TWO RESULTS ON THE ABELIAN IDEALS OF A BOREL SUBALGEBRA

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Abstract. Let $\mathfrak{b}$ a fixed Borel subalgebra of a finite-dimensional complex simple Lie algebra $\mathfrak{g}$. The Shi bijection associates to every ad-nilpotent ideal $\mathfrak{i}$ of $\mathfrak{b}$ a region $V_{\mathfrak{i}}$. In this paper, we show that $\mathfrak{i}$ is abelian if and only if $V_{\mathfrak{i}} \cap 2A$ is nonempty, if and only if the volume of $V_{\mathfrak{i}} \cap 2A$ equals to that of A, where A is the fundamental alcove of the affine Weyl group. We also determine the maximal eigenvalue m_{r-1} of the Casimir operator on $\Lambda^{r-1}\mathfrak{g}$ and the corresponding eigenspace M_{r-1}, where r is the number of positive roots.

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1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank l. Let $\mathfrak{b}$ a fixed Borel subalgebra of $\mathfrak{g}$. Two results of the abelian ideals of $\mathfrak{b}$ will be given in this paper. On one hand, we will describe the regions in the Shi arrangement corresponding to the abelian ideals of $\mathfrak{b}$. On the other hand, we will determine the maximal eigenvalue m_{r-1} of the Casimir operator on $\Lambda^{r-1}\mathfrak{g}$ and the corresponding eigenspace M_{r-1}, where r is the number of positive roots.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then we have the root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. Let $V = \mathfrak{h}^*_{\mathbb{R}}$ be the real vector space spanned by Δ. Let $(\,|\,)$ be the canonical inner product induced on V by the Killing form. Let $\Pi = \{\alpha_1, ..., \alpha_l\} \subseteq \Delta^+$ be a fixed choice of simple and positive root systems of Δ, respectively. Let $\mathfrak{n} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space relative to α. Then $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$. Let $W = W(\mathfrak{g}, \mathfrak{h})$ be the affine Weyl group.

Here, recall that an ideal $\mathfrak{i}$ of $\mathfrak{b}$ is called *ad-nilpotent* if $\mathfrak{i} \subseteq \mathfrak{n}$. Let us denote by $\mathcal{I}$ the set of all the ad-nilpotent ideals of $\mathfrak{b}$. Shi [10] associated a dominant region $V_\mathfrak{i}$ of the now called Shi arrangement $\mathtt{Shi}$ to any $\mathfrak{i} \in \mathcal{I}$. To state his results, let us recall some notations concerning hyperplane arrangements.

A *hyperplane arrangement* is a finite collection of affine hyperplanes in a Euclidean space. For example, the *Shi arrangement* associated with Δ^+ is the arrangement in V defined by

$$\mathtt{Shi} := \{H_{\alpha,0} \mid \alpha \in \Delta^+\} \cup \{H_{\alpha,1} \mid \alpha \in \Delta^+\}. \tag{1.1}$$

Here for $\alpha \in \Delta^+$ and $k \in \mathbb{Z}$, we define an affine hyperplane

$$H_{\alpha,k} := \{v \in V \mid (v|\alpha) = k\}. \tag{1.2}$$

This arrangement was defined by Shi in the study of the Kazhdan-Lusztig cellular structure of the affine Weyl group of type A, see Chapter 7 of [9]. In general, if $\mathcal{A}$ is a hyperplane arrangement in V, the connected components of $V - \bigcup_{H\in\mathcal{A}} H$ are called *regions*. We refer the reader to Section 3.11 of [11] for basics about hyperplane arrangements. For $\mathtt{Shi}$, a region is called *dominant* if it is contained in V_∞, where

$$V_\infty := \{v \in V \mid (v|\alpha) > 0, \forall \alpha \in \Delta^+\}. \tag{1.3}$$

Now let us recall the *Shi bijection* from Theorem 1.4 of [10] as follows:

Theorem 1.1 (**Shi**) *There exists a natural bijective map from the set of all the ad-nilpotent ideals of* $\mathfrak{b}$ *to the set of all the dominant regions of the hyperplane arrangement* $\mathtt{Shi}$. *The map sends* $\mathfrak{i}$ *to* $V_\mathfrak{i} := \{v \in V \mid (v|\beta) > 1, \forall \beta \in \Phi(\mathfrak{i}); 0 < (v|\beta) < 1, \forall \beta \in \Delta^+ \setminus \Phi(\mathfrak{i})\}$. *Here* $\Phi(\mathfrak{i})$ *denotes the set of roots in* $\mathfrak{i}$.

