MINIMAL INTUITIONISTIC GENERAL $L$-FUZZY AUTOMATA

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Abstract. In this paper we present an intuitionistic general $L$-fuzzy automaton (IGLFA) based on lattice valued intuitionistic fuzzy sets [2]. In this note, we define $(\alpha, \beta)$-language, $(\alpha, \beta)$-complete, $(\alpha, \beta)$-accessible, $(\alpha, \beta)$-reduced for an IGLFA over a bounded complete lattice $L$, where $\alpha, \beta \in L$ and $\alpha \leq L N(\beta)$. In particular, we prove a theorem which is generalization of Myhill-Nerode theorem in ordinary deterministic automata. In other words for any recognizable $(\alpha, \beta)$-language over a bounded complete lattice $L$, there exist minimal $(\alpha, \beta)$-complete and deterministic IGLFA, which preserve $(\alpha, \beta)$-language, where $\alpha, \beta \in L$ and $\alpha \leq L N(\beta)$. Also, we show that for any given $(\alpha, \beta)$-language $\mathcal{L}$, the minimal $(\alpha, \beta)$-complete and deterministic IGLFA recognizing $\mathcal{L}$ is isomorphic with threshold $(\alpha, \beta)$ to any $(\alpha, \beta)$-complete, $(\alpha, \beta)$-accessible, deterministic, $(\alpha, \beta)$-reduced IGLFA recognizing $\mathcal{L}$. Moreover, we give some examples to clarify these notions. Finally, by using these notions, we give some theorems and obtain some results.

Keywords: Max-min intuitionistic general $L$-fuzzy automata; $(\alpha, \beta)$-language; $(\alpha, \beta)$-reduced; Minimal intuitionistic general $L$-fuzzy automata; $(\alpha, \beta)$-isomorphic.

1. Introduction

The theory of fuzzy sets was introduced by L.A. Zadeh in 1965 [40]. W.G. Wee [35] introduced the idea of fuzzy automata. E.T. Lee and L.A. Zadeh [21] in 1969 gave the concept of fuzzy finite state automata. Thereafter, there were a considerable number of authors, such as Mordeson and Malik [22], [23], Topencharov and Peeva [33] and others having contributed to this field. Fuzzy finite automata have many important applications such as in learning system, pattern recognition, neural networks and data base theory [14], [15], [22], [25], [26], [29], [36]. Atanasov [1] has extended the notion of fuzzy sets to the intuitionistic fuzzy sets (IFS) by
adding non-membership value, which may express more accurate and flexible information as compared with fuzzy sets. Intuitionistic fuzzy set theory has many applications in several subjects, see [5], [6], [9], [11], [16], [18], [19], [34], [7]. Using the notion of intuitionistic fuzzy sets, W.L. Jun [17] introduced the notion of intuitionistic fuzzy finite state machines as a generalization of fuzzy finite state machines. Based on the papers [17], [18], Zhang and Li [41] discussed intuitionistic fuzzy recognizers and intuitionistic fuzzy finite automata. K. Atanassov and S. Stoeva generalized the concept of IFS to intuitionistic $L$-fuzzy sets [2] where $L$ is an appropriate lattice. A. Tepavcevic and T. Gerstenkorn gave a new definition of lattice valued intuitionistic fuzzy sets in [32]. Thus, on the basis of lattice valued intuitionistic fuzzy sets, Yang et al. [38] introduced the concepts of lattice-valued intuitionistic fuzzy finite state machines. In 2004, M. Doostfatemeh and S.C. Kremer [10] extended the notion of fuzzy automata and gave the notion of general fuzzy automata. In 2014, M. Shamsizadeh and M.M. Zahedi [31] gave the notion of max-min intuitionistic general fuzzy automata. We will use bounded complete lattices as the structures of truth values. Note that usually the real unit interval $[0, 1]$ is used in the literature (the reader not familiar with lattices may, without any harm, substitute $[0, 1]$ for bounded complete lattices throughout the paper). Our paper deals with intuitionistic general fuzzy automata over a bounded complete lattice $L$, where endowed with a $t$-norm $T$, a $t$-conorm $S$, the least element 0 and the greatest element 1, denoted by $L = (L, \leq_L, T, S, 0, 1)$. State minimization is a fundamental problem in automata theory. There are many papers on the minimization problem of fuzzy finite automata. For example, minimization of mealy type of fuzzy finite automata in discussed in [4], minimization of fuzzy finite automata with crisp final states without outputs in studied in [3], minimizing the deterministic finite automaton with fuzzy (final) states in [24] and minimization of fuzzy machines becomes the subject of [28], [27], [30], [33]. Myhill-Nerode’s theorem has been extended to fuzzy regular language and also an algorithm is given for minimizing the deterministic finite automaton with fuzzy (final) states in [20], [24]. It is important to find the minimal intuitionistic general $L$-fuzzy automata that recognizes the same language as a given language. In this note, for a given complete lattice $L = (L, \leq_L, T, S, 0, 1)$, we define an $(\alpha, \beta)$-language, where $\alpha, \beta \in L, \alpha <_L N(\beta)$. Furthermore, we show that for any max-min IGLFA $\tilde{F}^*$, there exist $(\alpha, \beta)$-complete, $(\alpha, \beta)$-accessible and deterministic max-min IGLFA recognizing $\mathcal{L}(\tilde{F}^*)$, where $\alpha, \beta \in L, \alpha <_L N(\beta)$. Also, we prove a theorem which is generalization of Myhill-Nerode theorem in ordinary deterministic automata. In other words, we have shown that for any $(\alpha, \beta)$-language, there exist minimal $(\alpha, \beta)$-complete and deterministic intuitionistic general $L$-fuzzy automata (IGLFA). Also, we define an $(\alpha, \beta)$-reduced IGLFA. We show that for any given $(\alpha, \beta)$-language $\mathcal{L}$, the minimal $(\alpha, \beta)$-complete and deterministic IGLFA recognizing $\mathcal{L}$ is isomorphic with threshold $(\alpha, \beta)$ to any $(\alpha, \beta)$-complete, $(\alpha, \beta)$-accessible, deterministic, $(\alpha, \beta)$-reduced IGLFA recognizing $\mathcal{L}$. Moreover we give some new notions and results as mentioned in the abstract and some examples to clarify these new notions.
2. Preliminaries

In this section we give some definitions that we need in the sequel. Assume that \( E \) is an universal set. A fuzzy set \( A \) on \( E \) is characterized by the same symbol \( A \) as a function \( A : E \to [0, 1] \) where \( A(u) \in [0, 1] \) is the membership degree of the element \( u \in E \) [40].

