A GENERALIZATION OF SIMPSON TYPE INEQUALITY VIA DIFFERENTIABLE FUNCTIONS USING (s, m)-CONVEX FUNCTIONS

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Abstract. We derive a differentiable mapping integral identity, which is involved with two parameters. By using this result, we establish new inequalities of Simpson type based on (s, m)-convexity for differentiable mappings. This contributes to new better estimates than the earlier results. Finally, some applications to special means of positive real numbers have also been presented.

Keywords: Simpson's inequality; Höder inequality; (s, m)-convex function.

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1. Introduction

Throughout this paper, we consider a real number interval $I \subseteq \mathbb{R}$ and I_0 is the interior of I. We also use the notation: $\mathbb{R} = (-\infty, \infty), \mathbb{R}_0 = [0, \infty)$. We first recite some definitions concerning various kinds of generalized convex functions.

Definition 1.1 ([6]) Let $s \in (0, 1]$ be a real number. A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is said to be s-convex(in the second sense)if

$$f\big(\lambda x + (1 - \lambda)y\big) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

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Definition 1.2 ([15]) For $f : [0, b] \to \mathbb{R}$ and $m \in (0, 1]$, if

$$f\big(\lambda x + m(1-\lambda)y\big) \le \lambda f(x) + m(1-\lambda)f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is m-convex function on $[0, b]$.

Definition 1.3 ([11]) Let $(s, m) \in (0, 1]^2$ be a real number. A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is said to be (s, m)-convex (in the second sense) if

$$f\big(\lambda x + m(1-\lambda)y\big) \le \lambda^s f(x) + m(1-\lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

It can be easily seen that $(s, m) \in \big\{(1, 1), (s, 1), (1, m)\big\}$. One receives the following classes of functions respectively: convex, s-convex and m-convex.

The following inequality is very remarkable and well known in the literature as Simpson type inequality, which plays important roles in analysis. Particularly, it is well applied in numerical integration.

Theorem 1.1 ([3]) *Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $||f^{(4)}||_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\Big[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})\Big] - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2880}||f^{(4)}||_\infty (b-a)^4. \tag{1.1}$$

In recent decades, a lot of inequalities of Simpson type and Hadamard type for various kinds of convex functions have been established and developed by many scholars, some of them may be reformulated as follows.

Theorem 1.2 ([4]) *Suppose that $f : \mathbb{R}_0 \to \mathbb{R}_0$ is an s-convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in \mathbb{R}_0, a < b$. If $f \in L^1[a, b]$, then*

$$2^{s-1} f(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

Theorem 1.3 ([5]) *Let $f : [a, b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a, b]$. Then*

$$\left|\int_a^b f(x)dx - \frac{b-a}{3}\Big[\frac{f(a)+f(b)}{2} + 2f\big(\frac{a+b}{2}\big)\Big]\right| \le \frac{5}{36}L(b-a)^2. \tag{1.3}$$

Theorem 1.4 ([7]) *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \tag{1.4}$$

For some generalizations of (1.4), please refer to [8], [10], [12].

Theorem 1.5 *([13]) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable on I°, where $a, b \in I$ such that $a < b$. If the mapping $|f'|^{p/(p-1)}$ is $(1, m)$-convex on $[a, b]$, for $m \in (0, 1]$ with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}
&\left|\frac{1}{6}\Big[f(ma) + 4f\big(\frac{b+ma}{2}\big) + f(b)\Big] - \frac{1}{b-ma}\int_{ma}^{b} f(x)dx\right| \\
&\le (b-ma)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}} \qquad (1.5)\\
&\times\left[\left(|f'(\frac{b+ma}{2})|^q + |f'(ma)|^q\right)^{\frac{1}{q}} + \left(|f'(\frac{b+ma}{2})|^q + |f'(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

For recent results and generalizations involving Simpson type inequalities and Hadamard's inequalities, we refer to [1], [2], [9], [14].

In the present paper, we explore new inequalities of Simpsons type based on (s, m)-convexity for differentiable mappings. This contributes to new better estimates than the earlier results. Finally, we apply these inequalities to deduce some inequalities of special means.

