ON CONVEXITY OF FUZZY MAPPINGS AND FUZZY OPTIMIZATIONS

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Abstract. In this paper, based on a new order and a new metric on the set of fuzzy numbers, we present the concepts of convexity of the fuzzy mappings and give characterization theorems for the convex fuzzy mappings and quasi-convex fuzzy mappings. Finally, we discuss the properties of convex fuzzy optimizations.

Keywords: convexity, fuzzy mapping, fuzzy optimization.

1. Introduction

In many scientific and engineering applications the fuzzy set concept plays an important role. The fuzziness appears when we need to perform, on manifold, calculations with imprecision variables. The fuzzy set theory was introduced initially by Zadeh [25] in 1965. In the theory and applications of fuzzy sets convexity is a most useful concept. In fact, in the basic and classical paper [25], Zadeh paid special attention to the investigation of the convex fuzzy sets which covers nearly the second half of the space of the paper.

Following the seminal work of Zadeh, a lot of scholars have discussed various aspects of the theory and applications of fuzzy convex analysis. Nanda and Kar [9] proposed a concept of convex fuzzy mapping and proved that a fuzzy mapping is convex if and only if its epigraph is a convex set. Yan and Xu [22] discussed the convexity and quasi-convexity of fuzzy mappings by considering the concept of ordering proposed by Goetschel and Voxman [6]. Noor [11] introduced the concept of fuzzy preinvex functions over the field of real numbers $\mathbb{R}$, and obtained some properties of fuzzy preinvex functions. In [16], Syau introduced the concepts of pseudo-convexity, invexity and pseudo-invexity for fuzzy mappings of one

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variable by using notion of differentiability and the results proposed by Goetschel and Voxman [6]. Syau [17] introduced and investigated a new kind of generalized convex fuzzy mapping known as a B-vex fuzzy mapping. Wang and Wu [19] by establishing the fuzzy subdifferential of a fuzzy mapping, considered the application to convex fuzzy programming. Panigrahi [12] extended and generalized these concepts to fuzzy mappings of several variables using Buckley and Feurings [2] approach for fuzzy differentiation and derived a K-K-T condition for a constrained fuzzy minimization problem. In [20], [21], Wu and Xu introduced the concepts of fuzzy pseudo-convex, fuzzy invex, fuzzy pseudo-invex and fuzzy preinvex mapping from $\mathbb{R}^n$ to the set of fuzzy numbers based on the concept of differentiability of fuzzy mapping due to Wang and Wu [19]. In order to solve the open problem in fuzzy analysis that we proposed in [13], we introduced some new and more general definitions in the area of fuzzy starshapedness, and developed several theorems on the shadows of starshaped fuzzy sets [14], which generalize the important results obtained by Liu [8].

In this paper, after defining a new order on the set of fuzzy numbers, which is finer than the usually used "fuzzy-max" order [1], we will present the concepts of convexity of the fuzzy mappings and give characterization theorems for the convex fuzzy mappings and quasi-convex fuzzy mappings in Section 3. Finally, we discuss the properties of convex fuzzy optimizations in Section 4.

2. Preliminaries

We now quote some concepts and results which will be needed in the sequel.

A fuzzy set $\tilde{u}$ of $\mathbb{R}$ is a function $\tilde{u} : \mathbb{R} \to [0, 1]$. For any fuzzy set $\tilde{u}$, we denote by $[\tilde{u}]^\alpha = \{x \in \mathbb{R} : \tilde{u}(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$, its a-level set. We define the set $[\tilde{u}]^0$ by $[\tilde{u}]^0 = \overline{\{x \in \mathbb{R} : \tilde{u}(x) > 0\}}$, where $\overline{A}$ denotes the closure of a crisp set A. A fuzzy set $\tilde{u}$ is said to be a fuzzy number if it satisfies the following conditions [3]:

(i) $\tilde{u}$ is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $\tilde{u}(x_0) = 1$;

(ii) $\tilde{u}$ is convex, i.e., $\tilde{u}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\tilde{u}(x_1), \tilde{u}(x_2)\}$, for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in (0, 1)$;

(iii) $\tilde{u}$ is upper semi-continuous;

(iv) $[\tilde{u}]^0$ is compact.

