ON SOME PROPERTIES OF M-HYPERCYCLIC C_0-SEMIGROUP

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Abstract. In this article, we justify that every separable infinite dimensional complex Banach space admits an M-hypercyclic C_0-semigroups. Also, we prove that if $(T_t)_{t\geq 0}$ is an M-hypercyclic C_0- semigroup of a generator A acting on such X then the point spectrum $\sigma_p(T_t^*)$ may be empty or not. On other hand we introduce the concept of diskcyclic, M-diskcyclic and M-disk transitive C_0-semigroup; we obtain some proprieties as well as an important characterization for these notions.

Keywords: hypercyclicity, M-hypercyclicity, C_0-semigroup, point spectrum, topologically transitive, diskcyclic operator, M-Diskcyclic operator.

1. Introduction

Let $B(X)$ be the complex Banach algebra of all bounded linear operators on a infinite dimensional complex separable Banach space X. Recall that an element $T \in B(X)$ is called hypercyclic if there exists a vector $x \in X$ such that the orbit $Orb(T,x) = \{T^n x : n \geq 0\}$ is dense in X (in this case x is called a hypercyclic vector). For more details about hypercyclic operators and their proprieties see [5] and [7].

Analogously, a C_0- semigroup $\mathcal{T} = (T_t)_{t\geq 0} \subset B(X)$ is called hypercyclic if there exists a vector $x \in X$ such that the orbit of $\mathcal{T}$ $Orb(\mathcal{T}, x) = \{T_t x : t \geq 0\}$ is dense in X (such a vector is called a hypercyclic vector for $\mathcal{T}$). The set of all hypercyclic vectors of $\mathcal{T}$ is denoted by $HC(\mathcal{T})$.

Desch, Schappacher and Webb [5] have shown that if $\mathcal{T}$ is a hypercyclic C_0-semigroup of a generator A, then the generator A^* (the adjoint of A) of the dual semigroup C_0-semigroup $\mathcal{T}^* = (T_t^*)_{t\geq 0}$ verifies the following conditions

1. If $\phi \in X^*\backslash\{0\}$, then $Orb(\mathcal{T}^*, \phi) = \{T_t^*\phi : t \geq 0\}$ is unbounded.
2. the point spectrum $\sigma_p(A^*) = \emptyset$.

T. Kalmes in [6] Showed that if $\mathcal{T}$ is a C_0-semigroup of a generator A such that the resolvent set $\rho(A) \neq \emptyset$ and if the injection $i : (D(A), \|.\|_A) \hookrightarrow (X, \|.\|)$ is compact then $\mathcal{T}$ is not hypercyclic.

At 1941, Oxtoby and Ulam [10] showed that, if $\mathcal{T}$ is a hypercyclic C_0-semigroup and $x \in HC(\mathcal{T})$, then there exists a G_δ-dense set I of $\mathbb{R}^+$ such that $x \in HC(T_t)$ for all $t \in I$ (by Baire's terminology a G_δ set is an intersection of countable collection of open subsets).

Recently, Conejero, Peris and Müller in [4] established a more general result. They proved that if $\mathcal{T}$ is a C_0-semigroup then we have the following equivalences:

1. $\mathcal{T}$ is hypercyclic.
2. There exists $t > 0$, such that T_t is hypercyclic.
3. For all $t > 0$, T_t is hypercyclic.

Recall that Peris (see [7]) proved that if $\mathcal{T}$ is a hypercyclic C_0-semigroup of a generator A then the range $R(T_t - \lambda I)$ is dense in X for every $t > 0$, $\lambda \in \mathbb{C}$, $\sigma_p(T_t^*) = \emptyset$ and $\sigma_p(A^*) = \emptyset$.

At 2011, Madore and Martinez-Avendano [9] introduced a new concept related of hypercyclicity it called subspace-hypercyclicity. We say that an operator $T \in B(X)$ is M-hypercyclic if there is a nonzero vector subspace M and a vector x of X such that $Orb(T, x) \cap M$ is dense in M, the same authors also gave the concept of M-transitivity and they showed that each M-transitive operator is M-hypercyclic and that the converse is false. Le in [8] and Talebi and Asadipour in [13] gave an example of operator which is M-hypercyclic but not M-transitive, for more details about relationship between M-hypercyclicity and M-transitivity and others properties see [8], [9], [11], [13].