2. Main results

Before proceeding toward our main theorems regarding of Simpson type inequality using (s, m)-convex function, we need the following integral identity, which will be used in the sequel.

Lemma 2.1 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable on I°, where $a, b \in I$ such that $a < b$. If $f' \in L[a, b]$ and $k, t \in \mathbb{R}$, then the following equality holds:*

$$\begin{aligned}
&tf(ma) + (1-k)f(b) + (k-t)f\big(\frac{b+ma}{2}\big) - \frac{1}{b-ma}\int_{ma}^{b} f(x)dx \\
&= (b-ma)\Big[\int_0^{\frac{1}{2}} (\lambda - t)f'\big(\lambda b + m(1-\lambda)a\big)d\lambda + \int_{\frac{1}{2}}^{1} (\lambda - k)f'\big(\lambda b + m(1-\lambda)a\big)d\lambda\Big]
\end{aligned}$$

for each $x \in [a, b]$.

Proof. We note that

$$J = \int_0^{\frac{1}{2}} (\lambda - t)f'\big(\lambda b + m(1-\lambda)a\big)d\lambda + \int_{\frac{1}{2}}^{1} (\lambda - k)f'\big(\lambda b + m(1-\lambda)a\big)d\lambda.$$

Integrating by parts, it yields that

$$\begin{aligned} J &= \frac{1}{b-ma}\Big[(\lambda-t)f\big(\lambda b+m(1-\lambda)a\big)\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} f\big(\lambda b+m(1-\lambda)a\big)d\lambda\Big] \\ &+ \frac{1}{b-ma}\Big[(\lambda-k)f\big(\lambda b+m(1-\lambda)a\big)\Big|_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} f\big(\lambda b+m(1-\lambda)a\big)d\lambda\Big] \\ &= \frac{1}{b-ma}\Big[tf(ma)+(1-k)f(b)+(k-t)f(\frac{b+ma}{2}) - \int_0^1 f\big(\lambda b+m(1-\lambda)a\big)d\lambda\Big] \end{aligned}$$

Let we substitute $x=\lambda b+m(1-\lambda)a$ and $dx=(b-ma)d\lambda$, which gives

$$\int_0^1 f\big(\lambda b+m(1-\lambda)a\big)d\lambda = \frac{1}{b-ma}\int_{ma}^{b} f(x)dx.$$

Therefore, we can show that

$$J=\frac{1}{b-ma}\Big[tf(ma)+(1-k)f(b)+(k-t)f(\frac{b+ma}{2}) - \frac{1}{b-ma}\int_{ma}^{b} f(x)dx\Big],$$

which is required.

Remark 2.1. Applying Lemma 2.1 for $t=\frac{1}{6}$ and $k=\frac{5}{6}$, we obtain Lemma 2.1 in [13].

In what follows, we establish another refinement of the Simpson's inequality for (s,m)-convex functions in the second sense.

Theorem 2.1 *Let f be defined as in Lemma 2.1. If the mapping $|f'|$ is (s,m)-convex on $[a,b]$, for $(s,m)\in(0,1]^2$ and $0\le k,t\le 1$, then the following inequality holds:*

$$\begin{aligned} &\left|tf(ma)+(1-k)f(b)+(k-t)f\left(\frac{b+ma}{2}\right)-\frac{1}{b-ma}\int_{ma}^{b} f(x)dx\right| \\ &\le (b-ma)\big[v_1|f'(b)|+mv_2|f'(a)|\big], \end{aligned} \tag{2.1}$$

where

$$v_1=\frac{2t^{s+2}+2k^{s+2}+\big[2(s+1)-2(s+2)(k+t)\big]\frac{1}{2^{s+2}}+(s+1-ks-2k)}{(s+1)(s+2)} \tag{2.2}$$

$$v_2=\frac{2(1-t)^{s+2}+2(1-k)^{s+2}+\big[2(k+t)(s+2)-2(s+3)\big]\frac{1}{2^{s+2}}+(ts+2t-1)}{(s+1)(s+2)}. \tag{2.3}$$