Equivalently, a fuzzy number $\tilde{u}$ is a fuzzy set with non-empty bounded closed level sets $[\tilde{u}]^\alpha = [\tilde{u}_L(\alpha), \tilde{u}_R(\alpha)]$ for all $\alpha \in [0, 1]$, where $[\tilde{u}_L(\alpha), \tilde{u}_R(\alpha)]$ denotes a closed interval with the left end point $\tilde{u}_L(\alpha)$ and the right end point $\tilde{u}_R(\alpha)$. We denote the class of fuzzy numbers by $\mathscr{F}$. Notice that the real numbers $\mathbb{R}$ can be embedded in $\mathscr{F}$ by defining a fuzzy number $\tilde{a}$ as

$$\tilde{a}(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise}, \end{cases}$$

for each $a \in \mathbb{R}$.

For any $\tilde{u}, \tilde{v} \in \mathscr{F}$ and $a \in \mathbb{R}$, owing to Zadeh's extension principle [26], scalar multiplication and addition are defined for any $x \in \mathbb{R}$ by

$$a \times \tilde{u}(x) = a\tilde{u}(x) = \begin{cases} \tilde{u}\left(\dfrac{x}{a}\right), & \text{if } a \neq 0, \\ \tilde{0}, & \text{if } a = 0, \end{cases}$$

and

$$(\tilde{u} + \tilde{v})(x) = \sup_{x_1, x_2 : x_1 + x_2 = x} \min\{\tilde{u}(x_1), \tilde{v}(x_2)\}.$$

For any $\tilde{u} \in \mathscr{F}$, we define the fuzzy number $-\tilde{u} \in \mathscr{F}$ by $-\tilde{u} = (-1) \times \tilde{u}$, i.e., $-\tilde{u}(x) = \tilde{u}(-x)$, for all $x \in \mathbb{R}$. By the level set representations of the fuzzy numbers $\tilde{u}, \tilde{v}, \tilde{w}$, we can get that

$$\begin{aligned} ([\tilde{u}]^\alpha + [\tilde{v}]^\alpha) + [\tilde{w}]^\alpha &= [\tilde{u}]^\alpha + ([\tilde{v}]^\alpha + [\tilde{w}]^\alpha) \\ &= [\tilde{u}_L(\alpha) + \tilde{v}_L(\alpha) + \tilde{w}_L(\alpha), \tilde{u}_R(\alpha) + \tilde{v}_R(\alpha) + \tilde{w}_R(\alpha)], \\ [\tilde{u}]^\alpha + [\tilde{v}]^\alpha &= [\tilde{v}]^\alpha + [\tilde{u}]^\alpha = [\tilde{u}_L(\alpha) + \tilde{v}_L(\alpha), \tilde{u}_R(\alpha) + \tilde{v}_R(\alpha)], \\ [-\tilde{u}]^\alpha &= -[\tilde{u}_L(\alpha), \tilde{u}_R(\alpha)] = [-\tilde{u}_R(\alpha), -\tilde{u}_L(\alpha)], \end{aligned}$$

which implies $\mathscr{F}$ is a commutative semigroup under addition.

Definition 2.1 [7] For a fuzzy number $\tilde{u}$, we define a function $M_{\tilde{u}} : [0,1] \to \mathbb{R}$ by assigning the midpoint of each α-level set to $M_{\tilde{u}}(\alpha)$ for all $\alpha \in [0,1]$, i.e.,

$$M_{\tilde{u}}(\alpha) = \frac{\tilde{u}_L(\alpha) + \tilde{u}_R(\alpha)}{2}.$$

It is interesting to note that the midpoint function $M_{\tilde{u}}$ is actually a gradual number and all of the gradual numbers form a group structure for addition [4, 5].

Lemma 2.1 [15] *For any $\tilde{u} \in \mathscr{F}$, the midpoint function $M_{\tilde{u}}$ is continuous from the right at* 0 *and continuous from the left on* $[0,1]$. *Furthermore it is a function of bounded variation on* $[0,1]$.

