ON THE EXCHANGE PROPERTY FOR THE HARTLEY TRANSFORM

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Abstract. In this paper we investigate the exchange property for the Hartley transform by using the relation between the Fourier transform and the Hartley transform. Simplified construction of tempered Boehmians is also presented.
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1. Introduction

The concept of Boehmians is motivated by the regular operator introduced by Boehme [2], which form a subalgebra of the field of Mikusiński operators and thus they include only such functions whose support is bounded from the left. The theory of Boehmians (quotient of sequences), its properties and different classes of Boehmian spaces can be studied in [1], [8], [9]. Tempered Boehmians is a natural extension of tempered distribution which, therefore, makes it possible to define an extension of the Fourier transform for this class of Boehmians. The Fourier transform of a tempered Boehmian is a distribution. A continuous function $f : \mathbb{R}^N \to \mathbb{C}$ is called slowly increasing if there is a polynomial p on $\mathbb{R}^N$ such that $|f(x)| \leq p(x)$ for all $x \in \mathbb{R}^N$. The space of slowly increasing function will be denoted by $J(\mathbb{R}^N)$ or simply by J. An infinitely differentiable function is called rapidly decreasing if

$$(1) \qquad \sup_{|\alpha|\leq m} \sup_{x\in\mathbb{R}^N} \left(1 + x_1^2 + x_2^2 + \cdots + x_N^2\right)^m |D^\alpha f(x)| < \infty,$$

for every non-negative integer m, where $x = (x_1, x_2, ..., x_N)$, $\alpha = (\alpha_1, ...\alpha_N)$, α_N's are non negative integer, $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$, and

$$(2) \qquad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \ .$$

The space of all rapidly decreasing functions is denoted by $S(\mathbb{R}^N)$ or simply by S. If $f \in J$ and $\varphi \in S$, then the convolution

$$(f * \varphi)(x) = \int_{\mathbb{R}^N} f(u)\varphi(x-u)du \tag{3}$$

is well defined and $f * \varphi \in J$. A sequence $\varphi_n \in S$ is called a delta sequence if it satisfies the following conditions

(i) $\int_{\mathbb{R}^N} \varphi_n dx = 1\ ,\ \ \forall n \in \mathbb{N}$

(ii) $\int_{\mathbb{R}^N} |\varphi_n|\, dx \le C$, for some constant M and for all $n \in \mathbb{N}$

(iii) $\lim_{n\to\infty} \int_{\|x\|\ge\epsilon} \|x\|^k |\varphi_n(x)|\, dx = 0$, for every $k \in \mathbb{N}$ and $\epsilon > 0$.

If $\varphi \in S$ and $\int \varphi = 1$, then the sequence of functions φ_n is a delta sequence. A complex-valued function f on $\mathbb{R}^N$ is called slowly increasing if there exists a polynomial p on $\mathbb{R}^N$ such that $f(x)/p(x)$ is bounded. The space of all increasing continuous functions on $\mathbb{R}^N$ is denoted by $\mathcal{I}$. If $f_n \in \mathcal{I}$, $\{\varphi_n\}$ is a delta sequence under usual notion, then the space of equivalence classes of quotients of sequence will be denoted by $\mathcal{B}_{\mathcal{I}}$, elements of which will be called tempered Boehmians. For $F = [f_n/\varphi_n] \in \mathcal{B}_{\mathcal{I}}$, define $D^\alpha F = [(f_n * D^\alpha\varphi_n)/(\varphi_n * \varphi_n)]$. If F is a Boehmian corresponding to differentiable function, then $D^\alpha F \in \mathcal{B}_{\mathcal{I}}$.

