MULTIPLICATION COMPONENTS OF GRADED MODULES

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Abstract. Let G be a group and $g \in G$. Let R be a commutative G-graded ring and M be a graded R-module. In this paper, we study some cases when R is strongly graded ring and the component M_e of M is multiplication R_e-module. Also, we prove that if R is strongly graded, then the components M_g of M are multiplication R_e-modules if and only if the component M_e is P-torsion or P-cyclic where P is a prime ideal of the component R_e of R.
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1. Introduction

A ring R with unity 1 graded by a group G will means that $R = \bigoplus_{g\in G} R_g$ where R_g is an additive subgroup of R and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. If the inclusion is an equality, then the ring is called strongly graded. Clearly, R_e is a subring of R with $1 \in R_e$. An R-module is said to be graded if $M = \bigoplus_{g\in G} M_g$ for a family of subgroups $\{M_g\}_{g\in G}$ of M such that $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. Clearly, M_g is an R_e-module for all $g \in G$. In a similar way, we define a strongly graded module. The ring R is strongly graded if and only if every graded R-module is strongly graded. For more details, we refer the readers to [4], as well as [6], and the references therein.

If M is an R-module and N is an R-submodule of M, then the ideal $\{r \in R : rM \subseteq N\}$ of R will be denoted by $(N : M)$. An R-module M is said to be multiplication module if for every R-submodule N of M, there exists an ideal of R such that $N = IM$. Moreover, if $N = IM$ for some ideal of R, then $N = (N : M)M$. Given a ring R and a multiplicative subset S of R, the ring of fractions $S^{-1}R$ is $\{\frac{r}{s} : r \in R, s \in S\}$. For more details, we refer the readers to [1], [2], [3], as well as [5], and the references therein.

Throughout this paper, unless stated otherwise, R is commutative nontrivially graded ring.

2. Results

In this section, we introduce and prove the main results of the paper.

Theorem 2.1 *Let R be a strongly G-graded ring and M be a torsion free graded R-module. If M_e is a multiplication R_e-module and A is a proper ideal of R_e such that $AM_e = M_e$, then $M = \{0\}$.*

Proof. Let $g \in G$, $x \in M_g$. Then $R_{g^{-1}}x$ is an R_e-submodule of M_e. Since M_e is multiplication R_e-module, $R_{g^{-1}}x = BM_e$ for some ideal B of R_e and then $R_{g^{-1}}x = BM_e = BAM_e = ABM_e = AR_{g^{-1}}x = R_{g^{-1}}Ax$ and so $R_ex = Ax$. Now, $x = 1.x \in R_ex = Ax$ and then there exists $a \in A$ such that $(1-a)x = 0$. Since M is torsion free, either $1 = a$ or $x = 0$. If $1 = a \in A$, then $A = R_e$ a contradiction. So, $x = 0$, i.e., $M_g = \{0\}$ for all $g \in G$ and hence $M = \{0\}$. ∎

Theorem 2.2 *Let R be a strongly G-graded ring and M be a graded R-module. If M_e is a multiplication R_e-module and S is a multiplicative subset of R_e, then $S^{-1}M_g$ is multiplication as an $S^{-1}R_e$-module for all $g \in G$.*

Proof. Let $g \in G$ and X be an $S^{-1}R_e$-submodule of $S^{-1}M_g$. Then $X = S^{-1}N$ for some R_e-submodule N of M_g and then $R_{g^{-1}}N$ is an R_e-submodule of M_e and it follows that $R_{g^{-1}}N = AM_e$ for some ideal A of R_e. So, $X = R_eX = R_gR_{g^{-1}}S^{-1}N = R_gS^{-1}R_{g^{-1}}N = R_gS^{-1}AM_e = S^{-1}AR_gM_e = S^{-1}AM_g = S^{-1}AS^{-1}M_g$ where $S^{-1}A$ is an ideal of $S^{-1}R_e$ and this completes the proof. ∎

