TWO RESULTS ON THE ABELIAN IDEALS
OF A BOREL SUBALGEBRA

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Abstract. Let $b$ a fixed Borel subalgebra of a finite-dimensional complex simple Lie algebra $g$. The Shi bijection associates to every ad-nilpotent ideal $i$ of $b$ a region $V_i$. In this paper, we show that $i$ is abelian if and only if $V_i \cap 2A$ is nonempty, if and only if the volume of $V_i \cap 2A$ equals to that of $A$, where $A$ is the fundamental alcove of the affine Weyl group. We also determine the maximal eigenvalue $m_{r-1}$ of the Casimir operator on $Λ^{r-1}g$ and the corresponding eigenspace $M_{r-1}$, where $r$ is the number of positive roots.

Keywords: abelian ideal, ad-nilpotent ideal, Borel subalgebra, Casimir operator, Shi bijection.

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1. Introduction

Let $g$ be a finite-dimensional complex simple Lie algebra of rank $l$. Let $b$ a fixed Borel subalgebra of $g$. Two results of the abelian ideals of $b$ will be given in this paper. On one hand, we will describe the regions in the Shi arrangement corresponding to the abelian ideals of $b$. On the other hand, we will determine the maximal eigenvalue $m_{r-1}$ of the Casimir operator on $Λ^{r-1}g$ and the corresponding eigenspace $M_{r-1}$, where $r$ is the number of positive roots.

Fix a Cartan subalgebra $h$ of $g$. Then we have the root system $Δ = Δ(g, h)$. Let $V = h_ℝ$ be the real vector space spanned by $Δ$. Let $( )$ be the canonical inner product induced on $V$ by the Killing form. Let $Π = \{α_1, ..., α_l\} \subseteq Δ^+$ be a fixed choice of simple and positive root systems of $Δ$, respectively. Let $n = \bigoplus_{α \in Δ^+} g_α$, where $g_α$ is the root space relative to $α$. Then $b = h \oplus n$ is a Borel subalgebra of $g$. Let $W = W(g, h)$ be the affine Weyl group.

Here, recall that an ideal $i$ of $b$ is called ad-nilpotent if $i \subseteq n$. Let us denote by $I$ the set of all the ad-nilpotent ideals of $b$. Shi [10] associated a dominant
region $V_i$ of the now called Shi arrangement $\text{Shi}$ to any $i \in \mathcal{I}$. To state his results, let us recall some notations concerning hyperplane arrangements.

A hyperplane arrangement is a finite collection of affine hyperplanes in a Euclidean space. For example, the Shi arrangement associated with $\Delta^+$ is the arrangement in $V$ defined by

(1.1) \[ \text{Shi} := \{ H_{\alpha,0} \mid \alpha \in \Delta^+ \} \cup \{ H_{\alpha,1} \mid \alpha \in \Delta^+ \}. \]

Here for $\alpha \in \Delta^+$ and $k \in \mathbb{Z}$, we define an affine hyperplane

(1.2) \[ H_{\alpha,k} := \{ v \in V \mid (v|\alpha) = k \}. \]

This arrangement was defined by Shi in the study of the Kazhdan-Lusztig cellular structure of the affine Weyl group of type $A$, see Chapter 7 of [9]. In general, if $\mathcal{A}$ is a hyperplane arrangement in $V$, the connected components of $V - \bigcup_{H \in \mathcal{A}} H$ are called regions. We refer the reader to Section 3.11 of [11] for basics about hyperplane arrangements. For $\text{Shi}$, a region is called dominant if it is contained in $V_\infty$, where

(1.3) \[ V_\infty := \{ v \in V \mid (v|\alpha) > 1, \forall \alpha \in \Delta^+ \} \]

Now let us recall the Shi bijection from Theorem 1.4 of [10] as follows:

**Theorem 1.1 (Shi)** There exists a natural bijective map from the set of all the ad-nilpotent ideals of $\mathfrak{b}$ to the set of all the dominant regions of the hyperplane arrangement $\text{Shi}$. The map sends $i$ to $V_i := \{ v \in V \mid (v|\beta) > 1, \forall \beta \in \Phi(i); 0 < (v|\beta) < 1, \forall \beta \in \Delta^+ \setminus \Phi(i) \}$. Here $\Phi(i)$ denotes the set of roots in $i$.