**Definition 2.1** [1] Let \( A \) be a given subset on \( E \). An intuitionistic fuzzy set (IFS) \( A^+ \) on \( E \) is an object of the following form

\[
A^+ = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in E \},
\]

where the functions \( \mu_A : E \to [0, 1] \) and \( \nu_A : E \to [0, 1] \) define the value of membership and the value of non-membership of the element \( x \in E \) to the set \( A \), respectively, and for every \( x \in E, \ 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \).

Obviously, every ordinary fuzzy set \( \{ (x, \mu_A(x)) \mid x \in E \} \) has an intuitionistic form \( \{ (x, \mu_A(x), 1 - \mu_A(x)) \mid x \in E \} \). If \( \pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \), then \( \pi_A(x) \) is the value of non-determinacy (uncertainty) of the membership of element \( x \in E \) to the set \( A \). In the case of ordinary fuzzy sets, where \( \nu_A(x) = 1 - \mu_A(x) \), we have \( \pi_A(x) = 0 \) for every \( x \in E \).

In the rest of this paper, we denote the set \( A^+ \) by \( A \).

Let \( L = (L, \leq_L, 0, 1) \) be a bounded (complete) lattice. By an \( L \)-fuzzy set \( A \) on \( E \) we mean a function \( A : E \to L \) [12].

**Definition 2.2** [37] Let \( L = (L, \leq_L, 0, 1) \) be a bounded lattice. A binary operation \( T : L \times L \to L \) is a lattice \( t \)-norm (Lt-norm) if it satisfies the following conditions:

1. \( T(1, x) = x \),
2. \( T(x, y) = T(y, x) \),
3. \( T(x, T(y, z)) = T(T(x, y), z) \),
4. if \( w \leq_L x \) and \( y \leq_L z \), then \( T(w, y) \leq_L T(x, z) \).

**Definition 2.3** [37] Let \( L = (L, \leq_L, 0, 1) \) be a bounded lattice. A binary operation \( S : L \times L \to L \) is a lattice \( t \)-conorm (Lt-conorm) if it satisfies the following conditions:

1. \( S(0, x) = x \),
2. \( S(x, y) = S(y, x) \),
3. \( S(x, S(y, z)) = S(S(x, y), z) \),
4. if \( w \leq_L x \) and \( y \leq_L z \), then \( S(w, y) \leq_L S(x, z) \).
In the rest of this paper, we denote \( D(x_1, D^{n-1}(x_2, ..., x_{n+1})) \) by \( D^n(x_1, x_2, ..., x_{n+1}) \) where \( D \) is an \( Lt \)-norm or an \( Lt \)-conorm on lattice \( L \), \( D^0(x) = x \) and \( D^1(x_1, x_2) = D(x_1, x_2) \), for any \( n \geq 1 \).

We recall from [8] that \( L^* : \{(x, y) \in [0, 1]^2 \mid 0 \leq x + y \leq 1\} \) is a complete lattice with the order defined by

\[
(x_1, x_2) \leq (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \leq y_2.
\]

**Definition 2.4** [2] Let \( X \) be a nonempty set and \( L \) be a complete lattice with an involutive order reversing unary operation \( N : L \to L \). An intuitionistic \( L \)-fuzzy set is an object of the form \( A = \{(x, \mu(x), \nu(x)) \mid x \in E\} \), where \( \mu \) and \( \nu \) are functions \( \mu : E \to L, \nu : E \to L \), such that for all \( x \in X \), \( \mu(x) \leq N(\nu(x)) \). We use the abbreviation ILFS for intuitionistic \( L \)-fuzzy set.

**Definition 2.5** [13] Let \( L \subseteq X^* \) and consider the relation \( \sim_L \) on \( X \), where \( x \sim_L y \) if and only if, for all \( z \in X^* \), \( xz \in L \iff yz \in L \).

From now on, we let \( L = (L, \leq_L, T, S, 0, 1) \) be a complete lattice, where endowed with an \( Lt \)-norm \( T \), an \( Lt \)-conorm \( S \), the least element \( 0 \) and the greatest element \( 1 \), also with an involutive order reversing unary operation \( N : L \to L \).

**Note 2.6** Let \( A, B \in L \). In this note we assume that \( A <_L B \) if and only if \( A \leq_L B \) and \( A \neq B \). We also assume that \( A \geq_L B \) if and only if \( B \leq_L A \).

