MULTIPLIERS ON SPACES OF VECTOR VALUED ENTIRE DIRICHLET SERIES OF TWO COMPLEX VARIABLES

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Abstract. In this paper, we study a class of sequence spaces defined by using the type of an entire function represented by vector valued Dirichlet series of two complex variables. The main results concern with obtaining the nature of the dual spaces of this sequence space and coefficient multipliers for some classes of vector valued Dirichlet series.

Keywords: vector valued Dirichlet series, analytic function, entire function, type, dual space, norm.

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1. Introduction

Let

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2),\ (s_j = \sigma_j + it_j\,,\, j = 1, 2) \tag{1.1}$$

be a Dirichlet series of two complex variables s_1, s_2; $a_{m,n}{}'s$ belong to a commutative complex Banach algebra $(E, ||.||)$ with the unit element ω and

$$0 < \lambda_1 < ... < \lambda_m \to \infty \ \text{ as } m \to \infty\,; 0 < \mu_1 < ... < \mu_n \to \infty \text{ as } n \to \infty. \tag{1.2}$$

Further, let

$$\limsup_{m,n\to\infty} \frac{\ln(m+n)}{\lambda_m + \mu_n} = D < +\infty\,, \tag{1.3}$$

and

$$\limsup_{m,n\to\infty} \frac{\ln(||a_{m,n}||)}{\lambda_m + \mu_n} = -\infty\ . \tag{1.4}$$

Then $f(s_1, s_2)$ represented by the vector valued Dirichlet series (VVDS) in (1.1) is an entire function (see [2]). We define the maximum modulus of $f(s_1, s_2)$ as

$$M(\sigma_1, \sigma_2) = \text{lub}\, ||f(\sigma_1 + it_1, \sigma_2 + it_2)||; -\infty < t_j < \infty \ (j = 1, 2).$$

The entire function $f(s_1, s_2)$ is said to be of order ρ where ρ is defined as

$$\rho = \limsup_{\sigma_1,\sigma_2 \to \infty} \frac{\ln \ln M(\sigma_1, \sigma_2)}{\ln(e^{\sigma_1} + e^{\sigma_2})}, \quad (0 \leqslant \rho \leqslant \infty)\,. \tag{1.5}$$

When $0 < \rho < \infty$,in order to further classify the growth, we define the type T of $f(s_1, s_2)$ as

$$T = \limsup_{\sigma_1,\sigma_2 \to \infty} \frac{\ln M(\sigma_1, \sigma_2)}{e^{\rho\sigma_1} + e^{\rho\sigma_2}}\ , \qquad (0 \leqslant T \leqslant \infty)\ . \tag{1.6}$$

The coefficient characterizations of order and type of generalized vector valued Dirichlet series were obtained by Srivastava and Sharma [2]. Thus, if $f(s_1, s_2)$ is an entire function of order ρ, then

$$\rho = \limsup_{m,n \to \infty} \frac{\ln \left(\lambda_m^{\lambda_m} \mu_n^{\mu_n}\right)}{\ln ||a_{m,n}||^{-1}} \tag{1.7}$$

and, if $f(s_1, s_2)$ is entire function of order ρ $(0 < \rho < \infty)$, then it is of type T if and only if

$$e\rho T = \limsup_{m,n \to \infty} \left[{\lambda_m}^{\lambda_m} {\mu_n}^{\mu_n} ||a_{m,n}||^{\rho}\right]^{1/(\lambda_m + \mu_n)}. \tag{1.8}$$

Let E_T denote the space of all entire functions $f(s_1, s_2)$ defined by VVDS (1.1) and satisfying

$$\limsup_{m,n \to \infty} \left[{\lambda_m}^{\lambda_m} {\mu_n}^{\mu_n} ||a_{m,n}||^{\rho}\right]^{1/(\lambda_m + \mu_n)} \leqslant e\rho T \tag{1.9}$$

Further, the sequences $\{\lambda_m\}$ and $\{\mu_n\}$ satisfy the stronger condition

$$\limsup_{m,n \to \infty} \frac{\ln(m + n)}{\lambda_m + \mu_n} = 0 \tag{1.10}$$

From equation (1.9), for a given $\varepsilon > 0$ there exists a positive integer n_0 such that for $m, n > n_0$,

$$\left[{\lambda_m}^{\lambda_m} {\mu_n}^{\mu_n} ||a_{m,n}||^{\rho}\right]^{1/(\lambda_m + \mu_n)} < e\rho(T + \varepsilon)\,.$$

In this paper, we have obtained various properties of the space E_T. In analogy with Khoi [1], we give some definitions regarding dual spaces in reference to double sequences.