Next, Recall that if $T \in B(X)$ and $S \in B(X)$ then the operator $T \oplus S$ is defined on the Banach spaces

$$X \oplus X := \{(x, y) : x \in X, y \in X\} = \{x \oplus y : x \in X, y \in X\}$$

endowing by the product topology by

$$(T \oplus S)(x \oplus y) = Tx \oplus Sy = (Tx, Sy).$$

We have the following properties [7]:

1. For all $n \in \mathbb{N}, (T \oplus S)^n = T^n \oplus S^n$.
2. $(T \oplus S)^* = T^* \oplus S^*$.
3. If $T \oplus S$ is hypercyclic, then T and S are hypercyclic.

On other hand, Zeana in [15] introduced and studied the notion of a diskcyclicity. An operator $T \in B(X)$ is called diskcyclic if there is a vector $x \in X$ such that the disc orbit $\mathbb{D}Orb(T, x) = \{\alpha T^n x : \alpha \in \mathbb{C}, |\alpha| \leq 1, n \geq 0\}$ is dense in X, in this case the vector x is called diskcyclic for T. For a systematic investigation, in [3] Bamerni and al introduced the concept of M-diskcyclicity and M-disk transitivity. An operator $T \in B(X)$ is called M-diskcyclic if there is non-trivial subspace M of X and a vector $x \in X$ such that $\mathbb{D}Orb(T, x) \cap M$ is dense in M, Such a vector x is called a M-diskcyclic vector for T.

More recently, Azimi in [2] studied the concept of diskcyclicity and its relevant criteria for a sequence $(T_n)_{n\geq 0}$ of bounded operators between separable Banach spaces and M-diskcyclicity for $(T_n)_{n\geq 0} \subset B(X)$ where X is a separable infinite dimensional Hilbert space.

In this article, we continue [12] to study some proprieties for the M-hypercyclicity of C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$. In section two, we know that if $(T_t)_{t\geq 0}$ is a hypercyclic C_0- semigroup of a generator A then T_t is hypercyclic for every $t > 0$, $\sigma_p(T_t^*) = \emptyset$ and $\sigma_p(A^*) = \emptyset$. We will show that it is not the case when $(T_t)_{t\geq 0}$ is a M-hypercyclic C_0-semigroup, precisely we give some examples proving that if $\mathcal{T}$ is M-hypercyclic then $\sigma_p(T_t^*)$ may be empty or not. Again, we justify that every infinite dimensional separable Banach space admits a M -hypercyclic C_0-semigroup. In section 3 we introduce the notion of diskcyclicity, M-diskcyclicity and M-disk transitivity for C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$. We shows that if $\mathcal{T}$ is M-disk transitive then $\mathcal{T}$ is M-diskcyclic and we prove that the set $\mathcal{DC}(\mathcal{T}, M)$ of all M-diskcyclic vectors of $\mathcal{T}$ is dense in X. Finally some characterizations of M-disk transitivity for $\mathcal{T}$ are given.

2. Point spectrum of M-hypercyclic C_0-semigroup

Let us recall some definitions and theorem from [12].

Definition 2.1. [12] Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup and M is a nonzero subspace of X. $\mathcal{T}$ is called *M-hypercyclic* if there is a vector x of X such that $Orb(\mathcal{T}, x) \cap M$ is dense in M with $Orb(\mathcal{T}, x) = \{T_t x : t \geq 0\}$.

Definition 2.2. [12] Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup on X and M is a nonzero subspace of X. We say that $\mathcal{T}$ is *M-transitive* if for every non-empty open sets U and V of M there is $t \geq 0$ such that $T_t^{-1}(U) \cap V$ is a non-empty open set of M.

Theorem 2.1. [12] *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup and $M \neq \{0\}$ a subspace of X. If $\mathcal{T}$ is M-transitive, then it is M-hypercyclic.*

In the sequel, we will need the following important propositions.