3. Intuitionistic general \( L \)-fuzzy automata

**Definition 1.1** An intuitionistic general \( L \)-fuzzy automaton (IGLFA) \( \tilde{F} \) is a ten-tuple machine denoted by \( \tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4) \), where

- \( Q \) is a set of states,
- \( X \) is a finite set of input symbols, \( X = \{a_1, a_2, ..., a_m\} \),
- \( \tilde{R} \) is the ILFS of start states, \( \tilde{R} = \{(q, \mu^o(q), \nu^o(q)) \mid q \in R\} \), where \( R \) is a finite subset of \( Q \),
- \( Z \) is a finite set of output symbols, \( Z = \{b_1, b_2, ..., b_l\} \),
- \( \tilde{\delta} : (Q \times L \times L) \times X \times Q \to L \times L \) is the augmented transition function,
- \( \tilde{\omega} : (Q \times L \times L) \times Z \to L \times L \) is the output function,
- \( F_1 = (F_1^T, F_1^S) \), where \( F_1^T : L \times L \to L \) is an \( Lt \)-norm and it is called the membership assignment function. \( F_1^T(\mu, \delta_1) \) as is seen, is motivated by two parameters \( \mu \) and \( \delta_1 \), where \( \mu \) is the membership value of a predecessor and \( \delta_1 \) is the membership value of a transition. Moreover, \( F_1^S : L \times L \to L \) is an \( Lt \)-conorm, where is the dual of \( F_1^T \) respect to the involutive negation and it is called non-membership assignment function. \( F_1^S(\nu, \delta_2) \) as is seen, is motivated by two
parameters $\nu$ and $\delta_2$, where $\nu$ is the non-membership value of a predecessor and $\delta_2$ is the non-membership value of a transition.

In this definition, the process that takes place upon the transition from the state $q_i$ to $q_j$ on an input $a_k$ is given by:

\begin{align}
\mu^{t+1}(q_j) &= \tilde{\delta}_1((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j) = F^T_1(\mu^t(q_i), \delta_1(q_i, a_k, q_j)), \\
\nu^{t+1}(q_j) &= \tilde{\delta}_2((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j) = F^S_1(\nu^t(q_i), \delta_2(q_i, a_k, q_j)),
\end{align}

thus

\begin{equation}
\tilde{\delta}((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j) = (\tilde{\delta}_1((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j), \tilde{\delta}_2((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j)),
\end{equation}

where, $\delta(q_i, a_k, q_j) = (\delta_1(q_i, a_k, q_j), \delta_2(q_i, a_k, q_j))$, which $\delta$ is an ILFS. It means that the membership value of the state $q_j$ at time $t + 1$ is computed by the function $F^T_1$ using both the membership value of $q_i$ at time $t$ and the membership value of the transition. Also, the non-membership value of the state $q_j$ at time $t + 1$ is computed by function $F^S_1$ using both the non-membership value of $q_i$ at time $t$ and the non-membership value of the transition.

Considering (1.1), (1.2) and Definitions 2.2, 2.3 and 2.4, $\tilde{\delta}$ is an ILFS.

- $F_2 = (F^T_2, F^S_2)$, where $F^T_2 : L \times L \to L$ is an $L_t$-norm and it is called the membership assignment output function. $F^T_2(\mu, \omega_1)$ as is seen, is motivated by two parameters $\mu$ and $\omega_1$, where $\mu$ is the membership value of present state and $\omega_1$ is the membership value of an output function. Moreover, $F^S_2 : L \times L \to L$ is an $L_t$-conorm where it is the dual of $F^T_2$ respect to the involutive negation and is called non-membership assignment output function. $F^S_2(\nu, \omega_2)$ as is seen, is motivated by two parameters $\nu$ and $\omega_2$, where $\nu$ is the non-membership value of present state and $\omega_2$ is the non-membership value of an output function. Then, we have

\begin{align}
\tilde{\omega}_1((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) &= F^T_2(\mu^t(q_i), \omega_1(q_i, b_k)), \\
\tilde{\omega}_2((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) &= F^S_2(\nu^t(q_i), \omega_2(q_i, b_k)),
\end{align}

thus

\begin{equation}
\tilde{\omega}((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) = (\tilde{\omega}_1((q_i, \mu^t(q_i), \nu^t(q_i)), b_k), \tilde{\omega}_2((q_i, \mu^t(q_i), \nu^t(q_i)), b_k)),
\end{equation}

where, $\omega(q_i, b_k) = (\omega_1(q_i, b_k), \omega_2(q_i, b_k))$, which $\omega$ is an ILFS. It means that the output membership value of the state $q_i$ at time $t$ is computed by the function $F^T_2$ using both the membership value of $q_i$ at time $t$ and the membership value of the output function, also output non-membership value of the state $q_i$ at time $t$ is computed by function $F^S_2$ using both the non-membership value of $q_i$ at time $t$ and the non-membership value of the output function.
Considering (1.4), (1.5) and Definitions 2.2, 2.3 and 2.4, \( \tilde{\omega} \) is an ILFS.

- \( F_3 = (F_3^{TS}, F_3^{ST}) \), where \( F_3^{ST} : L^* \to L \) is an \( Lt \)-norm and it is called the multi-non-membership function. The multi-non-membership resolution function resolves the multi-non-membership active states and assigns a single non-membership value to them. Moreover, \( F_3^{TS} : L^* \to L \) is an \( Lt \)-conorm, where it is the dual of \( F_3^{ST} \) with respect to the involutive negation and it is called the multi-membership function. The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

- \( F_4 = (F_4^{TS}, F_4^{ST}) \), where \( F_4^{ST} : L^* \to L \) is an \( Lt \)-norm and it is called the multi-non-membership output function. The multi-non-membership resolution output function resolves the output multi-non-membership active state and assigns a single output non-membership value to it. Moreover, \( F_4^{TS} : L^* \to L \) is an \( Lt \)-conorm, where it is the dual of \( F_4^{ST} \) with respect to the involutive negation and is multi-membership output function. The multi-membership resolution output function resolves the output multi-membership active state and assigns a single output membership value to it.

Let \( Q_{act}(t_i) \) be the set of all active states at time \( t_i \) for all \( i \geq 0 \). We have \( Q_{act}(t_0) = R \) and \( Q_{act}(t_i) = \{ (q, \mu^i(q), \nu^i(q)) \mid \exists (q', \mu^{i-1}(q'), \nu^{i-1}(q')) \in Q_{act}(t_{i-1}), \exists a \in X, \delta(q', a, q) \in \Delta, \mu^i(q) > L 0 \} \) for all positive integer \( i \).