A sequence $(u_{m,n})$ is said to be multiplier from a sequence space A into a sequence space B if $(u_{m,n} a_{m,n}) \in B$ whenever $(a_{m,n}) \in A$. The space of multipliers from a sequence space A into a sequence space B is denoted by (A, B). If D is a fixed sequence space then the D-dual of a sequence space A is defined to be (A, D), the multipliers from A to D and denoted by A^D. Some duals are defined with some conditions such as Kothe dual, Abel dual. The Kothe dual is obtained when $D = l^1$,and will be denoted by A^α (it is also denoted by A^K).

In what follows, we shall always consider E to be a complex Banach algebra and the sequences $\{\lambda_m\}$ and$\{\mu_n\}$ satisfy the condition (1.10). We denote by E_T the sequence space

$$E_T = \{(a_{m,n}) \in E;\ (a_{m,n}) \text{ satisfies } (1.9)\}.$$

The Kothe dual of the space E_T is defined as

$$E_T^\alpha = \left\{(u_{m,n});\ \sum_{m,n=1}^{\infty} ||u_{m,n}a_{m,n}|| \text{ converges } \forall\,(a_{m,n}) \in E_T\right\}.$$

Now, we introduce another sequence space E_T^β defined as

$$E_T^\beta = \left\{(u_{m,n});\ \sum_{m,n=1}^{\infty} u_{m,n}a_{m,n} \text{ converges } \forall\,(a_{m,n}) \in E_T\right\}.$$

2. Main Results

We first study properties of some dual spaces of the space E_T. Later, we characterize the multipliers on E_T. It can be easily verified that, for the spaces defined above, $E_T^\alpha \subseteq E_T^\beta$. Now, we find the criteria for the reverse inclusion relation to be true.

We prove

Theorem 1. *For every T, $0 < T < \infty$, we have $E_T^\alpha = E_T^\beta$. Moreover, $(u_{m,n}) \in E_T^\beta$, if and only if*

$$\liminf_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} > e\rho T. \tag{2.1}$$

Proof. Let us assume that $(u_{m,n}) \in E_T^\beta$, but (2.1) is not satisfied, i.e.,

$$\liminf_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho T.$$

For a given $\varepsilon > 0$, there exist increasing sequences (m_k) and (n_l) of positive integers such that

$$\left[{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho(T+\varepsilon),\ \forall\, m = m_k, n = n_l,\ k, l = 1, 2, ...$$

Let $(a_{m,n})$ be a sequence defined as

$$a_{m,n} = \begin{cases} \omega/||u_{m,n}||, & \text{if } m = m_k \text{ and } n = n_l;\ k,l = 1,2,..., \\ 0, & \text{for other values of } m \text{ and } n. \end{cases}$$

Then, we have

$$\begin{aligned} \limsup_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m} {\mu_n}^{\mu_n} ||a_{m,n}||^{\rho}\right]^{1/(\lambda_m+\mu_n)} &= \lim_{k,l\to\infty} \left[{\lambda_{m_k}}^{\lambda_{m_k}} {\mu_{n_l}}^{\mu_{n_l}} ||a_{m_k,n_l}||^{\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})} \\ &= \lim_{k,l\to\infty} \left[{\lambda_{m_k}}^{\lambda_{m_k}} {\mu_{n_l}}^{\mu_{n_l}} ||u_{m_k,n_l}||^{-\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})} \\ &\leqslant e\rho T. \end{aligned}$$

It follows that $(a_{m,n}) \in E_T$. But $||a_{m_k,n_l} u_{m_k,n_l}|| = 1$, $(k,l = 1,2,...)$, that is, $\lim_{m,n\to\infty} ||a_{m,n}u_{m,n}|| \neq 0$. So, the series $\sum_{m,n=1}^{\infty} u_{m,n}a_{m,n}$ does not converge, therefore our assumption is not valid. Hence, if $(u_{m,n}) \in E_T^{\beta}$, then (2.1) will always be satisfied.