If $F = [f_n/\varphi_n] \in \mathcal{B}_{\mathcal{I}}$ and $f_n \in S$, for all $n \in \mathbb{N}$, then F is called a rapidly decreasing Boehmian. The space of all rapidly decreasing Boehmian is denoted by $\mathcal{B}_S$. If $F = [f_n/\varphi_n] \in \mathcal{B}_{\mathcal{I}}$ and $G = [g_n/\psi_n] \in \mathcal{B}_S$, then the convolution is

$$F * G = [(f_n * g_n)/(\varphi_n * \psi_n)] \in \mathcal{B}_{\mathcal{I}}.$$

In what follows, we will denote by S' the space of tempered distributions, that is, the space of continuous linear functional on S. If $f \in S'$ and $\varphi \in S$, then the convolution $f * \varphi$ is defined as $(f * \varphi)(x) = f(\varphi_x)$, where $\varphi_x(z) = \varphi(x-z)$. The Hartley transform of a tempered distribution f, denoted by $\mathcal{H}f$, is defined by $\mathcal{H}f(\varphi) = f(\mathcal{H}\varphi)$, where $\mathcal{H}\varphi$ is the Hartley transform of φ defined by Nait *et al.* [11]. Further, Hartley transform on generalized functions and spaces of Boehmians is studied in [6], [7], [13], [14].

The Hartley transform is an integral transformation that maps a real-valued temporal or spacial function into a real-valued frequency function via the kernel, $cas(\nu x) \equiv cos(\nu x) + sin(\nu x)$. Hartley [5] studied a more symmetrical Fourier integral, which leads to a parallelism that exists between the function of the original variable and that of its transform.

The Hartley transform of a function $f(x)$ is defined by Olejniczak [12]:

$$\mathcal{H}(\nu) = \mathcal{H}[f](\nu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) cas(\nu x) dx, \tag{4}$$

$$\text{or} \quad \mathcal{H}(f) = \int_{\mathbb{R}} f(x) cas(2\pi f x) dx, \tag{5}$$

where the angular frequency variable ν is related to the frequency variable f by $\nu = 2\pi f$, and

$$\mathcal{H}(f) = \sqrt{2\pi}\mathcal{H}(2\pi f) = \sqrt{2\pi}\mathcal{H}(\nu). \tag{6}$$

The integral kernel, known as the cosine and sine function, is define by

$$cas(\nu x) \equiv cos(\nu x) + sin(\nu x) = \sqrt{2}\sin\left(\nu x + \frac{\pi}{4}\right) = \sqrt{2}\cos\left(\nu x - \frac{\pi}{4}\right). \tag{7}$$

The inverse Hartley transform is defined by

$$\mathcal{H}^{-1}(\mathcal{H}(f)) = f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{H}(\nu)cas(\nu x)d\nu = \int_{\mathbb{R}} \mathcal{H}(f)cas(2\pi fx)df. \tag{8}$$

The relation between the Hartley and Fourier transform is given by Olejniczak [12]

$$\mathcal{H}(\nu) = [Re\{F(\omega)\} - Im\{F(\omega)\}]_{\omega=\nu}, \tag{9}$$

$$\text{where} \quad Re\{F(\omega)\} = Re(\omega) = \mathcal{H}^e(\nu) = \mathcal{H}^e(-\nu), \tag{10}$$

$$Im\{F(\omega)\} = Im(\omega) = -\mathcal{H}^o(\nu) = \mathcal{H}^o(-\nu), \tag{11}$$

$\mathcal{H}$ is the Hartley transform and $\mathcal{H}^e(\nu)$ and $\mathcal{H}^o(\nu)$ are, respectively, even and odd parts of the Hartley transform. $Re\{F(\omega)\}$ is the real part and $Im\{F(\omega)\}$ is the imaginary part of the Fourier transform. We mention elementary properties of the Hartley transform as follows [7]:

1. Linearity: If $f_1(x)$ and $f_2(x)$ have the Hartley transforms $\mathcal{H}_1(f)$ and $\mathcal{H}_2(f)$, respectively, α and β are positive constants, $\alpha, \beta > 0$, then

$$\int_{\mathbb{R}} [\alpha f_1(x) + \beta f_2(x)]cas(2\pi fx)dx = \alpha\mathcal{H}_1(f) + \beta\mathcal{H}_2(f). \tag{12}$$