Since \( Q_{act}(t_i) \) is an ILFS, to say that a state \( q \) belongs to \( Q_{act}(t_i) \), we write \( q \in \text{Domain}(Q_{act}(t_i)) \) and for simplicity of notation we denote it by \( q \in Q_{act}(t_i) \). The combination of the operations of functions \( F_i^T \) and \( F_i^{TS} \) (\( F_i^{ST} \) and \( F_i^T \)) on a multi-membership (multi-non-membership) state \( q_j \) will lead to the multi-membership (multi-non-membership) resolution algorithm. Also, the set of all transition of IGLFA \( \tilde{F} \) is denoted by \( \Delta \).

**Algorithm:** Multi-membership resolution (multi-non-membership resolution) for transition function.

If there are several simultaneous transitions to the active state \( q_j \) at time \( t+1 \), then the following algorithm will assign a unified membership value (non-membership value) to that

1. Each transition membership value (transition non-membership value)
   \( \delta_1(q_i, a_k, q_j) \) (\( \delta_2(q_i, a_k, q_j) \)) together with \( \mu^i(q_i) \) (\( \nu^i(q_i) \)), will be processed by the membership (non-membership) assignment function \( F_i^T \) (\( F_i^S \)) and will produce a new membership value (non-membership value) as follows:

   \[
   \tilde{\delta}_1((q_i, \mu^i(q_i), \nu^i(q_i), a_k, q_j)) = F_i^T(\mu^i(q_i), \tilde{\delta}_1(q_i, a_k, q_j)),
   \]

   \[
   (\tilde{\delta}_2((q_i, \mu^i(q_i), \nu^i(q_i), a_k, q_j)) = F_i^S(\nu^i(q_i), \tilde{\delta}_2(q_i, a_k, q_j))).
   \]

2. These new membership (non-membership) values are not necessarily equal. Hence, they will be processed by \( F_3^{TS} \) (\( F_3^{ST} \)), which is called the multi-membership (multi-non-membership) resolution function. By some manipulation on the product results by \( F_i^T \) (\( F_i^S \)), we obtain just one element, say
the instantaneous membership value (non-membership value) of the active state \( q_j \)

\[
\mu^{t+1}(q_j) = (F^TS)^{n-1}(x_1, x_2, ..., x_n),
\]

\[
(\nu^{t+1}(q_j) = (F^{ST})^{n-1}(x_1, x_2, ..., x_n)),
\]

where

- \( n \) is the number of simultaneous transitions to the active state \( q_j \) at time \( t + 1 \) and \( x_i = F^T_1(\mu^t(q_i), \delta_1(q_i, a_k, q_j))(x_i = F^S_1(\nu^t(q_i), \delta_2(q_i, a_k, q_j))) \), \( 1 \leq i \leq n \),

- \( \delta_1(q_i, a_k, q_j)(\delta_2(q_i, a_k, q_j)) \) is the membership (non-membership) value of transition from \( q_i \) to \( q_j \) upon input \( a_k \),

- \( \mu^t(q_i) (\nu^t(q_i)) \) is the membership (non-membership) value of \( q_i \) at time \( t \),

**Algorithm:** Multi-membership resolution (Multi-non-membership resolution) for output function

If there are several simultaneous output to the active state \( q_i \) at time \( t \), the following algorithm will assign a unified membership value (non-membership value) to that

1. Each output membership value (non-membership value) \( \omega_1(q_i, b_k) \) \( (\omega_2(q_i, b_k)) \) together with \( \mu^t(q_i) (\nu^t(q_i)) \), will be processed by the membership (non-membership) assignment function \( F^T_2 (F^S_2) \) and will produce a new output membership value (non-membership value) as follows:

\[
\tilde{\omega}_1((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) = F^T_2(\mu^t(q_i), \omega_1(q_i, b_k)),
\]

\[
(\tilde{\omega}_2((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) = F^S_2(\nu^t(q_i), \omega_2(q_i, b_k))).
\]

2. These new output membership (non-membership) values are not necessarily equal. Hence, they will be processed by \( F^{TS}_4 (F^{ST}_4) \), which is called the output multi-membership (multi-non-membership) resolution function. By some manipulation on the product results by \( F^T_2 (F^S_2) \), we obtain just one element, say the instantaneous output membership value (non-membership value) of the active state \( q_i \).

\[
\omega^t_1(q_i) = (F^{TS}_4)^{n-1}(x_1, x_2, ..., x_n),
\]

\[
(\omega^t_2(q_i) = (F^{ST}_4)^{n-1}(x_1, x_2, ..., x_n)),
\]

where

- \( n \) is the number of simultaneous outputs to the active state \( q_i \) at time \( t \), \( x_j = F^T_2(\mu^t(q_i), \omega_1(q_i, b_j))(x_j = F^S_2(\nu^t(q_i), \omega_2(q_i, b_j))) \), \( 1 \leq j \leq n \), for some \( a_k \in X \) and \( q_i \) is an active state at time \( t \),

- \( \omega_1(q_i, b_j)(\omega_2(q_i, b_j)) \) is the membership (non-membership) value of output from \( q_i \) on \( b_k \),
\begin{itemize}
    \item $\mu^t(q_i)$ ($\nu^t(q_i)$) is the membership (non-membership) value of $q_i$ at time $t$,
    \item $\omega^t_i(q_i)$ ($\omega^t_{2i}(q_i)$) is the output membership (non-membership) value of $q_i$ at time $t$.
\end{itemize}

**Remark 1.2** We let for all $q \in Q$ such that $q \notin \tilde{R}$, $\mu^0(q) = 0$ and $\nu^0(q) = 1$ and for all $q \in \tilde{R}$, $\mu^0(q) > L 0$.