Proposition 2.1. *Let $(A_t)_{t\geq 0}$ be a C_0-semigroup of a generator A on X. Let $T_t = A_t \oplus I$ for all $t \geq 0$. Then we have*

1. *$(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus X$.*
2. *The generator of $(T_t)_{t\geq 0}$ is the operator T defined on $D(T) = D(A) \oplus X$ such that $T(x \oplus y) = Ax \oplus 0$, for all $x \in D(A)$, $y \in X$.*

Proof. 1. Since $(A_t)_{t\geq 0}$ is a C_0-semigroup, then

$$T_0 = A_0 \oplus I = I \oplus I = I_{X\oplus X}.$$

Let $x \oplus y \in X \oplus X$ and $t, s \geq 0$, we have

$$T_{t+s}(x \oplus y) = A_{t+s}x \oplus y = A_t A_s(x) \oplus y = T_t(A_s x \oplus y) = T_t T_s(x \oplus y).$$

On the other hand,

$$\lim_{t\to s} T_t(x \oplus y) = \lim_{t\to s}(A_t x \oplus y) = A_s x \oplus y = T_s(x \oplus y).$$

Therefore, $(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus X$.

2. Let $x \in D(A)$ and $y \in X$ we have:

$$\begin{aligned} \lim_{t\to 0} \frac{T_t(x \oplus y) - (x \oplus y)}{t} &= \lim_{t\to 0} \frac{A_t x \oplus y - x \oplus y}{t} \\ &= \lim_{t\to 0} \frac{(A_t x - x) \oplus 0}{t} \\ &= Ax \oplus 0. \end{aligned}$$

Then $D(T) = D(A) \oplus X$ and $T(x \oplus y) = A(x) \oplus 0$, for all $x \in D(A)$. ∎

Proposition 2.2. *Let $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ be two C_0-semigroups on X of generators respectively A and B. We set $T_t = A_t \oplus B_t$ for all $t \geq 0$. Then we have*

1. *$(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus X$.*
2. *The generator of $(T_t)_{t\geq 0}$ is the the operator T defined on $D(T)=D(A)\oplus D(B)$ by $T(x \oplus y) = Ax \oplus By$.*

Proof. 1. Let $x \oplus y \in X \oplus X$ and $t, s \geq 0$. Since $(A_t)_{t\geq 0}$ and $(B_t)_{\geq 0}$ are a C_0-semigroups then we have:

$$T_0(x \oplus y) = A_0 x \oplus B_0 y = Ix \oplus Iy = x \oplus y,$$

thus $T_0 = I \oplus I = I_{X\oplus X}$. Next,

$$\begin{aligned} T_{t+s}(x \oplus y) &= A_{t+s}x \oplus B_{t+s}y \\ &= A_t A_s x \oplus B_t B_s y \\ &= (A_t \oplus B_t)(A_s x \oplus B_s y) \\ &= T_t T_s(x \oplus y). \end{aligned}$$

Consequently $T_{t+s} = T_tT_s$. On the other hand,

$$\begin{aligned}\lim_{t\to s} T_t(x \oplus y) &= \lim_{t\to s}(A_tx \oplus B_ty) \\ &= A_sx \oplus B_sy \\ &= T_s(x \oplus y).\end{aligned}$$

This implies that $(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus X$.

2. Let $x \in D(A)$ and $y \in D(B)$. We have

$$\begin{aligned}\lim_{t\to 0} \frac{T_t(x \oplus y) - (x \oplus y)}{t} &= \lim_{t\to 0} \frac{(A_tx - x) \oplus (B_ty - y)}{t} \\ &= A(x) \oplus B(y).\end{aligned}$$

Then

$$D(T) = D(A) \oplus D(B) \text{ and } T(x \oplus y) = Ax \oplus By. \qquad \blacksquare$$

Remark 1. It is clear that $D(T)$ is dense in $X \oplus X$ either in Proposition 2.1 or in Proposition 2.2 since $D(A)$ and $D(B)$ are dense in X.

Remark 2. If $\mathcal{T} = (T_t)_{t\geq 0}$ is a hypercyclic C_0-semigroup then the point spectrum $\sigma_p(T_t^*) = \{\lambda \in \mathbb{C} : T_t^* - \lambda I \text{ is not injective }\} = \emptyset$ for every $t > 0$. We will show that it is not always the case when $\mathcal{T}$ is M-hypercyclic.

