&\le \frac{1}{4}\max\{\|\mid B+B^* \mid^2 + \mid B-B^* \mid^2\|, \|\mid B+B^* \mid^2 + \mid B-B^* \mid^2\|\} \\
&= \frac{1}{4}\|\mid B+B^* \mid^2 + \mid B-B^* \mid^2\| \\
&= \frac{1}{2}\| B^*B + BB^* \|,
\end{aligned}$$

which is the second inequality in (1.3).

Theorem 2.16 *Let* $R = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ *be a* 2×2 *operator matrix in* $B(H_1 \oplus H_2)$, *with the Cartesian decomposition* $R = S + iT$ *and* $r \geq 2$. *Then*

$$\omega^r(R) \leq 2^{\frac{-r}{2}-1} \max\{\| \mid C + B^* \mid^r + \mid C - B^* \mid^r \|, \| \mid B + C^* \mid^r + \mid B - C^* \mid^r \|\}.$$

Proof. For every unit vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (H_1 \oplus H_2)$, we have

$$\begin{aligned}
\frac{1}{\sqrt{2}} \mid \langle RX, X\rangle \mid &= \frac{1}{\sqrt{2}} \mid \langle (S + iT)X, X\rangle \mid \\
&= \frac{1}{\sqrt{2}} \mid \langle SX, X\rangle + i \langle TX, X\rangle \mid \\
&= \sqrt{\frac{\langle SX, X\rangle^2 + \langle TX, X\rangle^2}{2}} \\
&\leq \sqrt[r]{\frac{\mid \langle SX, X\rangle \mid^r + \mid \langle TX, X\rangle \mid^r}{2}} \qquad \text{(by Lemma 2.1)} \\
&\leq 2^{\frac{-1}{r}} (\langle \mid S \mid X, X\rangle^r + \langle \mid T \mid X, X\rangle^r)^{\frac{1}{r}} \qquad \text{(by Lemma 2.4)} \\
&\leq 2^{\frac{-1}{r}} (\langle \mid S \mid^r X, X\rangle + \langle \mid T \mid^r X, X\rangle)^{\frac{1}{r}} \qquad \text{(by Lemma (2.3)a)} \\
&= 2^{\frac{-1}{r}} \langle (\mid S \mid^r + \mid T \mid^r) X, X\rangle^{\frac{1}{r}} \\
&= 2^{\frac{-1}{r}-1} \left\langle \begin{bmatrix} \eta & 0 \\ 0 & \theta \end{bmatrix} X, X \right\rangle^{\frac{1}{r}}.
\end{aligned}$$

Thus,

$$\mid \langle RX, X\rangle \mid^r \leq 2^{\frac{-r}{2}-1} \left\langle \begin{bmatrix} \eta & 0 \\ 0 & \theta \end{bmatrix} X, X \right\rangle.$$

and so,

$$\begin{aligned}
\omega^r(R) &= \sup\{\mid \langle RX, X\rangle \mid^r : X \in (H_1 \oplus H_2), \| X \| = 1\} \\
&\leq 2^{\frac{-r}{2}-1} \sup\left\{\left\langle \begin{bmatrix} \eta & 0 \\ 0 & \theta \end{bmatrix} X, X \right\rangle : X \in (H_1 \oplus H_2), \| X \| = 1\right\} \\
&= 2^{\frac{-r}{2}-1} \max\{\| \eta \|, \| \theta \|\},
\end{aligned}$$

where

$$\eta = \mid C + B^* \mid^r + \mid C - B^* \mid^r$$

and

$$\theta = \mid B + C^* \mid^r + \mid B - C^* \mid^r,$$

as required. ∎

Remark 2.17 Let $B = C$ and $r = 2$ in Theorem (2.16). Then we get

$$\begin{aligned}
\omega^2\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) &= \omega^2(B) \qquad \text{(by Lemma (2.6)b)} \\
&\leq \frac{1}{4}\max\{\| | B{+}B^*|^2{+}|B{-}B^*|^2 \|, \| | B + B^*|^2{+}|B{-}B^*|^2 \|\} \\
&= \frac{1}{4} \| | B{+}B^* |^2 + | B{-}B^* |^2\| \\
&= \frac{1}{2} \| B^*B{+}BB^* \|,
\end{aligned}$$

which is the second inequality in (1.3).

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