**Note 1.3** In this paper, we assume that the max-min IGLFA has a finite number of states.

**Definition 1.4** Let $\tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be an IGLFA. We define the max-min intuitionistic general $L$-fuzzy automaton $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ such that $\tilde{\delta}^* : Q_{act} \times X^* \times Q \to L \times L$, where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \ldots \}$ and for every $i \geq 0$,

\begin{equation}
\tilde{\delta}_1^*((q, \mu^t(q), \nu^t(q)), \Lambda, p) = \begin{cases} 
1 & \text{if } p = q, \\
0 & \text{otherwise},
\end{cases}
\end{equation}

and

\begin{equation}
\tilde{\delta}_2^*((q, \mu^t(q), \nu^t(q)), \Lambda, p) = \begin{cases} 
0 & \text{if } p = q, \\
1 & \text{otherwise}.
\end{cases}
\end{equation}

Also for every $i \geq 0$, $\tilde{\delta}_1^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p) = \tilde{\delta}_1((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p)$ and $\tilde{\delta}_2^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p) = \tilde{\delta}_2((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p)$ and recursively,

\begin{equation}
\tilde{\delta}_1((q, \mu^0(q), \nu^0(q)), u_1u_2\ldots u_n, p) = \\
\vee \{\tilde{\delta}_1((q, \mu^0(q), \nu^0(q)), u_1, p_1) \land \tilde{\delta}_1((p_1, \mu^{t_i}(p_1), \nu^{t_i}(p_1)), u_2, p_2) \land \ldots \land \\
\tilde{\delta}_1((p_{n-1}, \mu^{t_{n-1}}(p_{n-1}), \nu^{t_{n-1}}(p_{n-1})), u_n, p) | \\
p_1 \in Q_{act}(t_1), \ldots, p_{n-1} \in Q_{act}(t_{n-1})\},
\end{equation}

\begin{equation}
\tilde{\delta}_2((q, \mu^0(q), \nu^0(q)), u_1u_2\ldots u_n, p) = \\
\vee \{\tilde{\delta}_2((q, \mu^0(q), \nu^0(q)), u_1, p_1) \lor \tilde{\delta}_2((p_1, \mu^{t_i}(p_1), \nu^{t_i}(p_1)), u_2, p_2) \lor \ldots \lor \\
\tilde{\delta}_2((p_{n-1}, \mu^{t_{n-1}}(p_{n-1}), \nu^{t_{n-1}}(p_{n-1})), u_n, p) | \\
p_1 \in Q_{act}(t_1), \ldots, p_{n-1} \in Q_{act}(t_{n-1})\},
\end{equation}

in which $u_i \in X$ for all $1 \leq i \leq n$ and assume that $u_{i+1}$ is the entered input at time $t_i$, for all $0 \leq i \leq n - 1$.

4. Minimal intuitionistic general $L$-fuzzy automata

**Definition 4.1** Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA. Suppose that $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. Then we say that
1. \( \tilde{F}^* \) is \((\alpha, \beta)\)-complete, if for any \( q \in Q, a \in X \) there exists \( p \in Q \) such that 
\[ \delta_1(q, a, p) >_L \alpha \quad \text{and} \quad \delta_2(q, a, p) <_L \beta, \]

2. \( q \in Q \) is \((\alpha, \beta)\)-accessible if there exist \( p \in \tilde{R}, x \in X^* \) such that 
\[ \tilde{\delta}_1((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) >_L \alpha, \quad \text{and} \quad \tilde{\delta}_2((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) <_L \beta, \]

3. \( \tilde{F}^* \) is \((\alpha, \beta)\)-accessible if for any \( q \in Q, a \in X \) there exists at most one \( p \in Q \) such that 
\[ \delta_2(q, a, p) <_L 1. \]

**Definition 4.2** Let \( \tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4) \) be a max-min IGLFA. Then we say that \( \tilde{F}^* \) is deterministic if there exists a unique \( p_0 \in \tilde{R} \) such that 
\[ \mu^{t_0}(p_0) >_L 0, \quad \text{and for any} \quad q \in Q, a \in X \] there exists at most one \( p \in Q \) such that 
\[ \delta_2(q, a, p) <_L 1. \]

**Example 4.3** Consider the complete lattice \( L = (L, \leq_L, T, S, 0, 1) \) as in Figure 1, where 
\[ \begin{align*}
L &= \{0, a, b, c, d, 1\}, \\
N(0) &= 1, \\
N(1) &= 0, \\
N(a) &= b, \\
N(b) &= a, \\
N(c) &= d, \\
N(d) &= c.
\end{align*} \]

![Figure 1: The complete lattice L of Example 4.3](image)

Let the max-min IGLFA \( \tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4) \) as in Figure 2, where 
\[ Q = \{q_0, q_1, q_2, q_3\}, X = \{u, v\}, \tilde{R} = \{(q_0, 1, 0)\}, Z = \{o\} \] and \( \delta : Q \times X \times Q : L \times L \) is defined as follows:
\[ \begin{align*}
\delta(q_0, u, q_1) &= (a, 0) \\
\delta(q_1, v, q_3) &= (a, b) \\
\delta(q_2, u, q_1) &= (1, 0) \\
\delta(q_2, v, q_0) &= (c, 0).
\end{align*} \]