Proof. From Lemma 2.1 and using the (s,m)-convexity of $|f'|$, we have

$$\begin{aligned}
&\left|tf(ma)+(1-k)f(b)+(k-t)f\big(\frac{b+ma}{2}\big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right|\\
&\leq (b-ma)\Big[\int_0^{\frac{1}{2}}|\lambda-t||f'\big(\lambda b+m(1-\lambda)a\big)|d\lambda+\int_{\frac{1}{2}}^{1}|\lambda-k||f'\big(\lambda b+m(1-\lambda)a\big)|d\lambda\Big]\\
&\leq (b-ma)\Big[\int_0^{\frac{1}{2}}|\lambda-t|\Big(\lambda^s|f'(b)|+m(1-\lambda)^s|f'(a)|\Big)d\lambda\Big]\\
&+(b-ma)\Big[\int_{\frac{1}{2}}^{1}|\lambda-k|\Big(\lambda^s|f'(b)|+m(1-\lambda)^s|f'(a)|\Big)d\lambda\Big]\\
&=(b-ma)\Big[\int_0^{\frac{1}{2}}|\lambda-t|\lambda^s d\lambda+\int_{\frac{1}{2}}^{1}|\lambda-k|\lambda^s d\lambda\Big]\big|f'(b)\big|\\
&+(b-ma)\Big[\int_0^{\frac{1}{2}}m|\lambda-t|(1-\lambda)^s d\lambda+\int_{\frac{1}{2}}^{1}m|\lambda-k|(1-\lambda)^s d\lambda\Big]\big|f'(a)\big|
\end{aligned}$$

By simple calculations, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}|\lambda-t|\lambda^s d\lambda+\int_{\frac{1}{2}}^{1}|\lambda-k|\lambda^s d\lambda\\
&=\frac{2t^{s+2}+2k^{s+2}+\big[2(s+1)-2(s+2)(k+t)\big]\frac{1}{2^{s+2}}+(s+1-ks-2k)}{(s+1)(s+2)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}}|\lambda-t|(1-\lambda)^s d\lambda+\int_{\frac{1}{2}}^{1}|\lambda-k|(1-\lambda)^s d\lambda\\
&=\frac{2(1-t)^{s+2}+2(1-k)^{s+2}+\big[2(k+t)(s+2)-2(s+3)\big]\frac{1}{2^{s+2}}+(ts+2t-1)}{(s+1)(s+2)}.
\end{aligned}$$

This proves as required.

Corollary 2.1 *Let f be defined as in Theorem 2.1. If the mapping $|f'|$ is (s,m)-convex on $[a,b]$, for $(s,m)\in(0,1]^2$, $t=\frac{1}{6}$ and $k=\frac{5}{6}$. Then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\Big[f(ma)+f(b)+4f\big(\frac{b+ma}{2}\big)\Big]-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right|\\
&\leq (b-ma)\big[w_1|f'(b)|+mw_2|f'(a)|\big],
\end{aligned}\tag{2.4}$$

where

$$w_1=w_2=\frac{6^{-s}-9(2)^{-s}+(5)^{s+2}(6)^{-s}+3s-12}{18(s+1)(s+2)}.\tag{2.5}$$

If $s=1$, *we can get:*

$$\left|\frac{1}{6}\Big[f(ma)+4f\Big(\frac{b+ma}{2}\Big)+f(b)\Big]-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right| \leq \frac{5(b-ma)}{72}\big[m|f'(a)|+|f'(b)|\big]. \tag{2.6}$$

The upper bound of the midpoint inequality for the first derivative is developed as follows:

Corollary 2.2 *By putting* $f(ma)=f\Big(\frac{ma+b}{2}\Big)=f(b)$ *in inequality* (2.1), *we have:*

$$\left|f\Big(\frac{b+ma}{2}\Big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right|\leq (b-ma)\big[v_1|f'(b)|+mv_2|f'(a)|\big], \tag{2.7}$$

where v_1 *and* v_2 *are defined in Theorem* 2.1.