Definition 2.2 For any $\tilde{u}, \tilde{v} \in \mathscr{F}$, we say that $\tilde{u} \preceq_m \tilde{v}$ if for all $\alpha \in [0,1]$, $M_{\tilde{u}}(\alpha) \leq M_{\tilde{v}}(\alpha)$; we say that $\tilde{u} \prec_m \tilde{v}$ if $\tilde{u} \preceq_m \tilde{v}$ and there exists $\alpha_0 \in [0,1]$ such that $M_{\tilde{u}}(\alpha_0) < M_{\tilde{v}}(\alpha_0)$; we say that $\tilde{u} =_m \tilde{v}$ if $\tilde{u} \preceq_m \tilde{v}$ and $\tilde{v} \preceq_m \tilde{u}$.

It is often convenient to write $\tilde{v} \succeq_m \tilde{u}$ (resp. $\tilde{v} \succ_m \tilde{u}$) in place of $\tilde{u} \preceq_m \tilde{v}$ (resp. $\tilde{u} \preceq_m \tilde{v}$). A subset A of $\mathscr{F}$ is said to be bounded above if there exists a fuzzy number $\tilde{u} \in \mathscr{F}$, called an upper bound of A, such that $\tilde{v} \preceq_m \tilde{u}$ for every $\tilde{v} \in \mathscr{F}$. Further, a fuzzy number $\tilde{u}_0 \in \mathscr{F}$ is called the least upper bound (sup, in short) for A if

(i) $\tilde{u}_0$ is an upper bound of A,
(ii) $\tilde{u}_0 \preceq_m \tilde{u}$ for every upper bound $\tilde{u}$ of A.

A lower bound and the greatest lower bound (inf, in short) are defined similarly.

Remark 2.1 In the field of fuzzy optimizations, ranking of fuzzy numbers is an important and prerequisite procedure. In general, we rank two fuzzy numbers by considering their left-hand functions and right-hand functions [10], [18], which is called the "fuzzy-max" order. We will show that two fuzzy numbers are not comparable by using the general method but they are comparable in the sense of Definition 2.2.

Example 2.1 Define fuzzy numbers $\widetilde{u}, \widetilde{v}$, respectively, by their level sets as

$$[\widetilde{u}]^{\alpha} = \left[\frac{1}{2}, \frac{3}{2} - \alpha\right] \quad \text{and} \quad [\widetilde{v}]^{\alpha} = [0, 2 - 2\alpha], \alpha \in [0,1].$$

Then we can get that the corresponding midpoint functions are respectively,

$$M_{\widetilde{u}}(\alpha) = 1 - \frac{\alpha}{2} \quad \text{and} \quad M_{\widetilde{v}}(\alpha) = 1 - \alpha, \ \alpha \in [0,1].$$

It is obvious that $\widetilde{u} \succ \widetilde{v}$ in the sense of Definition 2.2. However, the fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$ are not comparable with respect to the "fuzzy-max" order [10], [18].

We define a metric d_m on $\mathscr{F}$ by

$$d_m(\tilde{u}, \tilde{v}) = \sup_{0 \le \alpha \le 1} |M_{\tilde{u}}(\alpha) - M_{\tilde{v}}(\alpha)|$$

for any $u, v \in \mathscr{F}$. In general, this metric is not greater than the Hausdorff metric [15].

Throughout this paper, we suppose that V is a real vector space, and K is a convex subset of V. For any $x \in V$ and $\delta > 0$, let

$$B_\delta(x) = \{y \in V : \|x - y\| < \delta\},$$

where $\|\cdot\|$ is the norm on V. We present the definition of upper and lower semicontinuous fuzzy mappings, which is a variation of the one in [1] with respect to the order $\preceq_m$.

Definition 2.3 A fuzzy mapping $F : K \to \mathscr{F}$ is said to be:

(1) upper semicontinuous at $x_0 \in K$ if, for any $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$F(x) \preceq_m F(x_0) + \tilde{\varepsilon} \text{ whenever } x \in K \cap B_\delta(x_0).$$

F is upper semicontinuous if it is upper semicontinuous at each point of K.