\( \delta(q, x, q') = (0, 1) \) for all other \((q, x, q') \in Q \times X \times Q \) and \( \omega : Q \times Z : L \times L \) is defined by: 
\[ \omega(q_1, o) = (0, 1) \quad \text{and} \quad \omega(q, c) = (1, 0) \] for all other \((q, c) \in Q \times Z \). 
\( q_1 \) is \((0, \alpha)\)-accessible, \((d, \beta)\)-accessible, where \( \alpha \in \{a, b, c, d, 1\} \) and \( \beta \in \{a, b, c, d\} \), \( q_3 \) is \((0, 1)\)-accessible, \((0, c)\)-accessible, \((d, c)\)-accessible but \( q_3 \) is not \((d, 1)\)-accessible, since \( d \not\leq_L N(1) = 0 \). 
\( \tilde{F}^* \) is not \((\alpha, \beta)\)-accessible, not \((\alpha, \beta)\)-complete, for any \((\alpha, \beta) \in L \) in which \( \alpha \leq_L N(\beta) \). Clearly, it is deterministic.
**Theorem 4.4** Let $\tilde{F} = (Q, X, \{p\}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a deterministic $\max$-min IGLFA. If

$$\tilde{\delta}_1^*(\langle p, \mu^o(p), \nu^o(p) \rangle, x, q) \wedge \tilde{\delta}_1^*(\langle p, \mu^o(p), \nu^o(p) \rangle, x, q') >_{L} \alpha,$$

and

$$\tilde{\delta}_2^*(\langle p, \mu^o(p), \nu^o(p) \rangle, x, r) \lor \tilde{\delta}_1^*(\langle p, \mu^o(p), \nu^o(p) \rangle, x, r') <_{L} \beta,$$

then $q = q' = r = r'$, where $q, q', r, r' \in Q, x \in X^*, \alpha, \beta \in L$ and $\alpha \leq N(\beta)$.

**Proof.** First, let $x = \Lambda$. Then we have

$$\tilde{\delta}_1^*(\langle p, \mu^o(p), \nu^o(p) \rangle, \Lambda, q) \wedge \tilde{\delta}_1^*(\langle p, \mu^o(p), \nu^o(p) \rangle, \Lambda, q') >_{L} \alpha, \text{ and}$$

$$\tilde{\delta}_2^*(\langle p, \mu^o(p), \nu^o(p) \rangle, \Lambda, r) \lor \tilde{\delta}_2^*(\langle p, \mu^o(p), \nu^o(p) \rangle, \Lambda, r') <_{L} \beta,$$

which implies that $p = q' = q = r = r'$. Thus the theorem holds for $x = \Lambda$.

Now, we continue the proof for any $x \in X^*$ and $x \notin \Lambda$ by induction on $|x|$. Suppose that $|x| = 1$. Then

$$\tilde{\delta}_1^*(\langle p, \mu^o(p), \nu^o(p) \rangle, x, q) = F_1^T(\nu^o(p), \tilde{\delta}_1(p, x, q)) >_{L} \alpha \text{ and}$$

$$\tilde{\delta}_1^*(\langle p, \mu^o(p), \nu^o(p) \rangle, x, q') = F_1^T(\nu^o(p), \tilde{\delta}_1(p, x, q')) >_{L} \alpha$$

and

$$\tilde{\delta}_2^*(\langle p, \mu^o(p), \nu^o(p) \rangle, x, r) = F_1^S(\nu^o(p), \tilde{\delta}_1(p, x, r)) <_{L} \beta \text{ and}$$

$$\tilde{\delta}_2^*(\langle p, \mu^o(p), \nu^o(p) \rangle, x, r') = F_1^S(\nu^o(p), \tilde{\delta}_1(p, x, r')) <_{L} \beta.$$

These imply that

$$\tilde{\delta}_1(p, x, q) \wedge \tilde{\delta}_1(p, x, q') >_{L} \alpha \text{ and } \tilde{\delta}_2(p, x, r) \lor \tilde{\delta}_2(p, x, r') <_{L} \beta.$$

Since $\tilde{\delta}_1(p, x, q) \wedge \tilde{\delta}_1(p, x, q') >_{L} \alpha$, then $\tilde{\delta}_2(p, x, q) \lor \tilde{\delta}_2(p, x, q') <_{L} N(\alpha) \leq L 1$. Then by Definition 4.2, $q = q' = r = r'$.

Now, suppose the claim holds for all $y \in X^*$ such that $|y| = n - 1$ and $n > 1$. Let $x = ya$, where $y \in X^*, a \in X$ and $|y| = n - 1$. Then

$$\tilde{\delta}_1^*(\langle p, \mu^o(p), \nu^o(p) \rangle, x, q) = \lor \{\tilde{\delta}_1^*(\langle p, \mu^o(p), \nu^o(p) \rangle, y, p') \wedge \tilde{\delta}_1^*(\langle p', \mu^o+n-1(p'), \nu^o+n-1(p') \rangle, a, q) | p' \in Q \} >_{L} \alpha,$$
and
\[ \delta_1((p, \mu^{lo}(p), \nu^{lo}(p)), x, q') = \\bigvee \{ \delta_1((p, \mu^{lo}(p), \nu^{lo}(p)), y, p') \]
\[ \land \delta_1((p', \mu^{lo+n-1}(p'), \nu^{lo+n-1}(p')), a, q') \mid p' \in Q \} >_L \alpha. \]

So, there exist \( d, d' \in Q \) such that
\[ \bigvee \{ \delta_1((p, \mu^{lo}(p), \nu^{lo}(p)), y, p') \land \delta_1((p', \mu^{lo+n-1}(p'), \nu^{lo+n-1}(p')), a, q) \mid p' \in Q \}
\[ = \delta_1((p, \mu^{lo}(p), \nu^{lo}(p)), y, d) \land \delta_1((d, \mu^{lo+n-1}(d), \nu^{lo+n-1}(d)), a, q) >_L \alpha, \]
and
\[ \bigvee \{ \delta_1((p, \mu^{lo}(p), \nu^{lo}(p)), y, p') \land \delta_1((p', \mu^{lo+n-1}(p'), \nu^{lo+n-1}(p')), a, q') \mid p' \in Q \}
\[ = \delta_1((p, \mu^{lo}(p), \nu^{lo}(p)), y, d') \land \delta_1((d', \mu^{lo+n-1}(d'), \nu^{lo+n-1}(d')), a, q') >_L \alpha. \]