2. Scaling/Similarity: If $f(x)$ is the Hartley transform (4), then

$$\int_{\mathbb{R}} cas(2\pi fx)dx = \frac{1}{k}\mathcal{H}\left(\frac{f}{k}\right). \tag{13}$$

3. *nth* **Derivative of a Function** $f^n(x)$

$$f^n(x) = cas'\frac{n\pi}{2}(2\pi f)^n\mathcal{H}[(-1)^n f], \tag{14}$$

which is derived by recursive application of equation (9) to the Fourier transform of the function $\frac{df}{dx}$ and its higher order. Here $cas'(\xi) = cos(\xi) - sin(\xi)$ (complementary cas function) and $\frac{d}{d\tau}cas\tau = cas(-\tau) = cas'(\tau)$ (derivative relation), respectively.

4. Function Shift/Delay: If $f(x)$ is shifted in time by a constant T, then by substituting $x' = x - T$, the Hartley transform is

$$\mathcal{H}(f) = \int_{\mathbb{R}} f(x')cas[2\pi f(x'+T)]dx' = cos(2\pi fT)\mathcal{H}(f) + sin(2\pi fT)\mathcal{H}(-f). \tag{15}$$

5. Convolution: If $f_1(x)$ has the Hartley transform $\mathcal{H}_1(f)$ and $f_2(x)$ has the Hartley transform $\mathcal{H}_2(f)$, then

$$\begin{aligned} &f_1(x) * f_2(x) \\ &= \frac{1}{2}[\mathcal{H}_1(f)\mathcal{H}_2(f) + \mathcal{H}_1(-f)\mathcal{H}_2(f) + \mathcal{H}_1(f)\mathcal{H}_2(-f) - \mathcal{H}_1(-f)\mathcal{H}_2(-f)]. \end{aligned} \tag{16}$$

To obtain the result (16), simply substitute the convolution integral $f_1(x)*f_2(x) = \int_{\mathbb{R}} f_1(\lambda)f_1(x-\lambda)d\lambda$ in (5) and using shift /delay property, we get

$$\mathcal{H}(f) = \int_{\mathbb{R}} (f_1(x) * f_2(x))cas(2\pi f x)dx = \int_{\mathbb{R}} f_1(\lambda)[\int_{\mathbb{R}} f_2(x-\lambda)cas(2\pi f x)dx]d\lambda$$
$$= \int_{\mathbb{R}} f_1(\lambda)[cas(2\pi f\lambda)\mathcal{H}_2(f) + sin(2\pi f\lambda)\mathcal{H}_2(-f)]d\lambda, \quad \text{(by (12))}.$$

Factorizing the $\mathcal{H}_2(\cdot)$ of the right hand side, and using other properties of Hartley transform, equation (16) is defined. Simplification of equation (16) results into following symmetries in different forms, e.g.

(i) If $f_1(x)$ and (or) $f_2(x)$ is even, or if $f_1(x)$ is even and $f_2(x)$ is odd, or if $f_1(x)$ is odd and $f_2(x)$ is even, then $f_1(x) * f_2(x) = \mathcal{H}_1(f)\mathcal{H}_2(f)$.

(ii) If $f_1(x)$ is odd, then $f_1(x) * f_2(x) = \mathcal{H}_1(f)\mathcal{H}_2(-f)$.

(iii) If $f_2(x)$ is odd, then $f_1(x) * f_2(x) = \mathcal{H}_1(-f)\mathcal{H}_2(f)$.

(iv) (iv) If both functions are odd, then $f_1(x) * f_2(x) = -\mathcal{H}_1(f)\mathcal{H}_2(f)$.