It is easy to see that the dominant region of $\text{Shi}$ corresponding to the zero ideal coincides with the fundamental alcove of the affine Weyl group $\hat{W}$, which is defined as

(1.4) \[ A := \{ v \in V \mid (v|\alpha_i) > 0 \text{ for } i = 1, \ldots, l \text{ and } (v|\theta) < 1 \}, \]

where $\theta$ is the maximal positive root. Note that the Shi bijection is recently generalized to parabolic subalgebras in [3].

Theorem 1.1 gives us one way to visualize the set $\mathcal{I}$. It is then natural to ask for an according characterization of the abelian ideals among all the ad-nilpotent ones. A difficulty lies in that the region $V_i$ may be unbounded, even if $i$ is abelian. Indeed, we have

**Proposition 1.2** For any $i \in \mathcal{I}$, the following are equivalent:

(a) $i$ is abelian;

(b) $V_i \cap 2A$ is nonempty;

(c) $\text{vol}(V_i \cap 2A) = \text{vol}(A)$. Here "vol" means taking the volume.
Another theme of the current paper comes from [6], where Kostant found a connection between the commutative Lie subalgebras of \( \mathfrak{g} \) and the maximal eigenspaces of the Casimir operator \( \text{Cas} \in U(\mathfrak{g}) \) on certain degrees of \( \wedge \mathfrak{g} \). To be more precise, let \( p \) denote the maximal dimension of all the abelian subalgebras of \( \mathfrak{g} \). Let \( m_k \) be the maximal eigenvalue of \( \text{Cas} \) on \( \wedge^k \mathfrak{g} \) and \( M_k \) be the corresponding eigenspace. Then

\[
m_k \leq k,
\]

and equality holds if and only if \( k \leq p \). Moreover, in such a case, \( M_k \) is a multiplicity-free \( \mathfrak{g} \)-module whose highest weight vectors correspond to \( k \)-dimensional abelian subalgebras of \( \mathfrak{g} \) (or equivalently, \( k \)-dimensional abelian ideals of \( \mathfrak{b} \)). In (1.2) of [7], Kostant mentioned that

\[
m = \frac{1}{3} \dim \mathfrak{g},
\]

where \( m \) denotes the maximal eigenvalue of \( \text{Cas} \) on \( \wedge \mathfrak{g} \), i.e., \( m \) is the maximum of \( m_k \) over all \( k \). Recently, among other things, Han [4] determined the eigenspaces relative to \( m \). To state the result, let \( r \) denote the number of positive roots and put \( l = \text{rank} \mathfrak{g} \). For any dominant integral linear form \( \lambda \) on \( \mathfrak{h} \), let \( V_\lambda \) be the irreducible \( \mathfrak{g} \)-module with highest weight \( \lambda \). Then, it is proved in [4] that

(i) one has \( m_k \leq \frac{1}{3} \dim \mathfrak{g} \), and equality holds precisely for \( k = r, r+1, \cdots, r+l \). Moreover, for \( 0 \leq s \leq l \), \( M_{r+s} = C_s V_{2\rho} \);

(ii) For \( 0 \leq k \leq r \), one has \( m_k < m_{k+1} \) and \( \bigoplus_{k=0}^{r} M_k \) is a multiplicity-free \( \mathfrak{g} \)-module.