Given a prime ideal P of R_e, we consider the set

$$T_P(M_e) = \{m \in M_e : cm = 0 \text{ for some } c \in R_e - P\}.$$

It is easy to check that $T_P(M_e)$ is an R_e-submodule of M_e. If $T_P(M_e) = M_e$, then we will say that M_e is P-torsion. If there exists $x \in M_e$ and $c \in R_e - P$ such that $cM_e \subseteq R_ex$, we will say that M_e is P-cyclic. Now, we introduce the main result of our paper:

Theorem 2.3 *Let R be a strongly G-graded ring, and M be a graded R-module. Then for $g \in G$, M_g is multiplication R_e-module if and only if for every prime ideal P of R_e, either M_e is P-torsion or P-cyclic.*

Proof. Let $g \in G$. Suppose that M_g is multiplication R_e-module and P is a prime ideal of R_e. Firstly, we consider the case in which $PM_e = M_e$. Let $m \in M_e$. Then R_gm is an R_e-submodule of M_g and then there exists an ideal A of R_e such that $R_gm = AM_g$. So, $m = 1.m \in R_em = R_{g^{-1}}R_gm = R_{g^{-1}}AM_g = AR_{g^{-1}}M_g = AM_e = APM_e = APR_{g^{-1}}M_g = PR_{g^{-1}}AM_g = PR_{g^{-1}}R_gm = PR_em = R_ePm = Pm$ and then there exists $p \in P$ such that $(1-p)m = 0$ and it follows that $c = 1-p \in R_e - P$ such that $cm = 0$ and therefore, $M_e = T_P(M_e)$, i.e., M_e is P-torsion. Now, we consider that $PM_e \neq M_e$. Then there exists $x \in M_e - PM_e$ and since R_gx is an R_e-submodule of M_g, there exists an ideal B of R_e such that $R_gx = BM_g$. If $B \subseteq P$, then $x = 1.x \in R_ex = R_{g^{-1}}R_gx = R_{g^{-1}}BM_g = BM_e \subseteq PM_e$ a contradiction. Therefore, $B \nsubseteq P$, then there exists $c \in B - P$ such that $cM_e = R_{g^{-1}}cM_g \subseteq R_{g^{-1}}BM_g = R_{g^{-1}}R_gx = R_ex$, i.e., M_e is P-cyclic. Conversely, let $g \in G$ and N be an R_e-submodule of M_g. Suppose that $A = (R_{g^{-1}}N : M_e)$, $n \in R_{g^{-1}}N$ and $K = (AM_e : R_en)$. Assume that $K \neq R_e$. Then there exists a maximal ideal P of R_e containing K. If M_e is P-torsion, then there exists $c \in R_e - P$ such that $cn = 0$ and it follows that $c \in K - P$ a contradiction. So, M_e is P-cyclic, i.e., there exists $x \in M_e$ and $c \in R_e - P$ such that $cM_e \subseteq R_ex$. Thus, $R_{g^{-1}}cN$ is an R_e-submodule of R_ex, and then cN is R_e-submodule of R_gx

but R_gx is multiplication because it is cyclic, hence there exists $J = (cN : R_gx)$ such that $cN = Jx$. It holds that

$$\begin{aligned} cJM_e &= JcM_e \subseteq JR_ex = JR_{g^{-1}}R_gx = R_{g^{-1}}JR_gx \subseteq R_{g^{-1}}cN \\ &= cR_{g^{-1}}N \subseteq R_eR_{g^{-1}}N = R_{g^{-1}}N \end{aligned}$$

and hence, $cJ \subseteq A$. Now, the element $c^2n \in c^2N = cJx \subseteq Ax \subseteq AM_e$. As a result, $c^2 \in K \subseteq P$ which is a contradiction. It follows that $K = R_e$ and then $R_e = (AM_e : R_{g^{-1}}N)$ and therefore, $R_{g^{-1}}N \subseteq (R_{g^{-1}}N : M_e)M_e$ and then $N \subseteq (N : M_g)M_g$. Since the other inclusion is always true, the proof ends. ∎