Corollary 2.3 *Putting* $s=1$, $m=1$, $t=\frac{1}{6}$ *and* $k=\frac{5}{6}$ *in the above inequality* (2.7), *it yields that*

$$\left|f\Big(\frac{b+a}{2}\Big)-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq \frac{5(b-a)}{72}\big[|f'(b)|+|f'(a)|\big]. \tag{2.8}$$

Remark 2.2. It is noted that the above midpoint inequality (2.8) is better than the inequality (1.4) presented by Kiramic in [7].

Theorem 2.2 *Let* f *be defined as in Theorem 2.1 with* $\frac{1}{p}+\frac{1}{q}=1$. *If the mapping* $|f'|^{p/(p-1)}$ *is* (s,m)*-convex on* $[a,b]$, *for* $(s,m)\in(0,1]^2$ *and* $p>1$. *Then the following inequality holds:*

$$\begin{aligned}&\left|tf(ma)+(1-k)f(b)+(k-t)f\Big(\frac{b+ma}{2}\Big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right|\\ &\leq (b-ma)\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{s+1}\Big)^{\frac{1}{q}}\Bigg\{\Big[t^{p+1}+\Big(\frac{1}{2}-t\Big)^{p+1}\Big]^{\frac{1}{p}}\Big[|f'\Big(\frac{b+ma}{2}\Big)|^q+|f'(ma)|^q\Big]^{\frac{1}{q}}\\ &+\Big[\Big(k-\frac{1}{2}\Big)^{p+1}+(1-k)^{p+1}\Big]^{\frac{1}{p}}\Big[|f'\Big(\frac{b+ma}{2}\Big)|^q+|f'(b)|^q\Big]^{\frac{1}{q}}\Bigg\}.\end{aligned} \tag{2.9}$$

Proof. Using the the famous Höder integral inequality and by Lemma 2.1, we have

$$\Big|tf(ma)+(1-k)f(b)+(k-t)f\big(\frac{b+ma}{2}\big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\Big|$$
$$\leq(b-ma)\Big[\int_0^{\frac{1}{2}}|\lambda-t||f'\big(\lambda b+m(1-\lambda)a\big)|d\lambda+\int_{\frac{1}{2}}^{1}|\lambda-k||f'\big(\lambda b+m(1-\lambda)a\big)|d\lambda\Big]$$
$$\leq(b-ma)\Bigg\{\Big(\int_0^{\frac{1}{2}}|\lambda-t|^pd\lambda\Big)^{\frac{1}{p}}\Big[\int_0^{\frac{1}{2}}|f'\big(\lambda b+m(1-\lambda)a\big)|^qd\lambda\Big]^{\frac{1}{q}}$$
$$+\Big(\int_{\frac{1}{2}}^{1}|\lambda-k|^pd\lambda\Big)^{\frac{1}{p}}\Big[\int_{\frac{1}{2}}^{1}f'\big(\lambda b+m(1-\lambda)a\big)|^qd\lambda\Big]^{\frac{1}{q}}\Bigg\}.$$

By simple calculations, we acquire

$$\int_0^{\frac{1}{2}}|\lambda-t|^pd\lambda=\frac{t^{p+1}+(\frac{1}{2}-t)^{p+1}}{p+1} \tag{2.10}$$

and

$$\int_{\frac{1}{2}}^{1}|\lambda-k|^pd\lambda=\frac{(k-\frac{1}{2})^{p+1}+(1-k)^{p+1}}{p+1}. \tag{2.11}$$

Also the (s,m)-convexity of $|f'|^{p/(p-1)}$ implies that

$$\int_0^{\frac{1}{2}}|f'\big(\lambda b+m(1-\lambda)a\big)|^qd\lambda\leq\frac{|f'(\frac{b+ma}{2})|^q+|f'(ma)|^q}{s+1} \tag{2.12}$$

and

$$\int_{\frac{1}{2}}^{1}|f'\big(\lambda b+m(1-\lambda)a\big)|^qd\lambda\leq\frac{|f'(\frac{b+ma}{2})|^q+|f'(b)|^q}{s+1}. \tag{2.13}$$

Therefore, by combining (2.10), (2.11), (2.12) and (2.13), these lead to the desired result. The statement in Theorem 2.2 is proved.