We find that \( m_{r-1} \) and the corresponding eigenspace \( M_{r-1} \) can be explicitly determined. Let \( \Pi \) be the set of all short simple roots. We view every root as a short one if \( \mathfrak{g} \) is simply-laced, i.e., \( \mathfrak{g} \) is of type \( A, D, E \). Recall that the important integer

\[
h^\vee = \frac{1}{(\theta|\theta)}
\]

is the dual Coxeter number. Let \( \alpha \) be any short root, then

\[
k = \frac{(\theta|\theta)}{(\alpha|\alpha)}
\]

is a positive integer. Now we are able to state

**Theorem 1.3** Let \( \mathfrak{g} \) be a finite-dimensional complex simple Lie algebra. With notations as above, one has

\[
m_{r-1} = \frac{1}{3} \dim \mathfrak{g} - \frac{2}{kh^\vee},
\]

and

\[
M_{r-1} = \bigoplus_{\alpha \in \Pi} V_{2\rho - \alpha}.
\]

Moreover, for any \( \alpha \in \Pi \) which is not necessarily short, the irreducible \( \mathfrak{g} \)-module \( V_{2\rho - \alpha} \) occurs \( 2^l \) times in \( \wedge \mathfrak{g} \), which are distributed as follows:
Here the first row denotes dimension, and the second row denotes the according times of occurrence of $V_{2p-\alpha}$.

The paper is organized as follows: after collecting necessary preliminaries in Section 2, we will prove Proposition 1.2 in Section 3, and Theorem 1.3 in Section 4.

2. Preliminaries

We extend $V$ and its canonical inner product by setting $\hat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$, $(\delta|\delta) = (\delta|V) = (\lambda|\lambda) = (\lambda|V) = 0$, and $(\delta|\lambda) = 1$. The affine root system associated to $\Delta$ is $\hat{\Delta} = \Delta + \mathbb{Z}\delta$. The set of positive affine roots is $\hat{\Delta}^+ = (\Delta^+ + N\delta) \cup (\Delta^- + N^+\delta)$, where $\Delta^- = -\Delta^+$. We denote by $\alpha_0 = -\theta + \delta$ and set $\hat{\Pi} = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$.

For each $\alpha \in \hat{\Delta}^+$, let $s_\alpha$ be the corresponding reflection of $\hat{V}$. The affine Weyl group $\hat{W}$ associated to $\Delta$ is the group generated by $\{s_\alpha|\alpha \in \hat{\Delta}^+\}$. Now consider the $\hat{W}$-invariant affine subspace $E := V \oplus \mathbb{R}\delta + \lambda$. Let $\pi : E \rightarrow V$ be the natural projection. For $w \in \hat{W}$, set $\bar{w} = \pi \circ w|_E$. Then the map $w \mapsto \bar{w}$ gives an isomorphism of $\hat{W}$ to a group of affine translations $W_{af}$ of $V$. Thus we may and we will omit the bar, and use the same notation for the elements in $\hat{W}$ and the corresponding elements in $W_{af}$. We refer the reader to [1] and the references therein for more about affine Weyl groups.

For any $\alpha \in \Delta^+$, $k \in \mathbb{N}^+$, $h \in \mathbb{N}$, by (1.1) of [1], we have

$$
\begin{align*}
(2.1) \quad & w^{-1}(-\alpha + k\delta) < 0 \text{ if and only if } H_{\alpha,k} \text{ separates } A \text{ and } w(A), \\
(2.2) \quad & w^{-1}(\alpha + h\delta) < 0 \text{ if and only if } H_{\alpha,-h} \text{ separates } A \text{ and } w(A).
\end{align*}
$$

3. Proof of Proposition 1.2

This section is devoted to the proof of Proposition 1.2. Firstly, let us show that for any $i \in \mathcal{I}$, the following are equivalent:

(a') the index of nilpotence of $i$ equals to $k$;

(b') $V_i \cap kA$ is empty and $V_i \cap (k+1)A$ is non-empty.

Recall that the index of nilpotence of $i$, denoted by $n(i)$, is the number of nonzero terms of the descending central series of $i$:

$$
i^1 = i, i^2 = [i, i], \ldots, i^k = [i^{k-1}, i], \ldots
$$

For example, $n(0) = 0$ and $n(i) = 1$ for any nonzero abelian ideal $i \in \mathcal{I}$. For any $i \in \mathcal{I}$, let $\Phi(i^k)$ be the roots of $i^k$, $k \in \mathbb{N}^+$. Put

$$
L_i = \bigcup_{k \geq 1} (-\Phi(i^k) + k\delta).
$$
Recall that by Theorem 2.6 of [1], for any ad-nilpotent ideal $i$ of $b$, there exists a unique $w_i \in \hat{W}$ such that $L_i = N(w_i)$. Here, $N(w) := \{ \alpha \in \hat{\Delta}^+ | w^{-1}(\alpha) < 0 \}$ for any $w \in \hat{W}$. The equivalence of (a') and (b') will follow from the following two lemmas.