It is easy to see that the dominant region of $\mathtt{Shi}$ corresponding to the zero ideal coincides with the fundamental alcove of the affine Weyl group $\widehat{W}$, which is defined as

$$A := \{v \in V \mid (v|\alpha_i) > 0 \text{ for } i = 1, \dots, l \text{ and } (v|\theta) < 1\}, \tag{1.4}$$

where θ is the maximal positive root. Note that the Shi bijection is recently generalized to parabolic subalgebras in [3].

Theorem 1.1 gives us one way to visualize the set $\mathcal{I}$. It is then natural to ask for an according characterization of the abelian ideals among all the ad-nilpotent ones. A difficulty lies in that the region $V_\mathfrak{i}$ may be unbounded, even if $\mathfrak{i}$ is abelian. Indeed, we have

Proposition 1.2 *For any* $\mathfrak{i} \in \mathcal{I}$, *the following are equivalent:*

(a) $\mathfrak{i}$ *is abelian;*

(b) $V_\mathfrak{i} \cap 2A$ *is nonempty;*

(c) $\mathrm{vol}(V_\mathfrak{i} \cap 2A) = \mathrm{vol}(A)$. *Here "vol" means taking the volume.*

Another theme of the current paper comes from [6], where Kostant found a connection between the commutative Lie subalgebras of $\mathfrak{g}$ and the maximal eigenspaces of the Casimir operator $Cas \in U(\mathfrak{g})$ on certain degrees of $\wedge\mathfrak{g}$. To be more precise, let p denote the maximal dimension of all the abelian subalgebras of $\mathfrak{g}$. Let m_k be the maximal eigenvalue of Cas on $\wedge^k\mathfrak{g}$ and M_k be the corresponding eigenspace. Then

$$m_k \leq k,$$

and equality holds if and only if $k \leq p$. Moreover, in such a case, M_k is a multiplicity-free $\mathfrak{g}$-module whose highest weight vectors correspond to k-dimensional abelian subalgebras of $\mathfrak{g}$ (or equivalently, k-dimensional abelian ideals of $\mathfrak{b}$). In (1.2) of [7], Kostant mentioned that

$$m = \frac{1}{3}\dim \mathfrak{g}, \tag{1.5}$$

where m denotes the maximal eigenvalue of Cas on $\wedge\mathfrak{g}$, i.e., m is the maximum of m_k over all k. Recently, among other things, Han [4] determined the eigenspaces relative to m. To state the result, let r denote the number of positive roots and put $l = \operatorname{rank}\mathfrak{g}$. For any dominant integral linear form λ on $\mathfrak{h}$, let V_λ be the irreducible $\mathfrak{g}$-module with highest weight λ. Then, it is proved in [4] that

(i) one has $m_k \leq \frac{1}{3}\dim\mathfrak{g}$, and equality holds precisely for $k = r, r+1, \cdots, r+l$. Moreover, for $0 \leq s \leq l$, $M_{r+s} = C_l^s V_{2\rho}$;

(ii) For $0 \leq k < r$, one has $m_k < m_{k+1}$ and $\oplus_{k=0}^{r} M_k$ is a multiplicity-free $\mathfrak{g}$-module.

We find that m_{r-1} and the corresponding eigenspace M_{r-1} can be explicitly determined. Let Π_s be the set of all *short* simple roots. We view every root as a short one if $\mathfrak{g}$ is simply-laced, i.e., $\mathfrak{g}$ is of type A, D, E. Recall that the important integer

$$h^\vee = \frac{1}{(\theta|\theta)} \tag{1.6}$$

is the *dual Coxeter number.* Let α be any short root, then

$$k = \frac{(\theta|\theta)}{(\alpha|\alpha)} \tag{1.7}$$

is a positive integer. Now we are able to state

Theorem 1.3 *Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. With notations as above, one has*

$$m_{r-1} = \frac{1}{3}\dim \mathfrak{g} - \frac{2}{kh^\vee}, \tag{1.8}$$

and

$$M_{r-1} = \oplus_{\alpha\in\Pi_s} V_{2\rho-\alpha}. \tag{1.9}$$

Moreover, for any $\alpha \in \Pi$ which is not necessarily short, the irreducible $\mathfrak{g}$-module $V_{2\rho-\alpha}$ occurs 2^l times in $\wedge\mathfrak{g}$, which are distributed as follows:

$r-1$	r	$r+1$	$r+2$	$\cdots$	$r+l-2$	$r+l-1$	$r+l$	$r+l+1$
C_{l-1}^0	C_{l-1}^1	C_{l-1}^2	$C_{l-1}^3+C_{l-1}^{l-1}$	$\cdots$	$C_{l-1}^{l-1}+C_{l-1}^3$	C_{l-1}^2	C_{l-1}^1	C_{l-1}^0

Here the first row denotes dimension, and the second row denotes the according times of occurrence of $V_{2\rho-\alpha}$.

The paper is organized as follows: after collecting necessary preliminaries in Section 2, we will prove Proposition 1.2 in Section 3, and Theorem 1.3 in Section 4.

2. Preliminaries

We extend V and its canonical inner product by setting $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$, $(\delta|\delta) = (\delta|V) = (\lambda|\lambda) = (\lambda|V) = 0$, and $(\delta|\lambda) = 1$. The affine root system associated to Δ is $\widehat{\Delta} = \Delta + \mathbb{Z}\delta$. The set of positive affine roots is $\widehat{\Delta}^+ = (\Delta^+ + \mathbb{N}\delta) \cup (\Delta^- + \mathbb{N}^+\delta)$, where $\Delta^- = -\Delta^+$. We denote by $\alpha_0 = -\theta + \delta$ and set $\widehat{\Pi} = \{\alpha_0, \alpha_1, \cdots, \alpha_l\}$. For each $\alpha \in \widehat{\Delta}^+$, let s_α be the corresponding reflection of $\widehat{V}$. The affine Weyl group $\widehat{W}$ associated to Δ is the group generated by $\{s_\alpha | \alpha \in \widehat{\Delta}^+\}$. Now consider the $\widehat{W}$-invariant affine subspace $E := V \oplus \mathbb{R}\delta + \lambda$. Let $\pi : E \to V$ be the natural projection. For $w \in \widehat{W}$, set $\overline{w} = \pi \circ w|_E$. Then the map $w \mapsto \overline{w}$ gives an isomorphism of $\widehat{W}$ to a group of affine translations W_{af} of V. Thus we may and we will omit the bar, and use the same notation for the elements in $\widehat{W}$ and the corresponding elements in W_{af}. We refer the reader to [1] and the references therein for more about affine Weyl groups.

For any $\alpha \in \Delta^+$, $k \in \mathbb{N}^+$, $h \in \mathbb{N}$, by (1.1) of [1], we have

(2.1) $w^{-1}(-\alpha + k\delta) < 0$ if and only if $H_{\alpha,k}$ separates A and $w(A)$,

(2.2) $w^{-1}(\alpha + h\delta) < 0$ if and only if $H_{\alpha,-h}$ separates A and $w(A)$.

3. Proof of Proposition 1.2

This section is devoted to the proof of Proposition 1.2. Firstly, let us show that for any $\mathfrak{i} \in \mathcal{I}$, the following are equivalent:

(a') the index of nilpotence of $\mathfrak{i}$ equals to k;

(b') $V_{\mathfrak{i}} \cap kA$ is empty and $V_{\mathfrak{i}} \cap (k+1)A$ is non-empty.

Recall that the *index of nilpotence* of $\mathfrak{i}$, denoted by $n(\mathfrak{i})$, is the number of nonzero terms of the descending central series of $\mathfrak{i}$:

$$\mathfrak{i}^1 = \mathfrak{i}, \mathfrak{i}^2 = [\mathfrak{i}, \mathfrak{i}], \ldots, \mathfrak{i}^k = [\mathfrak{i}^{k-1}, \mathfrak{i}], \ldots.$$

For example, $n(0) = 0$ and $n(\mathfrak{i}) = 1$ for any nonzero abelian ideal $i \in \mathcal{I}$. For any $\mathfrak{i} \in \mathcal{I}$, let $\Phi(\mathfrak{i}^k)$ be the roots of $\mathfrak{i}^k$, $k \in \mathbb{N}^+$. Put

$$L_{\mathfrak{i}} = \bigcup_{k \geq 1} (-\Phi(\mathfrak{i}^k) + k\delta).$$

Recall that by Theorem 2.6 of [1], for any ad-nilpotent ideal $\mathfrak{i}$ of $\mathfrak{b}$, there exists a unique $w_\mathfrak{i} \in \widehat{W}$ such that $L_\mathfrak{i} = N(w_\mathfrak{i})$. Here, $N(w) := \{\alpha \in \widehat{\Delta}^+ | w^{-1}(\alpha) < 0\}$ for any $w \in \widehat{W}$. The equivalence of (a') and (b') will follow from the following two lemmas.