Corollary 2.4 *Let f be defined as in Theorem 2.2. If the mapping $|f'|^{p/(p-1)}$ is (s,m)-convex on $[a,b]$, for $(s,m)\in(0,1]^2$ with $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then we have the following inequalities:*

(1) *when $s=1$, we have*

$$\begin{aligned}&\Big|tf(ma)+(1-k)f(b)+(k-t)f\big(\frac{b+ma}{2}\big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\Big|\\&\leq(b-ma)\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big)^{\frac{1}{q}}\Bigg\{\Big[t^{p+1}+(\frac{1}{2}-t)^{p+1}\Big]^{\frac{1}{p}}\Big[|f'(\frac{b+ma}{2})|^q+|f'(ma)|^q\Big]^{\frac{1}{q}}\\&+\Big[(k-\frac{1}{2})^{p+1}+(1-k)^{p+1}\Big]^{\frac{1}{p}}\Big[|f'(\frac{b+ma}{2})|^q+|f'(b)|^q\Big]^{\frac{1}{q}}\Bigg\};\end{aligned} \tag{2.14}$$

(2) *when* $t=\frac{1}{6}$, $k=\frac{5}{6}$ *and* $s=1$, *we can deduce*

$$\begin{aligned}&\left|\frac{1}{6}\Big[f(ma)+4f\big(\frac{b+ma}{2}\big)+f(b)\Big]-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right|\\ &\le(b-ma)\left[\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right]^{\frac{1}{p}}\Big(\frac{1}{2}\Big)^{\frac{1}{q}}\\ &\times\left[\Big(|f'(\frac{b+ma}{2})|^q+|f'(ma)|^q\Big)^{\frac{1}{q}}+\Big(|f'(\frac{b+ma}{2})|^q+|f'(b)|^q\Big)^{\frac{1}{q}}\right],\end{aligned}\tag{2.15}$$

which is the same as inequality (1.5) *established by Qaisar et al. in* [13].

Corollary 2.5 *By putting* $|f'(ma)|=|f'(b)|=0$, $t=\frac{1}{6}$ *and* $k=\frac{5}{6}$ *in Theorem* 2.2, *we have*

$$\begin{aligned}&\left|\frac{1}{6}\Big[f(ma)+4f\big(\frac{b+ma}{2}\big)+f(b)\Big]-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right|\\ &\le 2\frac{(b-ma)}{(s+1)^{\frac{1}{q}}}\left[\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right]^{\frac{1}{p}}\Big|f'(\frac{b+ma}{2})\Big|.\end{aligned}\tag{2.16}$$

In the following corollary, we have the midpoint inequality for powers in terms of the first derivative.

Corollary 2.6 *By substituting* $f(ma)=f(b)=f(\frac{ma+b}{2})$, $t=\frac{1}{6}$ *and* $k=\frac{5}{6}$ *in Theorem* 2.2, *we have*

$$\begin{aligned}&\left|f\big(\frac{b+ma}{2}\big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right|\\ &\le(b-ma)\left[\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right]^{\frac{1}{p}}\Big(\frac{1}{s+1}\Big)^{\frac{1}{q}}\\ &\times\left[\Big(|f'(\frac{b+ma}{2})|^q+|f'(ma)|^q\Big)^{\frac{1}{q}}+\Big(|f'(\frac{b+ma}{2})|^q+|f'(b)|^q\Big)^{\frac{1}{q}}\right].\end{aligned}\tag{2.17}$$

In the following theorem, we obtain another form of Simpson type inequality for powers in term of the first derivative.