Corollary 2.4 *Let R be a strongly G-graded ring, and M be a graded R-module. Then for $g \in G$, M_g is multiplication R_e-module if and only if for every prime ideal P of R_e, either M_e is P-torsion or there exists an R_e-submodule N of M_e and $c \in R_e - P$ such that $cM_e \subseteq N$.*

Proof. To prove the sufficiency, let P be a prime ideal of R_e and suppose that M_e is not P-torsion. Then by hypothesis, there exists $c \in R_e - P$ such that $cM_e \subseteq N$ where N is an R_e-submodule of M_e. Since M_e is not P-torsion, N is not P-torsion. By Theorem 2.3, there exists $x \in N$ and $r \in R_e - P$ such that $rN \subseteq R_ex$. Thus, $crM_e = rcM_e \subseteq rN \subseteq R_ex$ and so M_e is P-cyclic and therefore, by Theorem 2.3, M_g is multiplication R_e-module for any $g \in G$. The necessity is obvious by Theorem 2.3. ∎

Corollary 2.5 *Let R be a strongly G-graded ring, and M be a graded R-module. If M_e is multiplication R_e-module and $Ann(M_e) = \{0\}$, then for $g \in G$,*

1. *$\bigcap_{k\in K}(I_kM_g) = (\bigcap_{k\in K} I_k)M_g$ for every family $I_k(k \in K)$ of ideals of R_e.*
2. *if N is an R_e-submodule of M_g and A is an ideal of R_e such that $N \subset AM_g$, then there exists an ideal B of R_e such that $B \subset A$ and $N \subseteq BM_g$.*

Proof. 1. Let $I_k(k \in K)$ be a family of ideals of R_e. We call $I = \bigcap_{k\in K} I_k$. Then it is always true that $IM_e \subseteq \bigcap_{k\in K}(I_kM_e)$. Let $x \in \bigcap_{k\in K}(I_kM_e)$ and $H = (IM_e : R_ex)$. Suppose that $H \neq R_e$. Then there exists a prime ideal P of R_e containing H. If $x \in T_P(M_e)$, then we find an element in $H - P$, so $x \notin T_P(M_e)$. By Theorem 2.3, M_e is P-cyclic and then there exists $m \in M_e$ and $c \in R_e - P$ such that $cM_e \subseteq R_em$ and so $cx \in \bigcap_{k\in K} I_km$. It follows that, for every $k \in K$, there exists $a_k \in I_k$ such that $cx = a_km$. Now, choose $k_0 \in K$ such that $cx \in I_{k_0}m$ and then $cx = a_{k_0}m$. Hence $a_{k_0}m = a_km$, i.e., $(a_{k_0} - a_k)m = 0$ for every $k \in K$. We have, $c(a_{k_0} - a_k)M_e = (a_{k_0} - a_k)cM_e \subseteq (a_{k_0} - a_k)R_em = R_e(a_{k_0} - a_k)m = \{0\}$. Since $Ann(M_e) = \{0\}$, $c(a_{k_0} - a_k) = 0$. Hence, $ca_{k_0} = ca_k \in I_k$ for every $k \in K$. As a consequence, $ca_{k_0} \in I$ and then $c^2x = ca_{k_0}m \in IM_e$ and so $c^2 \in H \subseteq P$ a contradiction. So, $H = R_e$ and hence $x \in IM_e$. Now, let $g \in G$. Then $\bigcap_{k\in K}(I_kM_g) = \bigcap_{k\in K}(I_kR_gM_e) = R_g\bigcap_{k\in K}(I_kM_e) = R_g(\bigcap_{k\in K} I_k)M_e = (\bigcap_{k\in K} I_k)R_gM_e = (\bigcap_{k\in K} I_k)M_g$.

2. Let $g \in G$, N be an R_e-submodule of M_g and A be an ideal of R_e such that $N \subset AM_g$. Then $R_{g^{-1}}N$ is an R_e-submodule of M_e such that $R_{g^{-1}}N \subset AM_e$.