Remark 3. It is clear that Proposition 2.1 is a particular case for Proposition 2.2 by taking $B_t = t$ for all t and its generator $B = 0$.

Example 1. Let $(A_t)_{t\geq 0}$ be a hypercyclic C_0-semigroup on X of a hypercyclic vector x (such a hypercyclic C_0-semigroup exists according to [1] and [14]). Then the C_0-semigroup $(T_t)_{t\geq 0} = (A_t \oplus I)_{t\geq 0}$ is M-hypercyclic with $M = X \oplus \{0\}$, in addition $\sigma_p(T_t^*) \neq \emptyset$ for every $t > 0$.

Indeed, according to Proposition 2.1, $(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus X$ and since x is a hypercyclic vector of $(A_t)_{t\geq 0}$ then $Orb((A_t), x)$ is dense in X, hence $Orb((A_t), x) \oplus \{0\}$ is dense in $X \oplus \{0\}$.

Since $Orb((T_t), x \oplus 0) \cap M = Orb((A_t), x) \oplus \{0\} \cap X \oplus \{0\} = Orb((A_t), x) \oplus \{0\}$ is dense in $X \oplus \{0\}$. Therefore, $(T_t)_{t\geq 0}$ is $(X \oplus \{0\})$-hypercyclic.

On the other hand, for all $t > 0$ and every $x^* \in X^* \backslash \{0\}$, we have

$$T_t^*(0 \oplus x^*) = A^*(0 \oplus x^*) = 0 \oplus x^*.$$

Consequently, $1 \in \sigma_p(T_t^*)$ for every $t > 0$ and we conclude that $\sigma_p(T_t^*) \neq \emptyset$. $\blacksquare$

Example 2. Let $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ be two hypercyclic C_0-semigroup on X and x be a hypercyclic vector of $(A_t)_{t\geq 0}$. Then the C_0-semigroup $(T_t)_{t\geq 0} = (A_t \oplus B_t)_{t\geq 0}$ is M-hypercyclic with $M = X \oplus \{0\}$. In addition, $\sigma_p(T_t^*) = \emptyset$ for all $t > 0$.

Indeed, by Proposition 2.2, $(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus X$ and we have $Orb((T_t), x \oplus 0) \cap M = Orb((T_t), x) \oplus \{0\} \cap X \oplus \{0\} = Orb((A_t), x) \oplus \{0\}$ which is dense in $X \oplus \{0\}$ because $Orb((A_t), x)$ is dense in X.

On the other hand, we have $\sigma_p(T_t^*) = \emptyset$ for all $t > 0$, otherwise:

Let $\lambda \in \sigma_p(T_t^*)$, then there exists $x^* \oplus y^* \in (X^* \oplus X^*)\backslash\{0\}$ such that $T_t^*(x^* \oplus y^*) = \lambda(x^* \oplus y^*) = \lambda x^* \oplus \lambda y^*$. Then we have $A_t^* x^* \oplus B_t^* y^* = \lambda x^* \oplus \lambda y^*$, on other word $A_t^* x^* = \lambda x^*$ and $B_t^* x^* = \lambda y^*$, therefore $\lambda \in \sigma_p(A_t^*)$ and $\lambda \in \sigma_p(B_t^*)$. Thus $\sigma_p(A_t^*) \neq \emptyset$ and $\sigma_p(B_t^*) \neq \emptyset$. This is a contradiction, because $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are hypercyclic, consequently $\sigma_p(T_t^*) = \emptyset$. ■

Corollairy 2.1. *If $\mathcal{T} = (T_t)_{t\geq 0}$ is a M-hypercyclic C_0-semigroup, then $\sigma_p(T_t^*)$ can be empty or not.*

Remark 4. In the two previous examples we have $\sigma_p(A^*) = \emptyset$ where A is the generator of $(A_t \oplus I)_{t\geq 0}$ in first example and $\sigma_p(A^* + B^*) = \emptyset$ where $A + B$ is the generator of $(A_t \oplus B_t)_{t\geq 0}$ in the second example.