Theorem 2.3 *Let* f *be defined as in Theorem* 2.2. *If the mapping* $|f'|^q$ *is* (s,m)*-convex on* $[a,b]$, *for* $(s,m)\in(0,1]^2$ *and* $q\ge 1$. *Then we have the following inequality:*

$$\begin{aligned}&\left|tf(ma)+(1-k)f(b)+(k-t)f\big(\frac{b+ma}{2}\big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right|\\ &\le(b-ma)\left\{\Big(t^2-\frac{1}{2}t+\frac{1}{8}\Big)^{1-\frac{1}{q}}\Big[u_1|f'(b)|^q+mu_2|f'(a)|^q\Big]^{\frac{1}{q}}\right.\\ &\left.+\Big(k^2-\frac{3}{2}k+\frac{5}{8}\Big)^{1-\frac{1}{q}}\Big[u_3|f'(b)|^q+mu_4|f'(a)|^q\Big]^{\frac{1}{q}}\right\},\end{aligned}\tag{2.18}$$

where

$$u_1 = \frac{2t^{s+2} + (s+1-2ts-4t)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$

$$u_2 = \frac{t(s+2) - 1 + 2(1-t)^{s+2} + (2ts+4t-s-3)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$

$$u_3 = \frac{2k^{s+2} + (s+1-2ks-4k)\frac{1}{2^{s+2}} + (s+1-ks-2k)}{(s+1)(s+2)}$$

and

$$u_4 = \frac{2(1-k)^{s+2} + (2ks+4k-s-3)\frac{1}{2^{s+2}}}{(s+1)(s+2)}.$$

Proof. From Lemma 2.1, and using power-mean inequality, it follows that

$$\left|tf(ma) + (1-k)f(b) + (k-t)f\big(\frac{b+ma}{2}\big) - \frac{1}{b-ma}\int_{ma}^{b} f(x)dx\right|$$
$$\leq (b-ma)\left[\int_0^{\frac{1}{2}} |\lambda - t||f'\big(\lambda b + m(1-\lambda)a\big)|d\lambda + \int_{\frac{1}{2}}^{1} |\lambda - k||f'\big(\lambda b + m(1-\lambda)a\big)|d\lambda\right]$$
$$\leq (b-ma)\left[\Big(\int_0^{\frac{1}{2}} |\lambda - t|d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^{\frac{1}{2}} |\lambda - t||f'\big(\lambda b + m(1-\lambda)a\big)|^q d\lambda\Big)^{\frac{1}{q}}\right.$$
$$\left. + \Big(\int_{\frac{1}{2}}^{1} |\lambda - k|d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_{\frac{1}{2}}^{1} |\lambda - k||f'\big(\lambda b + m(1-\lambda)a\big)|^q d\lambda\Big)^{\frac{1}{q}}\right].$$

By simple calculations, we can get

$$\int_0^{\frac{1}{2}} |\lambda - t|d\lambda = t^2 - \frac{1}{2}t + \frac{1}{8} \tag{2.19}$$

and

$$\int_{\frac{1}{2}}^{1} |\lambda - k|d\lambda = k^2 - \frac{3}{2}k + \frac{5}{8}. \tag{2.20}$$

The (s,m)-convexity of $|f'|^q$ implies that

$$\int_0^{\frac{1}{2}} |\lambda - t||f'\big(\lambda b + m(1-\lambda)a\big)|^q d\lambda \leq u_1|f'(b)|^q + mu_2|f'(a)|^q \tag{2.21}$$

and

$$\int_{\frac{1}{2}}^{1} |\lambda - k||f'\big(\lambda b + m(1-\lambda)a\big)|^q d\lambda \leq u_3|f'(b)|^q + mu_4|f'(a)|^q. \tag{2.22}$$

Thus, our desired result can be obtained by combining (2.19), (2.20), (2.21) and (2.22), the proof is completed.