Since M_e is multiplication, $R_{g^{-1}}N = CM_e$ for some ideal C of R_e and then $N = CM_g$. So, $N = AM_g \bigcap CM_g = (A \bigcap C)M_g$ by using (1). Hence, choose $B = A \bigcap C$. ∎

An R-module M is said to be finitely cogenerated if for every non-empty family of R-submodules $N_k (k \in K)$ of M such that $\bigcap_{k\in K} N_k = \{0\}$, there exists a finite subset F of K such that $\bigcap_{k\in F} N_k = \{0\}$. A ring R is said to be finitely cogenerated if it is finitely cogenerated as an R-module. We close the paper with the following result:

Theorem 2.6 *Let R be a strongly G-graded ring, and M be a graded R-module. If M_e is multiplication R_e-module and $Ann(M_e) = \{0\}$, then M_e is finitely cogenerated R_e-module if and only if M_g is finitely cogenerated R_e-module for all $g \in G$.*

Proof. Suppose that M_e is finitely cogenerated R_e-module. Firstly, we prove that R_e is finitely cogenerated. Let $I_k (k \in K)$ be a non-empty family of ideals of R_e such that $\bigcap_{k\in K} I_k = \{0\}$. Then by Corollary 2.5, $\bigcap_{k\in K}(I_k M_e) = \{0\}$. Since M_e is finitely cogenerated, there exists a finite subset F of K such that $\bigcap_{k\in F} I_k M_e = \{0\}$ and then by Corollary 2.5, $(\bigcap_{k\in F} I_k)M_e = \{0\}$. Since $Ann(M_e) = \{0\}$, $\bigcap_{k\in F} I_k = \{0\}$. Therefore, R_e is finitely cogenerated. Now, let $g \in G$ and $N_k (k \in K)$ be a non-empty family of R_e-submodules of M_g such that $\bigcap_{k\in K} N_k = \{0\}$. Then $R_{g^{-1}}N_k (k \in K)$ are R_e-submodules of M_e and then for $k \in K$, there exists an ideal A_k of R_e such that $R_{g^{-1}}N_k = A_k M_e$, it is clear that $(\bigcap_{k\in K} A_k)M_e = \bigcap_{k\in K}(A_k M_e) = \bigcap_{k\in K} R_{g^{-1}}N_k = R_{g^{-1}} \bigcap_{k\in K} N_k = \{0\}$. Since $Ann(M_e) = \{0\}$, $\bigcap_{k\in K} A_k = \{0\}$ and since R_e is finitely cogenerated, there exists a finite subset F of K such that $\bigcap_{k\in F} A_k = \{0\}$ and then

$$\begin{aligned}\textstyle\bigcap_{k\in F} N_k &= R_e(\textstyle\bigcap_{k\in F} N_k) = R_g R_{g^{-1}}(\textstyle\bigcap_{k\in F} N_k) \\ &= R_g(\textstyle\bigcap_{k\in F} R_{g^{-1}}N_k) = R_g(\textstyle\bigcap_{k\in F} A_k M_e) = R_g(\textstyle\bigcap_{k\in F} A_k)M_e = \{0\}.\end{aligned}$$

Hence, M_g is finitely cogenerated. The converse is obvious. ∎

References

[1] Atani, S.E., Atani, R.E., *Graded multiplication modules and the graded ideal $\theta_g(M)$*, Turkish Journal of Mathematics, 35 (2011), 1-9.

[2] Farzalipour, F., Ghiasvand, P., *On graded weak multiplication modules*, Tamkang J. Math., 43 (2) (2012), 171-177.

[3] Khaksari, A., Jahromi, F.R., *Multiplication graded modules*, International J. Algebra, 7 (1) (2013), 17-24.

[4] Nastasescu, C., Van Oystaeyen, F., *Graded ring theory*, Mathematical Library 28, North Holland, Amesterdam, 1982.

[5] Park, S., *Some Properties of Gr-multiplication modules*, Korean J. Math., 20 (3) (2012), 315-321.

[6] Refai, M., *Various types of strongly graded rings,* Abhath Al-Yarmouk Journal (Pure Sciences and Engineering Series), 4 (2) (1995), 9-19.

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