(2) lower semicontinuous at $x_0 \in K$ if, for any $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$F(x_0) - \tilde{\varepsilon} \preceq_m F(x) \text{ whenever } x \in K \cap B_\delta(x_0).$$

F is lower semicontinuous if it is lower semicontinuous at each point of K.

3. Main results

Definition 3.1 A fuzzy mapping $F : K \to \mathscr{F}$ is said to be:

(1) convex if, for every $\lambda \in [0,1]$ and $x, y \in K$,

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda F(x) + (1-\lambda)F(y),$$

(2) concave if, for every $\lambda \in [0,1]$ and $x, y \in K$,

$$F(\lambda x + (1-\lambda)y) \succeq_m \lambda F(x) + (1-\lambda)F(y).$$

Theorem 3.1 *A fuzzy mapping $F : K \to \mathscr{F}$ is convex if and only if for all $x, y \in K$, $\lambda \in (0,1)$ and all $\tilde{u}, \tilde{v} \in \mathscr{F}$ such that $F(x) \preceq_m \tilde{u}, F(y) \preceq_m \tilde{v}$,*

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda\tilde{u} + (1-\lambda)\tilde{v}.$$

Proof. Suppose $F : K \to \mathscr{F}$ is a convex fuzzy mapping. For any $\lambda \in (0,1)$ and $\tilde{u}, \tilde{v} \in \mathscr{F}$, we have that

$$M_{(\lambda\tilde{u}+(1-\lambda)\tilde{v})}(\alpha) = \frac{\lambda(\tilde{u}_L(\alpha) + \tilde{u}_R(\alpha)) + (1-\lambda)(\tilde{v}_L(\alpha) + \tilde{v}_R(\alpha))}{2},$$

for all $\alpha \in [0,1]$. Now for any $x, y \in K$, $\lambda \in (0,1)$ and $\tilde{u}, \tilde{v} \in \mathscr{F}$ such that $F(x) \preceq_m \tilde{u}, F(y) \preceq_m \tilde{v}$, we have

$$\begin{aligned} M_{F(\lambda x+(1-\lambda)y)}(\alpha) &\leq M_{\lambda F(x)+(1-\lambda)F(y)}(\alpha) \\ &= \frac{\lambda(F(x)_L(\alpha) + F(x)_R(\alpha)) + (1-\lambda)(F(y)_L(\alpha) + F(y)_R(\alpha))}{2} \\ &\leq \frac{\lambda(\tilde{u}_L(\alpha) + \tilde{u}_R(\alpha)) + (1-\lambda)(\tilde{v}_L(\alpha) + \tilde{v}_R(\alpha))}{2} \\ &= M_{(\lambda\tilde{u}+(1-\lambda)\tilde{v})}(\alpha), \end{aligned}$$

for all $\alpha \in [0,1]$, i.e.,

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda\tilde{u} + (1-\lambda)\tilde{v}.$$

Conversely, suppose the conditions hold. Then for any $x, y \in K$, let $\tilde{u_0} =_m F(x)$ and $\tilde{v_0} =_m F(y)$. Thus we get that

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda\tilde{u_0} + (1-\lambda)\tilde{v_0} =_m \lambda F(x) + (1-\lambda)F(y). \qquad \blacksquare$$

Theorem 3.2 *A fuzzy mapping $F : K \to \mathscr{F}$ is convex if and only if the set*

$$A = \{(x, \tilde{u}) : x \in K, \tilde{u} \in \mathscr{F}, F(x) \preceq_m \tilde{u}\}$$

is convex.

Proof. Suppose F is a convex fuzzy mapping. For any $(x, \tilde{u}), (y, \tilde{v}) \in A$, we have $F(x) \preceq_m \tilde{u}$ and $F(y) \preceq_m \tilde{v}$. By Theorem 3.1, we have that

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda\tilde{u} + (1-\lambda)\tilde{v},$$

which implies that

$$(\lambda x + (1-\lambda)y, \lambda\tilde{u} + (1-\lambda)\tilde{v}) = \lambda(x, \tilde{u}) + (1-\lambda)(y, \tilde{v}) \in A,$$

for all $\lambda \in (0,1)$.