**Lemma 3.1** Let $i \in I$ with $n(i) = k$. Then $V_i \cap kA$ is empty.

**Proof.** Take any $x \in V_i$, it suffices to show that $(x|\theta) > k$. Indeed, since $n(i) = k$, we can find $k$ roots $\varphi_1, \ldots, \varphi_k$ of $i$ such that $\varphi_1 + \cdots + \varphi_k$ is still a root of $i$. Then $(x|\theta) \geq (x|\varphi_1 + \cdots + \varphi_k) = (x|\varphi_1) + \cdots + (x|\varphi_k) > k$. \hfill \qed

**Lemma 3.2** Let $i \in I$ with $n(i) = k$. Then

$$(3.1) \quad w_i(A) \subseteq V_i \cap (k+1)A.$$  

In particular, $V_i \cap (k+1)A$ is nonempty.

**Proof.** Take an arbitrary $\alpha \in \Delta^+$. Since $n(i) = k$, we have

$$-\alpha, -\alpha + (k+1)\delta \notin L(i) = N(w_i).$$

Thus, neither $H_{a,0}$ nor $H_{a,k+1}$ separates $A$ and $w_i(A)$ by (2.2). Therefore, $w_i(A) \subseteq (k+1)A$. On the other hand,

$$\alpha \in \Phi(i) \iff -\alpha + \delta \in L_i = N(w_i) \iff H_{a,1} \text{ separates } A \text{ and } w_i(A).$$

Therefore, $w_i(A) \subseteq V_i$. \hfill \qed

Specializing $k$ to be 0 and 1, the equivalence of (a') and (b') implies that of (a) and (b) in Proposition 1.2. The equivalence of (a) and (c) in Proposition 1.2 follows from the following

**Lemma 3.3** Let $a \in I$ be abelian. Then $\text{vol}(V_a \cap 2A) = \text{vol}(A)$.

**Proof.** By (3.1), we have

$$\sum_{a \text{ abelian}} \text{vol}(w_a(A)) \leq \sum_{a \text{ abelian}} \text{vol}(V_a \cap 2A).$$

We will show that both sides equal to $\text{vol}(2A)$. Then for each abelian $a$, we would have $\text{vol}(V_a \cap 2A) = \text{vol}(w_a(A)) = \text{vol}(A)$, as desired.

For the RHS, since $V_i \cap 2A$ is empty when $n(i) > 1$, we have

$$\sum_{a \text{ abelian}} \text{vol}(V_a \cap 2A) = \sum_{i \in I} \text{vol}(V_i \cap 2A) = \text{vol}(V_\infty \cap 2A) = \text{vol}(2A).$$

For the LHS, the proof of Theorem 2.9 [1] shows that for any $w \in \hat{W}$, $w(A) \subseteq 2A$ if and only if $w = w_a$ for some abelian ideal $a$ of $b$. Thus, we have

$$\sum_{a \text{ abelian}} \text{vol}(w_a(A)) = \text{vol}(2A).$$ \hfill \qed
4. The maximal eigenvalue of \( \text{Cas} \) on \( \wedge^{r-1} \mathfrak{g} \)

This section is devoted to determining the maximal eigenvalue of the Casimir operator on \( \wedge^{r-1} \mathfrak{g} \), where \( r \) is the number of positive roots. We begin with some preliminaries.

Let \( \wedge \mathfrak{g} \) be the exterior algebra. Then the adjoint action of \( \mathfrak{g} \) extends naturally to \( \wedge \mathfrak{g} \). One knows that, as \( \mathfrak{g} \)-modules, \( \wedge^k \mathfrak{g} \) is isomorphic to \( \wedge^{n-k} \mathfrak{g} \) for each \( k \).