Problem. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a M-hypercyclic C_0-semigroup with $M \neq \{0\}$ a subspace of X. Do we have $\sigma_p(A^*) = \emptyset$ where A is the generator of $\mathcal{T}$?*

Theorem 2.2. [3] *If E is a dense subset of X, then there is a non-trivial closed subspace M of X such that $E \cap M$ is dense in M.*

Corollairy 2.2. *If $\mathcal{T}$ is hypercyclic C_0-semigroup on X, then there is a closed subspace M of X such that $\mathcal{T}$ is M-hypercyclic C_0-semigroup.*

Proof. Since $\mathcal{T}$ is hypercyclic, then there is a vecteur $x \in X$ such that $E = Orb(\mathcal{T}, x)$ is dense in X, by Theorem 2.2 there is a non trivial subspace M of X such that $E \cap M = Orb(\mathcal{T}, x) \cap M$ is dense in M. Then $\mathcal{T}$ is M-hypercyclic. ■

Theorem 2.3. [14] *Every separable infinite dimensional Banach space admits a hypercyclic uniformly continuous semigroup.*

Corollairy 2.3. *Every infinite dimensional separable Banach space admits a subspace-hypercyclic C_0-semigroup.*

Proof. Theorem 2.2 gives in every infinite dimensional separable Banach admits a C_0-semigroup hypercyclic, applying Corollary 2.2 this C_0-semigroup is M-hypercyclic for some M. ■

3. The diskcyclicity and subspace-diskcyclicity of C_0-semigroup

Definition 3.1. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup. We say that $\mathcal{T}$ *is diskcyclic* if there exists a vecteur $x \in X$, such that

$$\mathcal{D}Orb(\mathcal{T}, x) = \{\lambda T_t x : t \geq 0, \lambda \in \mathbb{C}, |\lambda| \leq 1\}$$

is dense in X. The vector x is called a *diskcyclic vector* for $\mathcal{T}$.

We denote the set of all diskcyclic vectors of $\mathcal{T}$ by $\mathcal{DC}(\mathcal{T})$.

Remark 5. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup.

1. If there is a $t_0 \geq 0$ such that T_{t_0} is diskcyclic, then $\mathcal{T}$ is diskcyclic.
2. If $\mathcal{T}$ is hypercyclic then $\mathcal{T}$ is diskcyclic.

Proposition 3.1. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup on X. Then*

$$\mathcal{DC}(\mathcal{T}) = \bigcap_{m\geq 0} \bigcup_{t\geq 0} \bigcup_{|\lambda|\geq 1} T_t^{-1}(\lambda B_m),$$

where $(B_m)_{m\geq 0}$ is a countable open basis for X.

Proof. Of course, if $(B_m)_{m\geq 0}$ is a countable open basis for X then we have

$$\begin{aligned} x \in \mathcal{DC}(\mathcal{T}) &\iff \mathcal{D}Orb(\mathcal{T}, x) \text{ is dense in} X \\ &\iff \mathcal{D}Orb(\mathcal{T}, x) \cap B_m \neq \phi \text{ for all } \ m \geq 0 \\ &\iff \forall m \geq 0, \exists\, t \geq 0,\ \exists \lambda \in \mathbb{C}\ \ |\lambda| \leq 1 \text{ such that } \lambda T_t x \in B_m \\ &\iff \forall m \geq 0, \exists\, t \geq 0,\ \exists \lambda \in \mathbb{C}\ \ |\lambda| \geq 1 \text{ such that } x \in T_t^{-1}(\lambda B_m) \\ &\iff x \in \bigcap_{m\geq 0} \bigcup_{t\geq 0} \bigcup_{|\lambda|\geq 1} T_t^{-1}(\lambda B_m). \end{aligned}$$

■

Proposition 3.2. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup, and let $(\beta_t)_{t\geq 0}$ such that $\beta_t \geq 0$. If $(\beta_t T_t)_{t\geq 0}$ is diskcyclic then $(\alpha_t T_t)_{t\geq 0}$ is diskcyclic for all $\beta_t \leq \alpha_t$.*