Conversely, suppose that (2.1) holds, i.e.,

$$\liminf_{m,n\to\infty} \frac{\left[{\lambda_m}^{\lambda_m} {\mu_n}^{\mu_n} ||u_{m,n}||^{-\rho}\right]^{1/\lambda_m+\mu_n}}{\rho e} = M > T.$$

Then, for a given $\delta > 0$, there exist positive integers M_1 and N_1 and such that, $\forall\, m \geq M_1$ and $\forall\, n \geq N_1$, we have

$$\left[{\lambda_m}^{\lambda_m} {\mu_n}^{\mu_n} ||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} > e\rho(M-\varepsilon),$$

or

$$||u_{m,n}||^{\rho} < \frac{{\lambda_m}^{\lambda_m} {\mu_n}^{\mu_n}}{[\rho e(M-\varepsilon)]^{(\lambda_m+\mu_n)}}.$$

Also, for every sequence $(a_{m,n}) \in E_T$, there exist M_2 and N_2 such that, $\forall\, m \geq M_2$, $n \geq N_2$,

$$||a_{m,n}||^{\rho} < \frac{[\rho e(T+\varepsilon)]^{(\lambda_m+\mu_n)}}{{\lambda_m}^{\lambda_m} {\mu_n}^{\mu_n}}.$$

Therefore, for all $m \geq \max\{M_1, M_2\}$ and $n \geq \max\{N_1, N_2\}$,

$$||a_{m,n}u_{m,n}|| < \left(\frac{T+\varepsilon}{M-\varepsilon}\right)^{(\lambda_m+\mu_n)/\rho}.$$

Since $M > T$, we can choose $\varepsilon > 0$ such that $M - \varepsilon > T + \varepsilon$. Then, from the above inequality, we can see that the series $\sum_{m,n=1}^{\infty} ||u_{m,n}a_{m,n}||$ converges. Hence $(u_{m,n}) \in E_T^{\alpha}$ and, therefore, $E_T^{\beta} \subseteq E_T^{\alpha}$. This completes the proof of Theorem 1. ∎

Next, we prove

Theorem 2. *The space E_T is perfect, i.e., $E_T^{\alpha\alpha} = E_T$.*

Proof. Let the sequence $(a_{m,n}) \notin E_T$. Then we have

$$\limsup_{m,n\to\infty} \left[\lambda_m{}^{\lambda_m} \mu_n{}^{\mu_n} ||a_{m,n}||^{\rho}\right]^{1/(\lambda_m+\mu_n)} \geqslant e\rho T.$$

We denote by $e\rho T^*$ the left hand side limit if it is finite, and a number $> e\rho T$ if the limit is infinite. Then, for arbitrary small $\delta > 0$, there exist infinitely increasing sequences (m_k) and (n_l) of positive integers such that

$$||a_{m,n}||^{\rho} \geqslant \frac{[\rho e(T^* - \delta)]^{(\lambda_m+\mu_n)}}{\lambda_m^{\lambda_m} \mu_n^{\mu_n}}, \; m = m_k, \; n = n_l.$$

Let us define a sequence

$$u_{m,n} = \begin{cases} \omega /||a_{m_k,n_l}|| & \text{if } m = m_k, \; n = n_l, \text{ where } k, l = 1, 2, ... \\ 0 & \text{for other values of } m \text{ and } n. \end{cases}$$

Then, we have

$$\begin{aligned} \liminf_{m,n\to\infty} \left[\lambda_m^{\lambda_m} \mu_n^{\mu_n} ||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} &= \lim_{k,l\to\infty} \left[\lambda_{m_k}^{\lambda_{m_k}} \mu_{n_l}^{\mu_{n_l}} ||u_{m_k,n_l}||^{-\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})} \\ &= \lim_{k,l\to\infty} \left[\lambda_{m_k}^{\lambda_{m_k}} \mu_{n_l}^{\mu_{n_l}} ||a_{m_k n_l}||^{\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})} \\ &\geqslant e\rho T^* > e\rho T. \end{aligned}$$

Hence, from Theorem 1, $(u_{m,n}) \in E_T^{\alpha}$. But $||a_{m,n}u_{m,n}|| = 1$, $\forall\, m = m_k$, $n = n_l$, i.e., $\sum a_{m,n}u_{m,n}$ does not converge. Therefore, $(a_{m,n}) \notin E_T^{\alpha\alpha}$. Hence $E_T^{\alpha\alpha} \subseteq E_T$. The reverse inclusion relation $E_T^{\alpha\alpha} \supseteq E_T$ always holds. Hence the space E_T is perfect.