Lemma 3.1 *Let $\mathfrak{i} \in \mathcal{I}$ with $n(\mathfrak{i}) = k$. Then $V_\mathfrak{i} \cap kA$ is empty.*

Proof. Take any $x \in V_\mathfrak{i}$, it suffices to show that $(x|\theta) > k$. Indeed, since $n(\mathfrak{i}) = k$, we can find k roots $\varphi_1, \dots, \varphi_k$ of $\mathfrak{i}$ such that $\varphi_1 + \cdots + \varphi_k$ is still a root of $\mathfrak{i}$. Then

$$(x|\theta) \geq (x|\varphi_1 + \cdots + \varphi_k) = (x|\varphi_1) + \cdots + (x|\varphi_k) > k. \quad \blacksquare$$

Lemma 3.2 *Let $\mathfrak{i} \in \mathcal{I}$ with $n(\mathfrak{i}) = k$. Then*

$$w_\mathfrak{i}(A) \subseteq V_\mathfrak{i} \cap (k+1)A. \tag{3.1}$$

In particular, $V_\mathfrak{i} \cap (k+1)A$ is nonempty.

Proof. Take an arbitrary $\alpha \in \Delta^+$. Since $n(\mathfrak{i}) = k$, we have

$$-\alpha, -\alpha + (k+1)\delta \notin L(\mathfrak{i}) = N(w_\mathfrak{i}).$$

Thus, neither $H_{\alpha,0}$ nor $H_{\alpha,k+1}$ separates A and $w_\mathfrak{i}(A)$ by (2.2). Therefore, $w_\mathfrak{i}(A) \subseteq (k+1)A$. On the other hand,

$$\alpha \in \Phi(\mathfrak{i}) \iff -\alpha + \delta \in L_\mathfrak{i} = N(w_\mathfrak{i}) \iff H_{\alpha,1} \text{ separates } A \text{ and } w_\mathfrak{i}(A).$$

Therefore, $w_\mathfrak{i}(A) \subseteq V_\mathfrak{i}$. ■

Specializing k to be 0 and 1, the equivalence of (a') and (b') implies that of (a) and (b) in Proposition 1.2. The equivalence of (a) and (c) in Proposition 1.2 follows from the following

Lemma 3.3 *Let $\mathfrak{a} \in \mathcal{I}$ be abelian. Then* $\mathrm{vol}(V_\mathfrak{a} \cap 2A) = \mathrm{vol}(A)$.

Proof. By (3.1), we have

$$\sum_{\mathfrak{a} \text{ abelian}} \mathrm{vol}(w_\mathfrak{a}(A)) \leq \sum_{\mathfrak{a} \text{ abelian}} \mathrm{vol}(V_\mathfrak{a} \cap 2A).$$

We will show that both sides equal to $\mathrm{vol}(2A)$. Then for each abelian $\mathfrak{a}$, we would have $\mathrm{vol}(V_\mathfrak{a} \cap 2A) = \mathrm{vol}(w_\mathfrak{a}(A)) = \mathrm{vol}(A)$, as desired.

For the RHS, since $V_\mathfrak{i} \cap 2A$ is empty when $n(\mathfrak{i}) > 1$, we have

$$\sum_{\mathfrak{a} \text{ abelian}} \mathrm{vol}(V_\mathfrak{a} \cap 2A) = \sum_{\mathfrak{i} \in \mathcal{I}} \mathrm{vol}(V_\mathfrak{i} \cap 2A) = \mathrm{vol}(V_\infty \cap 2A) = \mathrm{vol}(2A).$$

For the LHS, the proof of Theorem 2.9 [1] shows that for any $w \in \widehat{W}$, $w(A) \subseteq 2A$ if and only if $w = w_\mathfrak{a}$ for some abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$. Thus, we have

$$\sum_{\mathfrak{a} \text{ abelian}} \mathrm{vol}(w_\mathfrak{a}(A)) = \mathrm{vol}(2A). \quad \blacksquare$$

4. The maximal eigenvalue of Cas on $\wedge^{r-1}\mathfrak{g}$

This section is devoted to determining the maximal eigenvalue of the Casimir operator on $\wedge^{r-1}\mathfrak{g}$, where r is the number of positive roots. We begin with some preliminaries.