In general, equation (16) can be written in the following form given by Loonker [6]:

$$\mathcal{H}[f * g] = \mathcal{H}[f]\mathcal{H}[f]. \tag{17}$$

6. Parseval relation: If $f_1(x)$ has the Hartley transform $\mathcal{H}_1(f)$ and $f_2(x)$ has the Hartley transform $\mathcal{H}_2(f)$, then

$$\int_{\mathbb{R}} f_1(x)f_2(x)dx = \int_{\mathbb{R}} \mathcal{H}_1(f)\mathcal{H}_2(f)df. \tag{18}$$

The Hartley transform of distributions is defined by Chaudhary and Thorat [4]. For the space of slow growth S', inversion theorems are proved by Nair and Banerji [11], using the Parseval equation for the Hartley transform

$$< \mathcal{H}(f), \varphi > = < f, \mathcal{H}(\varphi) >, \tag{19}$$

where function f is absolutely integrable and φ is a testing function of rapid descent, and for tempered distribution of Hartley transform, we have

$$|\nu^m \mathcal{H}^{(k)}(\nu)| \leq \Upsilon_{m,k}, \ \nu \in \mathbb{R} \tag{20}$$

where m and k are non-negative integers and Υ is a constant.

2. The exchange property

For a family $\{\varphi_i\}_{i\in I}$=$\{\varphi_i\}_I$, where I is an index set and $\varphi_i\in S$, $\forall i\in I$, we have [1]:

$$\Psi(\{\varphi_i\}_I) = \{x \in \mathbb{R}^N : \hat{\varphi}_i(x) = 0 \ \ \forall i \in I\}. \tag{21}$$

A family of pairs $\{(f_i, \varphi_i)\}_I$, where $f_i \in S'$ and $\varphi_i \in S$, is said to have the exchange property if

$$f_i * \varphi_k = f_k * \varphi_i, \qquad \forall i, k \in I. \tag{22}$$

We will denote by A the collection of all families of pairs $\{(f_i, \varphi_i)\}_I$, where I is an index set, $f_i \in S'$ and $\varphi_i \in S$ for all $i \in I$, satisfying the exchange property such that $\Psi(\{\varphi_i\}_I) = \phi$. If (φ_i) is a delta sequence, then $\Psi(\{\varphi_i\}_N) = \phi$.

Definition 1 If $\{(f_i, \varphi_i)\}_I \in A$, then the unique $F \in \mathcal{D}'(\mathbb{R}^N)$ such that $\mathcal{H}f_i = \mathcal{H}\varphi_i F$, $\forall i \in I$ will be denoted by $\mathcal{H}\{(f_i, \varphi_i)\}_I$.

Let $\{(f_i, \varphi_i)\}_I, \{(g_k, \psi_k)\}_K \in A$. If $f_i * \psi_k = g_k * \varphi_i$ for all $i \in I$ and $k \in K$, then we write $\{(f_i, \varphi_i)\}_I \sim \{(g_k, \psi_k)\}_K$. This relation is clearly symmetric and reflexive. We will show that it is also transitive.

Let $\{(f_i, \varphi_i)\}_I, \{(g_k, \psi_k)\}_K, \{(h_l, \gamma_l)\}_L \in A$. If $\{(f_i, \varphi_i)\}_I \sim \{(g_k, \psi_k)\}_K$ and $\{(g_k, \psi_k)\}_K \sim \{(h_l, \gamma_l)\}_L$, then

$$f_i * \psi_k = g_k * \varphi_i, \;\; g_k * \gamma_l = h_l * \psi_k, \;\; \forall i \in I, k \in K, l \in L. \tag{23}$$

Therefore,

$$f_i * \psi_k * \gamma_l = g_k * \varphi_i * \gamma_l, \;\; g_k * \gamma_l * \varphi_i = h_l * \psi_k * \varphi_i, \;\; \forall i \in I, k \in K, l \in L. \tag{24}$$

Since $*$ is commutative, we have

$$f_i * \gamma_l * \psi_k = h_l * \varphi_i * \psi_k. \tag{25}$$

Now, for $i \in I$ and $l \in L$. Since $\Psi(\{\psi_k\}_K) = \phi$ and (25) holds for every $k \in K$, we conclude that $f_i * \gamma_l = h_l * \varphi_i$ $\forall i \in I$, $l \in L$, which means $\{(f_i, \varphi_i)\}_I \sim \{(h_l, \gamma_l)\}_L$.