Corollary 2.7 *Let f be defined as in Theorem 2.3, if $s = 1$, $t = \frac{1}{6}$ and $k = \frac{5}{6}$, the inequality holds for m-convex functions:*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(ma) + 4f\big(\frac{b+ma}{2}\big) + f(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)dx\right|\\ &\le (b-ma)\Big(\frac{5}{72}\Big)^{1-\frac{1}{q}}\left[\Big(\frac{29}{1296}|f'(b)|^q + \frac{61}{1296}m|f'(a)|^q\Big)^{\frac{1}{q}}\right.\\ &\left.+\Big(\frac{61}{1296}|f'(b)|^q + \frac{29}{1296}m|f'(a)|^q\Big)^{\frac{1}{q}}\right].\end{aligned} \tag{2.23}$$

In particular, if $s = m = 1$, the inequality holds for convex function. If $|f'(x)| \le L$, $\forall x \in I$, then we have

$$\left|\frac{1}{6}\left[f(a) + 4f\big(\frac{b+a}{2}\big) + f(b)\right] - \frac{1}{b-a}\int_{a}^{b} f(x)dx\right| \le \frac{5(b-a)}{72}L. \tag{2.24}$$

Remark 2.3. It is observed that inequality (2.24) with $m = 1$ gives an improvement for inequality (1.3).

3. Applications to special means

For positive numbers, $\beta > \alpha > 0$ and $n \in \mathbb{Z} \setminus \{0, -1\}$, define $A(\alpha,\ \beta) = \frac{\alpha+\beta}{2}$ and $L_n(\alpha,\ \beta) = \frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta-\alpha)}$. These quantities are respectively called the arithmetic and generalized logarithmic means of two positive number α and β.

Now, considering $f : [a, b] \to \mathbb{R}$, $f(x) = x^s$, $s \in (0, 1]$. Then we have the following means:

$$\frac{1}{b-a}\int_{a}^{b} f(x)dx = L_s^s(a, b), \quad \frac{f(a)+f(b)}{2} = A(a^s, b^s) \ \ \textit{and} \ \ f\big(\frac{a+b}{2}\big) = A^s(a, b).$$

Now, using the results of the Section 2, some new inequalities are derived for the above means.

Proposition 3.1 *Let $f : [a, b] \to R$, $0 < a < b$ and $s \in (0, 1]$. Then we have*

$$\begin{aligned}&\left|\frac{1}{3}A(a^s, b^s) + \frac{2}{3}A^s(a, b) - L_s^s(a, b)\right|\\ &\le s(b-a)\frac{6^{-s} - 9(2)^{-s} + (5)^{s+2}(6)^{-s} + 3s - 12}{18(s+1)(s+2)}\left[|a|^{s-1} + |b|^{s-1}\right].\end{aligned} \tag{3.1}$$

Proof. The assertion follows by taking $m = 1$ and from inequality (2.4) applies to mapping $f(x) = x^s$, $x \in [a, b]$ with $s \in (0, 1]$. Moreover, by setting $s = 1$ in inequality (3.1), we obtain

$$\big|A(a, b) - L(a, b)\big| \le \frac{5(b-a)}{36}. \tag{3.2}$$

Proposition 3.2 *Let $f : [a,b] \to \mathbb{R}$, $0 < a < b$ and $s \in (0,1]$. Then we can conclude*

$$\begin{aligned} &\left|\frac{1}{3}A(a^s,b^s)+\frac{2}{3}A^s(a,b)-L_s^s(a,b)\right| \\ &\le s(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\frac{s}{(s+1)^{1/q}} \\ &\quad\times\left[\left(|a^{s-1}|^q+|A^{s-1}(a,b)|^q\right)^{\frac{1}{q}}+\left(|A^{s-1}(a,b)|^q+|b^{s-1}|^q\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{3.3}$$

Proof. The assertion follows by taking $m = 1$ and from inequality (2.17) applies to mapping $f(x) = x^s$, $x \in [a,b]$ with $s \in (0,1]$. Moreover, by setting $s = 1$ in inequality(3.3), we derive

$$\big|A(a,b)-L(a,b)\big| \le 2(b-a)\Big[\frac{1+2^{p+1}}{6^{p+1}(p+1)}\Big]^{\frac{1}{p}}. \tag{3.4}$$

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