Also there exist \( s, s' \in Q \) such that
\[ \delta_2((p, \mu^{lo}(p), \nu^{lo}(p)), x, r) \]
\[ = \bigwedge \{ \delta_2((p, \mu^{lo}(p), \nu^{lo}(p)), y, p') \lor \delta_2((p', \mu^{lo+n-1}(p'), \nu^{lo+n-1}(p')), a, r) \mid p' \in Q \}
\[ = \delta_2((p, \mu^{lo}(p), \nu^{lo}(p)), y, s) \lor \delta_2((s, \mu^{lo+n-1}(s), \nu^{lo+n-1}(s)), a, r) <_L \beta, \]
and
\[ \delta_2((p, \mu^{lo}(p), \nu^{lo}(p)), x, r') \]
\[ = \bigwedge \{ \delta_2((p, \mu^{lo}(p), \nu^{lo}(p)), y, p') \lor \delta_2((p', \mu^{lo+n-1}(p'), \nu^{lo+n-1}(p')), a, r') \mid p' \in Q \}
\[ = \delta_2((p, \mu^{lo}(p), \nu^{lo}(p)), y, s') \lor \delta_2((s', \mu^{lo+n-1}(s'), \nu^{lo+n-1}(s')), a, r') <_L \beta. \]

Therefore, by the induction hypothesis, \( s = s' = d = d' \). Hence
\[ \delta_1((d, \mu^{lo+n-1}(d), \nu^{lo+n-1}(d)), a, q) \land \delta_1((d, \mu^{lo+n-1}(d), \nu^{lo+n-1}(d)), a, q') >_L \alpha, \]
and
\[ \delta_2((s, \mu^{lo+n-1}(s), \nu^{lo+n-1}(s)), a, r) \lor \delta_2((s, \mu^{lo+n-1}(s), \nu^{lo+n-1}(s)), a, r') <_L \beta, \]

imply that \( \delta_1(d, a, q) \land \delta_1(d, a, q') >_L \alpha \) and \( \delta_2(s, a, r) \lor \delta_2(s, a, r') <_L \beta. \) Then
\[ q = q' = r = r'. \] Hence the claim is hold.

**Corollary 4.5** Let \( \tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4) \) be a max-min IGLFA. Notice that if \( \tilde{F}^* \) is an \((\alpha, \beta)\)-complete and deterministic max-min IGLFA, then for any \( p \in Q, a \in X \) there exists exactly one state \( q \in Q \) such that \( \delta_1(p, a, q) >_L \alpha \) and \( \delta_2(p, a, q) <_L \beta \), where \( \alpha, \beta \in \Lambda \) and \( \alpha \leq_L \beta \).

**Example 4.6** Consider the complete lattice \( L = (L, \leq_L, T, S, 0, 1) \) as in Example 4.3, where \( L = \{0, a, b, c, d, 1\} \). Let the max-min IGLFA \( F^* = (Q, X, \tilde{R}, Z, \delta^*, \tilde{\omega}, F_1, F_2, F_3, F_4) \) as in Figure 3, where \( Q = \{q_0, q_1, q_2, q_3\}, X = \{u\}, \tilde{R} = \{(q_0, a, b)\}, \)
\( Z = \{0\} \) and \( \delta : Q \times X \times Q : L \times L \) is defined as follows:
\(\delta(q_0, u, q_1) = (c, d)\)
\(\delta(q_1, u, q_2) = (1, 0)\)
\(\delta(q_2, u, q_1) = (c, 0)\)
\(\delta(q_3, u, q_2) = (1, 0)\)
\(\delta(q, x, q') = (0, 1)\) for all other \((q, x, q') \in Q \times X \times Q\) and \(\omega: Q \times Z \rightarrow L \times L\) is defined by: \(\omega(q_1, o) = (1, 0)\) and \(\omega(q, e) = (0, 1)\) for all other \((q, e) \in Q \times Z\). Then \(\tilde{F}^*\) is \((a, b)\)-complete and deterministic.

**Definition 4.7** Let \(\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)\) be a max-min IGLFA. Then the \((\alpha, \beta)\)-language recognized by \(\tilde{F}^*\) is a subset of \(X^*\) defined by:

\[
\mathcal{L}_{\alpha, \beta}(\tilde{F}^*) = \{ x \in X^* | \tilde{\delta}_1^*((p, \mu^o(p), \nu^o(p)), x, q) \\
\wedge \tilde{\omega}_1((q, \mu^{o+|x|}(q), \nu^{o+|x|}(q)), b) >_L \alpha, \\
\tilde{\delta}_2^*((p, \mu^o(p), \nu^o(p)), x, q) \\
\vee \tilde{\omega}_2((q, \mu^{o+|x|}(q), \nu^{o+|x|}(q)), b') <_L \beta, \\
\text{for some } p \in \tilde{R}, q \in Q, b, b' \in Z \}.
\] (4.1)

**Definition 4.8** Let \(X\) be a nonempty finite set. Then subset \(\mathcal{L}\) of \(X^*\) is called recognizable \((\alpha, \beta)\)-language, if there exists a max-min IGLFA \(\tilde{F}^*\) such that \(\mathcal{L} = \mathcal{L}_{\alpha, \beta}(\tilde{F}^*)\), where \(\alpha, \beta \in L\) and \(\alpha \leq_L N(\beta)\).

**Theorem 4.9** Let \(\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)\) be a max-min IGLFA. Then there exists an \((\alpha, \beta)\)-complete IGLFA \(\tilde{F}^{*c}\) such that \(\mathcal{L}_{\alpha, \beta}(\tilde{F}^*) = \mathcal{L}_{\alpha, \beta}(\tilde{F}^{*c})\), where \(\alpha, \beta \in L\) and \(\alpha \leq_L N(\beta)\).