Theorem 2 *If a family of pair $\{(f_i, \varphi_i)\}_I$ has the exchange property and $\Omega = \Psi(\{\varphi_i\}_I)^c$ (the complement of $\Psi(\{\varphi_i\}_I)$ in $\mathbb{R}^N$), then there exists a unique $F \in \mathcal{D}'(\Omega)$ such that*

$$\mathcal{H}[f_i] = F\mathcal{H}[\varphi_i], \;\; \forall i \in I. \tag{26}$$

Proof. For every $x \in \Omega$ there exists $i \in I$ and $\varepsilon > 0$ such that $|\mathcal{H}\varphi_i(x)| > \varepsilon$ in an open neighbourhood of x. Then we can define $F = \mathcal{H}f_i/\mathcal{H}\varphi_i$ in that neighbourhood. Let for some $\varepsilon > 0$, we have $|\mathcal{H}\varphi_i(x)| > \varepsilon$ for all $x \in U$ and $|\mathcal{H}\varphi_k(x)| > \varepsilon$ for all $x \in V$, where U and V are open sets. Since $f_i * \varphi_k = f_k * \varphi_i$, we have

$$\mathcal{H}f_i\mathcal{H}\varphi_k = \mathcal{H}f_k\mathcal{H}\varphi_i \;\text{ and }\; \frac{\mathcal{H}f_i}{\mathcal{H}\varphi_i} = \frac{\mathcal{H}f_k}{\mathcal{H}\varphi_k}, \tag{27}$$

on $U \cap V$. This shows that F is a unique function. ∎

Theorem 3 *$\{(f_i, \varphi_i)\}_I \in A$ if and only if there exists a unique $F \in \mathcal{D}'(\mathbb{R}^N)$ such that $\mathcal{H}[f_i] = \mathcal{H}[\varphi_i]F$ for all $i \in I$.*

Proof. For any $i, k \in I$, we have

$$\mathcal{H}f_i\mathcal{H}\varphi_k = F\mathcal{H}\varphi_i\mathcal{H}\varphi_k = F\mathcal{H}\varphi_k\mathcal{H}\varphi_i = \mathcal{H}f_k\mathcal{H}\varphi_i \text{ [by Theorem 1]}. \tag{28}$$

This proves that the existence of such a function $F \in \mathcal{D}'(\mathbb{R}^N)$ implies and justifies the exchange property. This completes the proof of the theorem. ■

Theorem 4 *There exists $\{(f_i,\varphi_i)\}_I \in A$ for every $F \in \mathcal{D}'(\mathbb{R}^N)$ and such that $F = \mathcal{H}(\{(f_i,\varphi_i)\}_I)$.*

Proof. Since $\mathcal{D}(\mathbb{R}^N)$ denotes the space of smooth function with compact support, there exists a total sequence $\{\varphi_i\}_N$ such that $\mathcal{H}\varphi_i \in \mathcal{D}(\mathbb{R}^N)$, $\forall I \in N$. Then for every $I \in N$, there is $f_i \in S'$ such that $\mathcal{H}f_i = \mathcal{H}\varphi_i F$. Clearly $\{(f_i,\varphi_i)\}_I \in A$ and $F = \mathcal{H}(\{(f_i,\varphi_i)\}_N)$. This completes the proof of theorem. ■

Definition 5 [1] *Let $\{U_i\}_I$ be an open covering of $\mathbb{R}^N$ and let $\{\varphi_i\}_I$ be such that $|\mathcal{H}\varphi_i(x)| > 0$ for $x \in U_i$. A family $\{\varphi_i\}_I$ such that $\Psi(\{\varphi_i\}_I)$=ϕ will be called total.*

Lemma 6 [1] *If $(\{\varphi_i\}_I)$ and $(\{\psi_k\}_K)$ are total, then $\{\varphi_i * \psi_k\}_{I\times K}$ is total.*