Conversely, suppose A is a convex set. For any $x, y \in K$, let $\tilde{u_0} = F(x)$ and $\tilde{v_0} = F(y)$. For all $\lambda \in (0,1)$, since $(x, \tilde{u_0}), (y, \tilde{v_0}) \in A$, we have

$$\lambda(x, \tilde{u_0}) + (1-\lambda)(y, \tilde{v_0}) = (\lambda x + (1-\lambda)y, \lambda\tilde{u_0} + (1-\lambda)\tilde{v_0}) \in A,$$

which implies that

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda\tilde{u_0} + (1-\lambda)\tilde{v_0} = \lambda F(x) + (1-\lambda)F(y).$$ ∎

Theorem 3.3 *A fuzzy mapping $F : K \to \mathscr{F}$ is convex if and only if for every $\alpha \in [0,1]$ the induced function $M_{F(x)}(\alpha)$ is convex with respect to x.*

Proof. By Definition 3.1, we have that $F : K \to \mathscr{F}$ is convex if and only for any $\lambda \in [0,1]$ and $x, y \in K$,

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda F(x) + (1-\lambda)F(y),$$

which is true if and only if for every $\alpha \in [0,1]$,

$$\begin{aligned} M_{F(\lambda x+(1-\lambda)y)}(\alpha) &= \frac{F(\lambda x + (1-\lambda)y)_L(\alpha) + F(\lambda x + (1-\lambda)y)_R(\alpha)}{2} \\ &\leq M_{\lambda F(x)+(1-\lambda)F(y)}(\alpha) \\ &= \frac{\lambda(F(x)_L(\alpha) + F(x)_R(\alpha)) + (1-\lambda)(F(y)_L(\alpha) + F(y)_R(\alpha))}{2} \\ &= \lambda\frac{(F(x)_L(\alpha) + F(x)_R(\alpha))}{2} + (1-\lambda)\frac{(F(y)_L(\alpha) + F(y)_R(\alpha))}{2} \\ &= \lambda M_{F(x)}(\alpha) + (1-\lambda)M_{F(y)}(\alpha), \end{aligned}$$

i.e., $M_{F(x)}(\alpha)$ is convex with respect to x. ∎

Since concave fuzzy mappings satisfy the opposite inequalities, by the similar proofs we can give the following three theorems for concave fuzzy mappings.

Theorem 3.4 *A fuzzy mapping $F : K \to \mathscr{F}$ is concave if and only if for all $x, y \in K$, $\lambda \in (0,1)$ and all $\tilde{u}, \tilde{v} \in \mathscr{F}$ such that $F(x) \succeq_m \tilde{u}, F(y) \succeq_m \tilde{v}$,*

$$F(\lambda x + (1-\lambda)y) \succeq_m \lambda\tilde{u} + (1-\lambda)\tilde{v}.$$

Theorem 3.5 *A fuzzy mapping $F : K \to \mathscr{F}$ is concave if and only if the set*

$$A = \{(x, \tilde{u}) : x \in K, \tilde{u} \in \mathscr{F}, F(x) \succeq_m \tilde{u}\}$$

is convex.

Theorem 3.6 *A fuzzy mapping $F : K \to \mathscr{F}$ is concave if and only if for every $\alpha \in [0,1]$ the induced function $M_{F(x)}(\alpha)$ is concave with respect to x.*

Theorem 3.7 *Let $F : K \to \mathscr{F}$ be a fuzzy mapping. Then F is upper semicontinuous at $x_0 \in K$ if and only if $M_{F(x)}(\alpha)$ are upper semicontinuous at x_0 uniformly in $\alpha \in [0,1]$; F is lower semicontinuous at $x_0 \in K$ if and only if $M_{F(x)}(\alpha)$ are lower semicontinuous at x_0 uniformly in $\alpha \in [0,1]$.*