Let $\wedge\mathfrak{g}$ be the exterior algebra. Then the adjoint action of $\mathfrak{g}$ extends naturally to $\wedge\mathfrak{g}$. One knows that, as $\mathfrak{g}$-modules, $\wedge^k\mathfrak{g}$ is isomorphic to $\wedge^{n-k}\mathfrak{g}$ for each k. Therefore,

$$m_k = m_{n-k} \tag{4.1}$$

and

$$M_k \cong M_{n-k}. \tag{4.2}$$

Recall that m_k is the maximal eigenvalue of Cas on $\wedge^k\mathfrak{g}$ and M_k is the corresponding eigenspace. Note also that, as $\mathfrak{g}$-modules one has (see [8])

$$\wedge\mathfrak{g} = 2^l\, V_\rho \otimes V_\rho. \tag{4.3}$$

Lemma 4.1 *For any simple root α, $V_{2\rho-\alpha}$ occurs exactly once in $V_\rho \otimes V_\rho$.*

Proof. For any simple root α, let s_α be the simple reflection relative to α. Then $s_\alpha\rho = \rho - \alpha$ is an extreme weight of V_ρ and it is easy to check that $2\rho - \alpha$ is dominant for Δ^+. Thus by the Steinberg formula (cf. Theorem 24.4 of [5]), the multiplicity of $V_{2\rho-\alpha}$ in $V_\rho \otimes V_\rho$ is equal to

$$\sum_{\sigma\in W}\sum_{\tau\in W} \operatorname{sign}(\sigma\tau) p(2\sigma\rho + 2\tau\rho - 4\rho + \alpha),$$

where p is the Kostant partition function. Note that

$$\sigma\rho = \rho - \langle\Phi(\sigma)\rangle,$$

where $\Phi(\sigma) = \sigma(-\Delta^+) \cap \Delta^+$ and $\langle\Phi(\sigma)\rangle$ denotes the sum of roots in $\Phi(\sigma)$. Similarly, we have $\tau\rho = \rho - \langle\Phi(\tau)\rangle$. Thus the multiplicity equals to

$$\sum_{\sigma\in W}\sum_{\tau\in W} \operatorname{sign}(\sigma\tau) p(\alpha - 2\langle\Phi(\sigma)\rangle - 2\langle\Phi(\tau)\rangle).$$

One sees easily that in the above sum the unique nonzero term occurs only when $\sigma = \tau = 1$. Thus the concerned multiplicity equals to one. ∎

Now let us recall some notation and results from [6]. An element $w \in \wedge^k\mathfrak{g}$ is called *decomposable* if $w = z_1 \wedge \cdots \wedge z_k$, where $z_i \in \mathfrak{g}$. In such a case, we let $\mathfrak{a}(w)$

be the subspace spanned by $z_1, \ldots, z_k$. Now a $\mathfrak{g}$-submodule $V_1 \subseteq \wedge^k \mathfrak{g}$ is called *decomposably-generated* if it is spanned by decomposable elements.

A subspace $\mathfrak{a} \subseteq \mathfrak{g}$ is called $\mathfrak{b}$-*normal* if $[\mathfrak{b}, \mathfrak{a}] \subseteq \mathfrak{a}$. In such a case, $\mathfrak{a}$ is stable under $ad\,\mathfrak{h}$ and there exists a unique set of roots $\Delta(\mathfrak{a}) \subseteq \Delta$ such that

$$\mathfrak{a} = \mathfrak{a} \cap \mathfrak{h} + \sum_{\alpha \in \Delta(\mathfrak{a})} \mathfrak{g}_\alpha.$$

Define $\langle \mathfrak{a} \rangle = \sum_{\alpha \in \Delta(\mathfrak{a})} \alpha$.