Therefore,

\[
m_k = m_{n-k}
\]

and

\[
M_k \cong M_{n-k}.
\]

Recall that \( m_k \) is the maximal eigenvalue of \( \text{Cas} \) on \( \wedge^k \mathfrak{g} \) and \( M_k \) is the corresponding eigenspace. Note also that, as \( \mathfrak{g} \)-modules one has (see [8])

\[
\wedge \mathfrak{g} = 2^l V_\rho \otimes V_\rho.
\]

Lemma 4.1 For any simple root \( \alpha \), \( V_{2\rho-\alpha} \) occurs exactly once in \( V_\rho \otimes V_\rho \).

Proof. For any simple root \( \alpha \), let \( s_\alpha \) be the simple reflection relative to \( \alpha \). Then \( s_\alpha \rho = \rho - \alpha \) is an extreme weight of \( V_\rho \) and it is easy to check that \( \rho - \alpha \) is dominant for \( \Delta^+ \). Thus by the Steinberg formula (cf. Theorem 24.4 of [5]), the multiplicity of \( V_{2\rho-\alpha} \) in \( V_\rho \otimes V_\rho \) is equal to

\[
\sum_{\sigma \in W} \sum_{\tau \in W} \text{sign}(\sigma \tau) p(2\sigma \rho + 2\tau \rho - 4\rho + \alpha),
\]

where \( p \) is the Kostant partition function. Note that

\[
\sigma \rho = \rho - \langle \Phi(\sigma) \rangle,
\]

where \( \Phi(\sigma) = \sigma(\Delta^+) \cap \Delta^+ \) and \( \langle \Phi(\sigma) \rangle \) denotes the sum of roots in \( \Phi(\sigma) \). Similarly, we have \( \tau \rho = \rho - \langle \Phi(\tau) \rangle \). Thus the multiplicity equals to

\[
\sum_{\sigma \in W} \sum_{\tau \in W} \text{sign}(\sigma \tau) p(\alpha - 2\langle \Phi(\sigma) \rangle - 2\langle \Phi(\tau) \rangle).
\]

One sees easily that in the above sum the unique nonzero term occurs only when \( \sigma = \tau = 1 \). Thus the concerned multiplicity equals to one.

Now let us recall some notation and results from [6]. An element \( w \in \wedge^k \mathfrak{g} \) is called decomposable if \( w = z_1 \wedge \cdots \wedge z_k \), where \( z_i \in \mathfrak{g} \). In such a case, we let \( a(w) \)
be the subspace spanned by \( z_1, \ldots, z_k \). Now a \( \mathfrak{g} \)-submodule \( V_1 \subseteq \wedge^k \mathfrak{g} \) is called \textit{decomposably-generated} if it is spanned by decomposable elements.

A subspace \( \mathfrak{a} \subseteq \mathfrak{g} \) is called \textit{\( \mathfrak{b} \)-normal} if \( [\mathfrak{b}, \mathfrak{a}] \subseteq \mathfrak{a} \). In such a case, \( \mathfrak{a} \) is stable under \( \text{ad} \mathfrak{h} \) and there exists a unique set of roots \( \Delta(\mathfrak{a}) \subseteq \Delta \) such that

\[
\mathfrak{a} = \mathfrak{a} \cap \mathfrak{h} + \sum_{\alpha \in \Delta(\mathfrak{a})} \mathfrak{g}_\alpha.
\]

Define \( (\mathfrak{a}) = \sum_{\alpha \in \Delta(\mathfrak{a})} \alpha \).

Proposition 4 of [6] asserts that each \( M_k \) is a direct sum of decomposably-generated simple \( \mathfrak{g} \)-submodules of \( \wedge^k \mathfrak{g} \). Proposition 6 of [6] establishes a one-one correspondence between the set of all \( k \)-dimensional \( \mathfrak{b} \)-normal subspaces of \( \mathfrak{g} \) and all decomposably-generated simple \( \mathfrak{g} \)-submodules of \( \wedge^k \mathfrak{g} \). The correspondence can be described as follows: starting with a \( k \)-dimensional \( \mathfrak{b} \)-normal subspace \( \mathfrak{a} \) of \( \mathfrak{g} \), then the cyclic submodule \( V_\mathfrak{a} \) of \( \wedge^k \mathfrak{g} \) generated by \( \wedge^k \mathfrak{a} \) is irreducible, decomposably-generated. Actually, \( V_\mathfrak{a} \cong V(\mathfrak{a}) \) and \( \wedge^k \mathfrak{a} \) is the highest weight vector. Thus, to determine \( m_k \) and \( M_k \), it suffices to consider all the \( k \)-dimensional \( \mathfrak{b} \)-normal subspaces of \( \mathfrak{g} \).