Proof. By Definition 2.1, 2.2 and 2.3, F is upper semicontinuous at $x_0 \in K$ if and only if, for any $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$M_{F(x)}(\alpha) \leq M_{F(x_0)}(\alpha) + \varepsilon \text{ whenever } x \in K \cap B_\delta(x_0),$$

for all $\alpha \in [0,1]$; F is lower semicontinuous at $x_0 \in K$ if and only if, for any $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$M_{F(x_0)}(\alpha) - \varepsilon \leq M_{F(x)}(\alpha) \text{ whenever } x \in K \cap B_\delta(x_0),$$

for all $\alpha \in [0,1]$. ∎

Theorem 3.8 *Let $F : K \to \mathscr{F}$ be a fuzzy mapping. Then F is upper semicontinuous and lower semicontinuous at $x_0 \in K$ if and only if F is continuous with respect to the metric d_m.*

Proof. By Theorem 3.7, F is upper semicontinuous and lower semicontinuous at $x_0 \in K$ if and only if for any $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$M_{F(x_0)}(\alpha) - \varepsilon \leq M_{F(x)}(\alpha) \leq M_{F(x_0)}(\alpha) + \varepsilon \text{ whenever } x \in K \cap B_\delta(x_0),$$

for all $\alpha \in [0,1]$, which implies that

$$d_m(F(x_0), F(x)) = \sup_{0 \leq \alpha \leq 1} |M_{F(x_0)}(\alpha) - M_{F(x)}(\alpha)| \leq \varepsilon \text{ whenever } x \in K \cap B_\delta(x_0).$$ ∎

Theorem 3.9 *Let $F : K \to \mathscr{F}$ be an upper semicontinuous fuzzy mapping. If there exists a $\lambda \in (0,1)$ such that*

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda F(x) + (1-\lambda)F(y),$$

for all $x, y \in K$. Then F is a convex fuzzy mapping on K.

Proof. For any fixed $\alpha \in [0,1]$, by Theorem 3.7, $M_{F(x)}(\alpha)$ is upper semicontinuous with respect to x. Since there exists a $\lambda \in (0,1)$ such that

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda F(x) + (1-\lambda)F(y),$$

for all $x, y \in K$, by Definition 2.1, we have that

$$M_{F(\lambda x+(1-\lambda)y)}(\alpha) \leq M_{\lambda F(x)+(1-\lambda)F(y)}(\alpha) = \lambda M_{F(x)}(\alpha) + (1-\lambda)M_{F(y)}(\alpha),$$

for all $x, y \in K$. Thus by Theorem 2.3 in [23], we get $M_{F(x)}(\alpha)$ is convex with respect to x on K. From Theorem 3.3, it follows that F is a convex fuzzy mapping on K. ■

Theorem 3.10 *Let $F : K \to \mathscr{F}$ be a lower semicontinuous fuzzy mapping. If, for any $x, y \in K$, there exists a $\lambda = \lambda(x, y) \in (0,1)$ such that*

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda F(x) + (1-\lambda)F(y).$$

Then F is a convex fuzzy mapping on K.

Proof. From the hypothesis, for any $x, y \in K$, by Definition 2.1, there exists a $\lambda = \lambda(x, y) \in (0,1)$ such that

$$M_{F(\lambda x+(1-\lambda)y)}(\alpha) \leq M_{\lambda F(x)+(1-\lambda)F(y)}(\alpha) = \lambda M_{F(x)}(\alpha) + (1-\lambda)M_{F(y)}(\alpha).$$

Thus by by Theorem 4 in [24], we get $M_{F(x)}(\alpha)$ is convex with respect to x on K for all $\alpha \in [0,1]$. From Theorem 3.3, it follows that F is a convex fuzzy mapping on K. ■

Theorem 3.11 *For any $\tilde{u}, \tilde{v} \in \mathscr{F}$, the set $\{\tilde{u}, \tilde{v}\}$ has the least upper bound and the greatest lower bound.*