Theorem 3. *For the sequence space E_T defined as above, we have*

$$(E_T, l^p) = E_T^{\alpha}, \; \forall\, 0 < p \leq \infty.$$

Proof. Suppose that a sequence$(u_{m,n}) \notin E_T^{\alpha}$. Then, from Theorem 1, we have

$$\liminf_{m,n\to\infty} \left[\lambda_m^{\lambda_m} \mu_n^{\mu_n} ||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho T.$$

Then, for an arbitrarily small $\varepsilon > 0$, there exist monotonically increasing sequences (m_k) and (n_l) of positive integers such that

$$\left[\lambda_m^{\lambda_m} \mu_n^{\mu_n} ||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} < e\rho(T + \varepsilon), \; m = m_k, \; n = n_l.$$

Let $0 < p < \infty$. We consider the sequence

$$a_{m,n} = \begin{cases} \omega /||u_{m_k,n_l}|| & \text{if } m = m_k, \; n = n_l \text{ and } k, l = 1, 2, ... \\ 0 & \text{for other values of } m \text{ and } n. \end{cases}$$

Then we have

$$\begin{aligned}\limsup_{m,n\to\infty}\left[\lambda_m^{\lambda_m}\,\mu_n^{\mu_n}||a_{m,n}||^{\rho}\right]^{1/(\lambda_m+\mu_n)} &= \lim_{k,l\to\infty}\left[\lambda_{m_k}^{\lambda_{m_k}}\,\mu_{n_l}^{\mu_{n_l}}||a_{m_k,n_l}||^{\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})}\\ &= \lim_{k,l\to\infty}\left[\lambda_{m_k}^{\lambda_{m_k}}\,\mu_{n_l}^{\mu_{n_l}}||u_{m_k,n_l}||^{-\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})}\\ &\leqslant e\rho T.\end{aligned}$$

Hence we get $(a_{m,n})\in E_T$. By the definition of (E_T,l^p) , $\sum\limits_{m,n=1}^{\infty}||a_{m,n}u_{m,n}||^p$ should be convergent . But $||a_{m_k,n_l}u_{m_k,n_l}||=1; k,l=1,2....$ This implies $(a_{m,n}u_{m,n})\notin l^p$.

For the case $p=\infty$, we consider a sequence

$$a_{m,n}=\begin{cases}\omega(m+n)^{1/\rho}||u_{m,n}||^{-1} & \text{if } m=m_k,\ n=n_l \text{ and } k,l=1,2,....,\\ 0 & \text{for other values of } m \text{ and } n.\end{cases}$$

Then we have

$$\begin{aligned}\limsup_{m,n\to\infty}\left[\lambda_m^{\lambda_m}\,\mu_n^{\mu_n}||a_{m,n}||^{\rho}\right]^{1/(\lambda_m+\mu_n)} &= \lim_{k,l\to\infty}\left[\lambda_{m_k}^{\lambda_{m_k}}\,\mu_{n_l}^{\mu_{n_l}}||a_{m_k,n_l}||^{\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})}\\ &= \lim_{k,l\to\infty}\left[\lambda_{m_k}^{\lambda_{m_k}}\,\mu_{n_l}^{\mu_{n_l}}||m_k n_l||^{\rho}||u_{m_k,n_l}||^{-\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})}\\ &\leqslant e\rho T\end{aligned}$$

using (1.10) and the inequality above. This shows that $(a_{m,n})\in E_\rho$. Since $\lim\limits_{k,l\to\infty}||a_{m_k,n_l}u_{m_k,n_l}||=+\infty$, this implies that $(a_{m,n},u_{m,n})\notin l^\infty$. Hence we conclude that, for $0<p\leq\infty$, $(u_{m,n})\notin E_T^\alpha\Rightarrow(u_{m,n})\notin(E_T,l^p)$. Thus $(E_T,l^p)\subseteq E_T^\alpha$, $0<p\leq\infty$.

Conversely, assume that $(u_{m,n})\in E_T^\alpha$. Then for a given $M>T$, there exist integers M_1 andN_1 such that $\forall\, m\geq M_1, n\geq N_1$,

$$||u_{m,n}||\leq\frac{{\lambda_m}^{\lambda_m/\rho}{\mu_n}^{\mu_n/\rho}}{[\rho eM]^{(\lambda_m+\mu_n)/\rho}}.$$

Suppose that $(a_{m,n})\in E_T$, then for $\delta\in(0,(M-T))$ there exist positive integers M_2 andN_2 such that$\forall\, m\geq M_2,\ n\geq N_2$,

$$||a_{m,n}||\leq\frac{[\rho e(T+\delta)]^{(\lambda_m+\mu_n)/\rho}}{{\lambda_m}^{\lambda_m/\rho}{\mu_n}^{\mu_n/\rho}}.$$

Consequently, for all $m\geq m_0=\max\{M_1,M_2\}, n\geq n_0=\max\{N_1,N_2\}$ we have

$$||a_{m,n}u_{m,n}||\leq||a_{m,n}||\,||u_{m,n}||<((T+\delta)/(M))^{(\lambda_m+\mu_n)/\rho}.$$

If $0<p<\infty$, then we have

$$\sum_{m=M,n=N}^{\infty}||a_{m,n}u_{m,n}||^p\leq\sum_{m=M,n=N}^{\infty}((T+\delta)/M)^{p(\lambda_m+\mu_n)/\rho}<\infty\,;$$

as $(T+\delta)/M < 1$, which implies that$(a_{m,n}u_{m,n}) \in l^p$.