\textbf{Proof of Theorem 1.3}. As mentioned above, to investigate \( m_{r-1} \) and \( M_{r-1} \), it suffices to focus on the \((r-1)\)-dimensional \( \mathfrak{b} \)-normal subspaces \( \mathfrak{a} \) of \( \mathfrak{g} \). As shown in the proof of Theorem 3.2(3) of [4], for \( V_\mathfrak{a} \) to lie in \( M_{r-1} \), we must have \( \Delta(\mathfrak{a}) \subseteq \Delta^+ \).

Since \( \dim \mathfrak{a} = r - 1 \) and \( \mathfrak{a} \) is \( \mathfrak{b} \)-normal, we must have \( \Delta(\mathfrak{a}) = \Delta^+ \setminus \{ \alpha \} \), where \( \alpha \) is a simple root. Since

\[
\text{Cas}(2\rho - \alpha) = \|3\rho - \alpha\|^2 - \|\rho\|^2 = 8\|\rho\|^2 - 2\|\alpha\|^2 = \frac{1}{3}\dim \mathfrak{g} - 2\|\alpha\|^2
\]

for any simple root \( \alpha \), we see that

\[
m_{r-1} = \frac{1}{3}\dim \mathfrak{g} - 2\|\alpha\|^2
\]

for any \textit{short} simple root \( \alpha \in \Pi_s \). Now (1.8) follows from (1.6) and (1.7).

Note also that \( V_{2\rho - \alpha}, \alpha \in \Pi_s \), are precisely the irreducible components of \( M_{r-1} \). Now for any \( \alpha \in \Pi_s \), it remains to figure out the multiplicity \( n_\alpha \) of \( V_{2\rho - \alpha} \) in \( M_{r-1} \), which is also the multiplicity of \( V_{2\rho - \alpha} \) in \( \wedge^{r-1}\mathfrak{g} \). One way to do this is to cite Han’s result [4], which says that \( M_{r-1} \) is multiplicity free. Thus \( n_\alpha = 1 \).

We give another approach. By (4.3) and Lemma 4.1, \( V_{2\rho - \alpha} \) occurs \( 2^l \) times in \( \wedge^r \mathfrak{g} \). Let us determine these occurrences explicitly. Let \( h_1, \ldots, h_{l-1} \) be a basis of \( \mathfrak{h}^* := \{ x \in \mathfrak{h} \mid \alpha(x) = 0 \} \). Let \( u_\alpha \) be the wedge product of all the \( r \) positive root vectors but \( e_\alpha \). Let \( v_\alpha = u_\alpha \wedge [e_\alpha, e_{-\alpha}] \wedge e_\alpha \wedge e_{-\alpha} \). Then it is apparent that, for \( 0 \leq s \leq l - 1 \),

\[
u_\alpha \wedge h_{j_1} \wedge \cdots \wedge h_{j_s}, v_\alpha \wedge h_{j_1} \wedge \cdots \wedge h_{j_s}, 1 \leq j_1 < \cdots < j_s \leq l - 1
\]

generate \( 2^l \) copies of \( V_{2\rho - \alpha} \) in \( \wedge^r \mathfrak{g} \). Thus they exhaust all the occurrences of \( V_{2\rho - \alpha} \) in \( \wedge^r \mathfrak{g} \). In particular \( n_\alpha = 1 \), we have (1.9), and the desired distribution of \( V_{2\rho - \alpha} \)
in \( \wedge \mathfrak{g} \) follows. Finally, it is easy to see that the same distribution also holds for any simple root which is not necessarily short.

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