Proof. For any $\tilde{u}, \tilde{v} \in \mathscr{F}$, By Lemma 2.1 we have that the midpoint functions $M_{\tilde{u}}$ and $M_{\tilde{v}}$ are continuous from the right at 0, continuous from the left on $[0,1]$, and are functions of bounded variation on $[0,1]$. Thus we define two functions $M_{\sup} : [0,1] \to \mathbb{R}$ and $M_{\inf} : [0,1] \to \mathbb{R}$ by

$$M_{\sup}(\alpha) = \max\{M_{\tilde{u}}(\alpha), M_{\tilde{v}}(\alpha)\}$$

and

$$M_{\inf}(\alpha) = \min\{M_{\tilde{u}}(\alpha), M_{\tilde{v}}(\alpha)\},$$

respectively. It is easy to see that the functions $M_{\sup}$ and $M_{\inf}$ are continuous from the right at 0 and continuous from the left on $[0,1]$. Furthermore, we have that

$$V_0^1(M_{\sup}) \leq V_0^1(M_{\tilde{u}}) + V_0^1(M_{\tilde{v}})$$

and

$$V_0^1(M_{\inf}) \leq V_0^1(M_{\tilde{u}}) + V_0^1(M_{\tilde{v}}),$$

where $V_0^1(f)$ represents the total variation of the function f. Thus $M_{\sup}$ and $M_{\inf}$ are functions of bounded variation on $[0,1]$. By Theorem 3.10 in [15], the functions $M_{\sup}$ and $M_{\inf}$ can determine two fuzzy numbers $\tilde{m}$ and $\tilde{n}$ such that

$$M_{\tilde{m}} = M_{\sup} \text{ and } M_{\tilde{n}} = M_{\inf}.$$

It is easy to see that $\tilde{m}$ and $\tilde{n}$ are $\sup\{\tilde{u},\tilde{v}\}$ and $\inf\{\tilde{u},\tilde{v}\}$, respectively. ∎

Definition 3.2 A fuzzy mapping $F : K \to \mathscr{F}$ is said to be:

(1) quasi-convex if for every $\lambda \in [0,1]$ and $x, y \in K$,

$$F(\lambda x + (1-\lambda)y) \preceq_m \sup\{F(x), F(y)\},$$

(2) quasi-concave if for every $\lambda \in [0,1]$ and $x, y \in K$,

$$F(\lambda x + (1-\lambda)y) \succeq_m \inf\{F(x), F(y)\}.$$

Theorem 3.12 *Let $F : K \to \mathscr{F}$ be a convex (resp. concave) fuzzy mapping. Then it is quasi-convex (resp. quasi-concave) fuzzy mapping on K.*

Proof. Let $\lambda \in [0,1]$ and $x, y \in K$. By Definition 2.1 and Theorem 3. 11, we have that

$$\inf\{F(x), F(y)\} \preceq_m \lambda F(x) + (1-\lambda)F(y) \preceq_m \sup\{F(x), F(y)\}.$$

If F is convex, it follows that

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda F(x) + (1-\lambda)F(y) \preceq_m \sup\{F(x), F(y)\};$$

If F is concave, it follows that

$$\inf\{F(x), F(y)\} \preceq_m \lambda F(x) + (1-\lambda)F(y) \preceq_m F(\lambda x + (1-\lambda)y).$$ ∎

Theorem 3.13 *A fuzzy mapping $F : K \to \mathscr{F}$ is quasi-convex (resp. quasi-concave) if and only if for every $\alpha \in [0,1]$ the induced function $M_{F(x)}(\alpha)$ is quasi-convex (resp. quasi-concave) with respect to x.*

Proof. By Definition 3.2, we have that $F : K \to \mathscr{F}$ is quasi-convex if and only for any $\lambda \in [0,1]$ and $x, y \in K$,

$$F(\lambda x + (1-\lambda)y) \preceq_m \sup\{F(x), F(y)\},$$

which is true if and only if for every $\alpha \in [0,1]$,

$$\begin{aligned} M_{F(\lambda x+(1-\lambda)y)}(\alpha) &\leq M_{\sup\{F(x),F(y)\}}(\alpha) \\ &= \max\{M_{F(x)}(\alpha), M_{F(y)}(\alpha)\}, \end{aligned}$$

i.e., $M_{F(x)}(\alpha)$ is quasi-convex with respect to x.