Theorem 7 *Let $\{(f_i,\varphi_i)\}_I, \{(g_k,\psi_k)\}_K \in A$. Then $\{(f_i,\varphi_i)\}_I \sim \{(g_k,\psi_k)\}_K$ if and only if $\mathcal{H}(\{(f_i,\varphi_i)\}_I) = \mathcal{H}(\{(g_k,\psi_k)\}_K)$.*

Proof. $F = \mathcal{H}(\{(f_i,\varphi_i)\}_I)$ and $G = \mathcal{H}(\{(g_k,\psi_k)\}_K)$. If $\{(f_i,\varphi_i)\}_I \sim \{(g_k,\psi_k)\}_K$, then $F\mathcal{H}\varphi_i\mathcal{H}\psi_k = \mathcal{H}f_i\mathcal{H}\psi_k = \mathcal{H}g_k\mathcal{H}\varphi_i$

$$\Rightarrow F\mathcal{H}\varphi_i\mathcal{H}\psi_k = G\mathcal{H}\psi_k\mathcal{H}\varphi_i, \quad \forall i \in I, k \in K. \tag{29}$$

Hence $F = G$, by Lemma 6. Now assume $F = G$. Then

$$\mathcal{H}f_i\mathcal{H}\psi_k = F\mathcal{H}\varphi_i\mathcal{H}\psi_k = G\mathcal{H}\psi_k\mathcal{H}\varphi_i = \mathcal{H}g_k\mathcal{H}\varphi_i, \quad \forall i \in I, k \in K. \tag{30}$$

Hence $\{(f_i,\varphi_i)\}_I \sim \{(g_k,\psi_k)\}_K$. This completes the proof of the theorem. ■

Theorem 8 *There exists a delta sequence φ_n such that, for every $T \in \mathcal{B}_j$, $T = (\{(f_n,\varphi_n)\}_N)$ for some $f_n \in J$.*

Proof. Let (ψ_n) be a delta sequence such that $\mathcal{H}\psi_n \in \mathcal{D}$. Then for any $T \in \mathcal{B}_j$, we have $\mathcal{H}T\mathcal{H}\psi_k \in S'$, since $\mathcal{H}T \in \mathcal{D}'$. Consequently, $\mathcal{H}T\mathcal{H}\psi_n = \mathcal{H}g_n$ for some $g_n \in S'$. It is easy to check that $T = [\{(g_n*\psi_n, \psi_n*\psi_n)\}_N]$. Since $f_n = g_n*\psi_n \in J$ and $\varphi_n = \psi_n * \psi_n$ is a delta sequence, where φ_n does not depend on T, hence the theorem is proved. ■

3. Algebric properties and convergence

$\mathcal{B}_j$ becomes a vector space with the addition operation, defined by

$$[\{(f_i,\varphi_i)\}_I] + [\{(g_k,\psi_k)\}_K] = [\{(f_i * \psi_k + g_k * \varphi_i, \varphi_i * \psi_k)\}_{I\times K}]. \tag{31}$$

Moreover, multiplication by a scalar is defined by

$$\lambda[\{(f_i,\varphi_i)\}_I] = [\{(\lambda f_i,\varphi_i)\}_I], \quad \lambda \in \mathbb{C}. \tag{32}$$

If $[\{(f_i,\varphi_i)\}_I]+[\{(g_k,\psi_k)\}_K] \in \mathcal{B}_j$ and $g_k \in S$ for all $k \in K$, then for the operation $*$ we can define

$$[\{(f_i,\varphi_i)\}_I] * [\{(g_k,\psi_k)\}_K] = [\{(f_i * g_k, \varphi_i * \psi_k)\}_{I\times K}]. \tag{33}$$

Definition 9 [1] Let $T_0, T_1, T_n, ... \in \mathcal{B}_j$. Then the sequence T_n is said to converge to T_0, which is written as $T_n \to T_0$ if there exists a total family $(\varphi_i)_I$ such that

(a) there exists tempered distribution $f_{i,n}$ where $i \in I$ and $n \in N$ such that $T_n = [\{(f_{i,n},\varphi_i)\}_I]$ for all $n = 0,1,2,...,$

(b) $f_{i,n} \to f_{i,0}$ in S' as $n \to \infty$ for every $i \in I$.