Now, let us take $p = \infty$. Then we have

$$||a_{m,n}u_{m,n}|| \leq ((T+\delta)/M)^{(\lambda_m+\mu_n)/\rho} < 1, \forall m \geq m_0,\ n \geq n_0,$$

which shows that $(a_{m,n}u_{m,n}) \in l^\infty$. Thus, in both the cases $(u_{m,n}) \in (E_T, l^p)$ and consequently $E_T^\alpha \subset (E_T, l^p)$, $0 < p \leq \infty$. This completes the proof of Theorem 3. ∎

In the next theorem, we obtain the sequence space of multipliers from l^p to E_T. We prove

Theorem 4. *For the sequence space E_T defined as above, we have*

$$(l^p, E_T) = E_T\ ,\ 0 < p \leq \infty.$$

Proof. First, we prove that $(l^p, E_T) \subseteq E_T$. Hence, for $0 < p < \infty$, let $(a_{m,n}) \in l^p$. Then $\sum_{m,n=1}^{\infty} |a_{m,n}|^p < \infty$ and, therefore,

$$\lim_{m,n\to\infty} |a_{m,n}|^p = 0 \tag{2.2}$$

Let $(u_{m,n}) \in (l^p, E_T)$. Then, $(a_{m,n}\, u_{m,n}) \in E_T$ and, using (1.9), we have

$$\limsup_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m} {\mu_n}^{\mu_n} ||a_{m,n}\, u_{m,n}||^\rho\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho T. \tag{2.3}$$

Hence from (2.2) and (2.3), we get

$$\limsup_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m} {\mu_n}^{\mu_n} ||\, u_{m,n}||^\rho\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho T$$

and hence $(u_{m,n}) \in E_T$. If $p = \infty$, then $(a_{m,n})$ is a bounded sequence and from (2.3) we have the above inequality and $(u_{m,n}) \in E_T$. Hence we get $(l^p, E_T) \subseteq E_T$, $0 < p \leqslant \infty$.

To prove the converse, assume that $(u_{m,n}) \in E_T$. Then we have

$$\limsup_{m,n\to\infty} \left[\lambda_m^{\lambda_m}\, \mu_n^{\mu_n} ||u_{m,n}||^\rho\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho T.$$

Let $(d_{m,n})$ be an arbitrary sequence such that $(d_{m,n}) \in l^p, 0 < p \leq \infty$. In both cases, i,e., $0 < p < \infty$ or $p = \infty$, there exists a constant P such that $|d_{m,n}| \leq P$, $\forall\, m, n \geq 1$. Hence we have

$$\begin{aligned}\limsup_{m,n\to\infty} \left[\lambda_m^{\lambda_m}\, \mu_n^{\mu_n} ||d_{m,n}u_{m,n}||^\rho\right]^{1/(\lambda_m+\mu_n)} &= \limsup_{m,n\to\infty} \left[\lambda_m^{\lambda_m}\, \mu_n^{\mu_n} |d_{m,n}|^\rho ||u_{m,n}||^\rho\right]^{1/(\lambda_m+\mu_n)} \\ &\leqslant \limsup_{m,n\to\infty} \left[\lambda_m^{\lambda_m}\, \mu_n^{\mu_n} P^\rho ||u_{m,n}||^\rho\right]^{1/(\lambda_m+\mu_n)} \\ &\leqslant e\rho T,\end{aligned}$$

which shows that $(d_{m,n}u_{m,n}) \in E_T$. Thus, $(u_{m,n}) \in (l^p, E_T)$ and, consequently, $E_T \subseteq (l^p, E_T)$, $\forall\, 0 < p \leq \infty$. Hence the result follows. ∎

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References

[1] Le Hai Khoi, it Holomorphic Dirichlet series in the half plane, Vietnam J. Math., 26 (1998), 259-271.

[2] Srivastava, G.S., Sharma, Archana, *Some growth properties of Entire functions represented by vector valued Dirichlet series in two variables*, Gen. Math. Notes, 2 (2011), 134-142.

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