Similarly, we can prove that $F : K \to \mathscr{F}$ is quasi-concave if and only if for every $\alpha \in [0,1]$ the induced function $M_{F(x)}(\alpha)$ is quasi-concave with respect to x. ∎

4. Applications to fuzzy optimization

Now we give some applications of the main results to fuzzy optimization.

Definition 4.1 For a fuzzy mapping $F : K \to \mathscr{F}$,

(1) an element $x_0 \in K$ is called a local minimizer of $F : K \to \mathscr{F}$ if there exists a $\delta > O$, such that

$$F(x_0) \preceq_m F(x) \text{ for all } x \in K \cap B_\delta(x_0);$$

(2) an element $x_0 \in K$ is called a strict local minimizer of $F : K \to \mathscr{F}$ if there exists a $\delta > O$, such that

$$F(x_0) \prec_m F(x) \text{ for all } x \neq x_0, x \in K \cap B_\delta(x_0);$$

(3) an element $x_0 \in K$ is called a global minimizer of $F : K \to \mathscr{F}$ if

$$F(x_0) \preceq_m F(x) \text{ for all } x \in K.$$

Theorem 4.1 *Let $F : K \to \mathscr{F}$ be a convex fuzzy mapping and $x_0 \in K$ is a local minimizer of F. Then x_0 is also a global minimizer of F.*

Proof. If there exists an $x' \in K$ such that $F(x') \prec_m F(x_0)$. Then since $F : K \to \mathscr{F}$ is convex, we have that

$$M_{F(\lambda x'+(1-\lambda)x_0)} \leq M_{\lambda F(x')+(1-\lambda)F(x_0)} = \lambda M_{F(x')} + (1-\lambda)M_{F(x_0)} < M_{F(x_0)},$$

which implies that

$$F(\lambda x' + (1-\lambda)x_0) \preceq_m \lambda F(x') + (1-\lambda)F(x_0) \prec_m F(x_0),$$

for all $\lambda \in (0, 1)$. Thus for arbitrary small positive number λ, we have

$$F(\lambda x' + (1-\lambda)x_0) \prec_m F(x_0),$$

which contradicts with the definition of the local minimizer at x_0. ∎

Theorem 4.2 *Let $F : K \to \mathscr{F}$ be a quasi-convex fuzzy mapping and $x_0 \in K$ is a global minimizer of F. Then the set*

$$A = \{x \in K : F(x) =_m F(x_0)\}$$

is the set of all global minimizers of F and it is convex.

Proof. Let $x' \in K$ be a global minimizer of F. Then we have that $F(x') \preceq_m F(x_0)$ and $F(x_0) \preceq_m F(x')$, i.e., $F(x) =_m F(x_0)$. Thus $x' \in A$.

Conversely, let $x' \in A$. Since x_0 is a global minimizer of F, we have $F(x') =_m F(x_0) \preceq_m F(x)$ for all $x \in K$, which implies x' is also a global minimizer of F.

Let $x, y \in A$ and $\lambda \in [0, 1]$, since F is a quasi-convex fuzzy mapping, we have that

$$F(\lambda x + (1-\lambda)y) \preceq_m \sup\{F(x), F(y)\} =_m F(x),$$

which implies $F(\lambda x + (1-\lambda)y) =_m F(x)$ because x is a global minimizer of F. Thus $\lambda x + (1-\lambda)y \in A$. ∎

5. Conclusions

In this present investigation, based on a new order and a new metric on the set of fuzzy numbers, we present the concepts of convexity of the fuzzy mappings and give characterization theorems for the convex fuzzy mappings and quasi-convex fuzzy mappings. Then, we discuss the properties of convex fuzzy optimizations.

The results of modern convex analysis research can be traced back to the work of Minkowski and Caratheodory. They investigated polyhedron and related convex problems, and funded the basic theories of convex analysis. During the last decades, the development of optimization theory has brought the fuzzy convexity into many theoretical and application problems. The fuzzy convex analysis has thus attracted more and more attention. Thus we hope our results would provide a background to ongoing work in the problems of those related fields.

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