Theorem 10 *The Hartley transform is an isomorphism from $\mathcal{B}_j$ to $\mathcal{D}'$.*

Proof. Since $T_n \to T_0$ in $\mathcal{B}_j$ if and only if $T_n - T_0 \to 0$, it suffices to prove the continuity at 0. Let $T_n \to 0$ is in $\mathcal{B}_j$. Then there exists tempered distribution $f_{i,n}$ where $i \in I$ and $n \in N$ such that $T_n = [\{(f_{i,n},\varphi_i)\}_I]$ for all $n = 1,2,...$ and $f_{i,n} \to 0$ in S' as $n \to \infty$ for every $i \in I$. If $\psi \in \mathcal{D}$, then there are $i_1,...i_k$ such that $\mathrm{supp}\psi \subset \bigcup_{m=1}^{k} \mathrm{supp}\mathcal{H}\varphi_{i,m}$. Then

$$\begin{aligned}\lim_{n\to\infty} \mathcal{H}T_n\psi &= \lim_{n\to\infty} \sum_{m=1}^{k} (\mathcal{H}T_n\mathcal{H}\varphi_{i,m}) \frac{\overline{\mathcal{H}\varphi_{i,m}\psi}}{\sum_{m=1}^{k} |\mathcal{H}\varphi_{i,m}|^2} \\ &= \sum_{m=1}^{k} (\lim_{n\to\infty} \mathcal{H}f_{im,n}) \frac{\overline{\mathcal{H}\varphi_{i,m}\psi}}{\sum_{m=1}^{k} |\mathcal{H}\varphi_{i,m}|^2} = 0,\end{aligned} \tag{34}$$

because $\lim_{n\to\infty} \mathcal{H}f_{i,n} = 0, \forall i \in I$, due to the continuity of the Hartley transform in S'. This proves the continuity of $\mathcal{H} : \mathcal{B}_j \to \mathcal{D}'$, because $\lim_{n\to\infty} \mathcal{H}T_n\psi = 0$ in S' for every $\psi \in \mathcal{D}$, implies $\lim_{n\to\infty} \mathcal{H}T_n = 0$ in $\mathcal{D}'$.

Now, assume $\lim_{n\to\infty} \mathcal{H}T_n = 0$ in $\mathcal{D}'$. By Theorem 8, there exists a delta sequence $(\varphi_i), i \in N$ such that for every $n \in N$, we have $T_n = [\{(f_{i,n},\varphi_i)\}_N]$ for some $f_{i,n} \in J$. Let $(\psi_k), k \in N$ be a delta sequence such that $\mathcal{H}\psi_k \in \mathcal{D}$ for every $k \in N$. Then

$$\lim_{n\to\infty} \mathcal{H}T_n\mathcal{H}\varphi_i.\mathcal{H}\psi_k = 0 \ \in S' \quad \text{for every } \ i,k \in N.$$

Since $\mathcal{H}T_n\mathcal{H}\varphi_i = f_{i,n} \quad \forall i,k \in N$ and $\lim_{n\to\infty} \mathcal{H}f_{i,n}\mathcal{H}\psi_k = 0 \in S'$, which implies $\lim_{n\to\infty} f_{i,n} * \psi_k = 0 \in S'$. But

$$T_n = [\{(f_{i,n},\varphi_i)\}_I] = [\{(f_{i,n} * \psi_k, \varphi_k * \psi_k)\}_{I\times K}]$$

for all $n = 0,1,2,....$ Thus we have $T_n \to 0$ in $\mathcal{B}_j$. This proves the theorem. ∎

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