Proposition 4 of [6] asserts that each M_k is a direct sum of decomposably-generated simple $\mathfrak{g}$-submodules of $\wedge^k \mathfrak{g}$. Proposition 6 of [6] establishes a one-one correspondence between the set of all k-dimensional $\mathfrak{b}$-normal subspaces of $\mathfrak{g}$ and all decomposably-generated simple $\mathfrak{g}$-submodules of $\wedge^k \mathfrak{g}$. The correspondence can be described as follows: starting with a k-dimensional $\mathfrak{b}$-normal subspace $\mathfrak{a}$ of $\mathfrak{g}$, then the cyclic submodule $V_{\mathfrak{a}}$ of $\wedge^k \mathfrak{g}$ generated by $\wedge^k \mathfrak{a}$ is irreducible, decomposably-generated. Actually, $V_{\mathfrak{a}} \cong V_{\langle \mathfrak{a} \rangle}$ and $\wedge^k \mathfrak{a}$ is the highest weight vector. Thus, to determine m_k and M_k, it suffices to consider all the k-dimensional $\mathfrak{b}$-normal subspaces of $\mathfrak{g}$.

Proof of Theorem 1.3. As mentioned above, to investigate m_{r-1} and M_{r-1}, it suffices to focus on the $(r-1)$-dimensional $\mathfrak{b}$-normal subspaces $\mathfrak{a}$ of $\mathfrak{g}$. As shown in the proof of Theorem 3.2(3) of [4], for $V_{\mathfrak{a}}$ to lie in M_{r-1}, we must have $\Delta(\mathfrak{a}) \subseteq \Delta^+$. Since $\dim \mathfrak{a} = r - 1$ and $\mathfrak{a}$ is $\mathfrak{b}$-normal, we must have $\Delta(\mathfrak{a}) = \Delta^+ \setminus \{\alpha\}$, where α is a simple root. Since

$$Cas(2\rho - \alpha) = \|3\rho - \alpha\|^2 - \|\rho\|^2 = 8\|\rho\|^2 - 2\|\alpha\|^2 = \frac{1}{3}\dim \mathfrak{g} - 2\|\alpha\|^2$$

for any simple root α, we see that

$$m_{r-1} = \frac{1}{3}\dim \mathfrak{g} - 2\|\alpha\|^2$$

for any *short* simple root $\alpha \in \Pi_s$. Now (1.8) follows from (1.6) and (1.7).

Note also that $V_{2\rho-\alpha}, \alpha \in \Pi_s$, are precisely the irreducible components of M_{r-1}. Now for any $\alpha \in \Pi_s$, it remains to figure out the multiplicity n_α of $V_{2\rho-\alpha}$ in M_{r-1}, which is also the multiplicity of $V_{2\rho-\alpha}$ in $\wedge^{r-1}\mathfrak{g}$. One way to do this is to cite Han's result [4], which says that M_{r-1} is multiplicity free. Thus $n_\alpha = 1$. We give another approach. By (4.3) and Lemma 4.1, $V_{2\rho-\alpha}$ occurs 2^l times in $\wedge\mathfrak{g}$. Let us determine these occurrences explicitly. Let $h_1, \ldots, h_{l-1}$ be a basis of $\mathfrak{h}^\alpha := \{x \in \mathfrak{h} \mid \alpha(x) = 0\}$. Let u_α be the wedge product of all the r positive root vectors but e_α. Let $v_\alpha = u_\alpha \wedge [e_\alpha, e_{-\alpha}] \wedge e_\alpha \wedge e_{-\alpha}$. Then it is apparent that, for $0 \leq s \leq l-1$,

$$u_\alpha \wedge h_{j_1} \wedge \cdots \wedge h_{j_s}, v_\alpha \wedge h_{j_1} \wedge \cdots \wedge h_{j_s}, 1 \leq j_1 < \cdots < j_s \leq l-1$$

generate 2^l copies of $V_{2\rho-\alpha}$ in $\wedge\mathfrak{g}$. Thus they exhaust all the occurrences of $V_{2\rho-\alpha}$ in $\wedge\mathfrak{g}$. In particular $n_\alpha = 1$, we have (1.9), and the desired distribution of $V_{2\rho-\alpha}$

in $\wedge\mathfrak{g}$ follows. Finally, it is easy to see that the same distribution also holds for any simple root which is not necessarily short. ∎

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