Proof. Indeed, since

$$\{\lambda \beta_t T_t x : t \geq 0, \lambda \in \mathbb{C}; |\lambda| \leq 1\} \subseteq \{\lambda \alpha_t T_t x : t \geq 0,\ \lambda \in \mathbb{C}, |\lambda| \leq 1\}.$$ ■

Definition 3.2. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup. We say that $\mathcal{T}$ is disk topologically transitive if for any pair U, V of nonempty open subset of X there are $t \geq 0$ and $\lambda \in \mathbb{C}$ $|\lambda| \leq 1$ such that $T_t(\lambda U) \cap V \neq \phi$.

Theorem 3.1. [3] *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup. $\mathcal{T}$ is diskcyclic if and only if $\mathcal{T}$ is disk topologically transitive.*

Definition 3.3. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup, and M be a nonzero subspace of X. We say that $\mathcal{T}$ is M-diskcyclic if there is a vector $x \in X$ such that $\mathcal{D}Orb(\mathcal{T}, x) \cap M$ is dense in M. The vector x is called the M-diskcyclic vector of $\mathcal{T}$.

The set of all M-dsikcyclic vectors is denote by $\mathcal{DC}(\mathcal{T}, M)$.

Remark 6. If $M = X$ then $\mathcal{T}$ is M-dsikcyclic and $\mathcal{T}$ is diskcyclic.

Example 3. Let $(A_t)_{t\geq 0}$ be a C_0-semigroup diskcyclic on X, with diskcyclic vector x, then the C_0-semigroup $T_t = A_t \oplus I$ is M-diskcyclic with $M = X \oplus \{0\}$. But $(T_t)_{t\geq 0}$ is not diskcyclic.

Proposition 3.3. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup on X and M is a nonzero subspace of X, then*

$$\mathcal{DC}(\mathcal{T}, M) = \bigcap_{m\geq 0} \bigcup_{|\alpha|\geq 1} \bigcup_{t\geq 0} T_t^{-1}(\alpha B_m),$$

where $(B_m)_{m\geq 0}$ is a countable open basis for the relative topology of M.

Proof. If we fix $(B_m)_{m\geq 0}$ a countable open basis for the relative topology of M then

$$\begin{aligned}
x \in \mathcal{DC}(\mathcal{T}, M) &\iff \{\alpha T_t x : |\alpha| \leq 1, t \geq 0\} \cap M \text{ is dense in } M \\
&\iff \forall\, m \geq 0; \{\alpha T_t x : |\alpha| \leq 1, t \geq 0\} \cap B_m \neq \phi \\
&\iff \forall\, m \geq 0; \exists \alpha, |\alpha| \leq 1 \text{ and } t \geq 0 \text{ such that } \alpha T_t x \in B_m \\
&\iff \forall\, m \geq 0; \exists \alpha, 0 < |\alpha| \leq 1 \text{ and } t \geq 0 \text{ such that } x \in T_t^{-1}(\frac{1}{\alpha} B_m) \\
&\iff \forall\, m \geq 0; \exists \alpha, |\alpha| \geq 1 \text{ and } t \geq 0 \text{ such that } x \in T_t^{-1}(\alpha B_m) \\
&\iff x \in \bigcap_{m\geq 0} \bigcup_{|\alpha|\geq 1} \bigcup_{t\geq 0} T_t^{-1}(\alpha B_m)
\end{aligned}$$

The proof is complete. ■

Definition 3.4. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup and M is a nonzero subspace of X. $\mathcal{T}$ is called M-disk transitive if for any nonempty open sets $U, V \subset M$ there exists $\alpha \in \mathbb{C}, |\alpha| \geq 1$ and $t \geq 0$ such that $T_t^{-1}(\alpha U) \cap V$ contains a nonempty open relatively subset of M.

Theorem 3.2. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup on X and M is a nonzero subspace of X. Then the following conditions are equivalent*

1. *$\mathcal{T}$ is M-disk transitive.*
2. *For any nonempty open sets $U, V \subset M$ there exists $\alpha \in \mathbb{C}$, $|\alpha| \geq 1$ and $t \geq 0$ such that $T_t^{-1}(\alpha U) \cap V$ is nonempty and $T_t(M) \subset M$.*
3. *For any nonempty open sets $U, V \subset M$ there exists $\alpha \in \mathbb{C}$, $|\alpha| \geq 1$ and $t \geq 0$ such that $T_t^{-1}(\alpha U) \cap V$ is nonempty open subset of M.*

Proof. (1) ⇒ (2). Let U, V to be nonempty subset of M. From (1) there exists $\alpha \in \mathbb{C}, |\alpha| \geq 1$ and $t \geq 0$ such that $T_t^{-1}(\alpha U) \cap V$ contains a relatively non empty open subset of M. Let $W \subset M$ a non empty open subset such that $W \subset T_t^{-1}(\alpha U) \bigcap V$. Then $T_t^{-1}(\alpha U) \cap V$ is non empty.

Now, we prove that $T_t(M) \subset M$. Indeed, since $W \subset T_t^{-1}(\alpha U) \cap V$ this implies that $T_t(W) \subset \alpha U$ and $W \subset V$, consequently $\frac{1}{\alpha} T_t(W) \subset U \subset M$ and $W \subset V \subset M$.

Next, let $x \in M$. We justify that a fortiori $T_t(x) \in M$. Since $W \neq \emptyset$, put $a \in W$. For $r > 0$ small enough we have $a + rx \in W$, it follows that

$$\frac{1}{\alpha} T_t(a + rx) = \frac{1}{\alpha} T_t(a) + \frac{r}{\alpha} T_t(x) \in \frac{1}{\alpha} T_t(W) \subset M.$$

From $a \in W$ we obtain that

$$\frac{1}{\alpha}T_t(a) \in M \text{ and } \frac{r}{\alpha}T_t(x) \in M,$$

therefore $T_t(x) \in M$, we obtain $T_t(M) \subset M$.

(2) $\Rightarrow$ (3). Let U, V be a non empty open subset of M. By (2) there exists $\alpha \in \mathbb{C}, |\alpha| \geq 1$ and $t \geq 0$ such that $T_t^{-1}(\alpha U) \cap V \neq \emptyset$ and $T_t(M) \subset M$. But $T_t : M \to M$ is continuous and since U is open in M then $T_t^{-1}(\alpha U)$ is open subset in M, thus $T_t^{-1}(\alpha U) \cap V$ is a non empty open subset of M.

(3) $\Rightarrow$ (1) Clear. ∎

Theorem 3.3. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup on X and M is a nonzero subspace of X. If $\mathcal{T}$ is M-disk transitive, then $\mathcal{DC}(\mathcal{T}, M)$ is a dense subset of M.*

Proof. Let $(B_m)_{m\geq 0}$ be a countable open basis for the relative topology of M, since $\mathcal{T}$ is M-disk transitive then from Theorem 3.2 for each i, j, there exists $t_{i,j} \geq 0$ and $\alpha_{i,j} \in \mathbb{C}$, $|\alpha_{i,j}| \geq 1$ such that the set $T_{t_{i,j}}^{-1}(\alpha_{i,j}B_i) \cap B_j \neq \emptyset$. Write $G_i = \bigcup_{j\geq 0} T_{t_{i,j}}^{-1}(\alpha_{i,j}B_i) \cap B_j$. G_i is a nonempty open set of M, by Baire's category theorem $\bigcap_{i\geq 0} G_i$ is dense in M. From Proposition 3.3, it follows that

$$\bigcap_{i\geq 0} G_i = \bigcap_{i} \bigcup_{j\geq 0} (T_{t_{i,j}}^{-1}(\alpha_{i,j}B_i) \bigcap B_j) \subseteq \bigcap_{i\geq 0} \bigcup_{t\geq 0} \bigcup_{|\alpha|\geq 1} T_t^{-1}(\alpha B_i) = \mathcal{DC}(\mathcal{T}, M).$$

We conclude that $\mathcal{DC}(\mathcal{T}, M)$ is dense in M. ∎

Corollairy 3.1. *If $\mathcal{T}$ is M-disk transitive, then $\mathcal{T}$ is M-diskcyclic.*

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