**Proof.** Let \(\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)\) does not be an \((\alpha, \beta)\)-complete max-min IGLFA. Let \(Q^c = Q \cup \{t\}\), where \(t\) is an element such that \(t \notin Q\). Consider \(\gamma, \eta \in L\), where \(\gamma >_L \alpha, \eta <_L \beta\) and \(\gamma \leq_L N(\eta)\). We now give an algorithm in which the output is an \((\alpha, \beta)\)-complete max-min IGLFA for a given \((\alpha, \beta)\)-incomplete max-min IGLFA as input.
Algorithm: (to make \((\alpha, \beta)\)-complete)

Step 1 input incomplete \(\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)\), where \(Q = \{q_1, q_2, ..., q_n\}\), \(X = \{a_1, a_2, ..., a_m\}\).

Step 2 \(i = 1\),

Step 3 \(j = 1\), if \(i \leq n\) go to the next step, else go to Step 7,

Step 4 if \(j \leq m\), then \(T = \{q \mid \delta_1(q, a_j, q) >_L \alpha\}\) else \(i = i + 1\) go to Step 3,

Step 5 if \(T = \emptyset\), then \(\delta_1^c(q, a_j, t) = \gamma, \delta_2^c(q, a_j, t) = \eta, j = j + 1\), go to Step 4, else go to Step 6,

Step 6 for \(q \in T\) if \(\delta_2(q, a_j, q) <_L \beta\), then \(j = j + 1\) and go to Step 4, else \(T = T - \{q\}\) and go to Step 5,

Step 7 \(\delta_1(q, a, q) = \delta_1(p, a, q)\), and \(\delta_2^c(p, a, q) = \delta_2(p, a, q)\), for all \(p, q \in Q, a \in X\), \(\delta_1^c(t, a, t) = \gamma, \delta_2^c(t, a, t) = \eta\) for all \(a \in X\),

Step 8 \(Q^c = Q \cup \{t\}\), \(\omega_1^c(p, b) = \begin{cases} \omega_1(p, b) & \text{if } p \neq t, \\ 0 & \text{if } p = t, \end{cases}\) and \(\omega_2^c(p, b) = \begin{cases} \omega_2(p, b) & \text{if } p \neq t, \\ 1 & \text{if } p = t, \end{cases}\)

Step 9 output \(\tilde{F}^{sc} = (Q^c, X, \tilde{R}, Z, \tilde{\delta}^{sc}, \tilde{\omega}_c, F_1, F_2, F_3, F_4)\).

It is easy to see that the max-min IGLFA \(\tilde{F}^{sc} = (Q^c, X, \tilde{R}, Z, \tilde{\delta}^{sc}, \tilde{\omega}_c, F_1, F_2, F_3, F_4)\) is \((\alpha, \beta)\)-complete. It is clear that \(L^{\alpha, \beta}(\tilde{F}^*) \subseteq L^{\alpha, \beta}(\tilde{F}^{sc})\). Let \(x = u_1u_2...u_{k+1} \in L^{\alpha, \beta}(\tilde{F}^{sc})\). Then there exist \(p \in \tilde{R}, q \in Q^c, b, b' \in Z\) such that

\[\tilde{\delta}_1^{sc}((p, \mu^\alpha(p), \nu^\alpha(p)), x, q) \land \tilde{\omega}_1^c((q, \mu^{1+|x|}(q), \nu^{1+|x|}(q)), b) >_L \alpha,\]

and

\[\tilde{\delta}_2^{sc}((p, \mu^\alpha(p), \nu^\alpha(p)), x, q) \lor \tilde{\omega}_2^c((q, \mu^{1+|x|}(q), \nu^{1+|x|}(q)), b') <_L \beta.\]

These imply that \(q \in Q\) and there exist \(p_1, p_2, ..., p_k, p'_1, p'_2, ..., p'_k \in Q\) such that

\[(4.2) \quad \tilde{\delta}_1^c((p, \mu^\alpha(p), \nu^\alpha(p)), u_1, p_1) \land ... \land \tilde{\delta}_1^c((p_k, \mu^{1+|x|}(p_k), \nu^{1+|x|}(p_k)), u_k, q) >_L \alpha,\]

and

\[(4.3) \quad \tilde{\delta}_2^c((p, \mu^\alpha(p), \nu^\alpha(p)), u_1, p'_1) \lor ... \lor \tilde{\delta}_2^c((p_k, \mu^{1+|x|}(p_k), \nu^{1+|x|}(p_k)), u_k, q) <_L \beta.\]

Definition 2.2 and (4.2) imply that \(\mu^\alpha(p) >_L \alpha, \delta_1^c(p, u_1, p_1) >_L \alpha, \delta_1^c(p_1, u_2, p_2) >_L \alpha, ..., \delta_1^c(p_k, u_k, q) >_L \alpha\). Now, suppose that \(p_j, 1 \leq j \leq k\), be the first state that \(\delta_1^c(p_j, u_j, p_{j+1}) >_L \alpha\) and \(\delta_1^c(p_j, u_j, p_{j+1})\) was undefined. Then \(p_{j+1} = t\). Therefore \(p_{j+1} = p_{j+2} = ... = q = t\), which is a contradiction. Hence

\[\tilde{\delta}_1^c((p, \mu^\alpha(p), \nu^\alpha(p)), x, q) \land \tilde{\omega}_1^c((q, \mu^{1+|x|}(q), \nu^{1+|x|}(q)), b) >_L \alpha.\]
In a similar manner we obtain
\[
\tilde{\delta}^*_2((p, \mu^l_{t_0}(p), \nu^l_{t_0}(p)), x, q) \lor \tilde{\omega}_2((q, \mu^l_{t_0+|x|}(q), \nu^l_{t_0+|x|}(q)), b') <_L \beta.
\]
Then the claim is hold.

**Example 4.10** Consider the complete lattice \( L = (L, \leq_L, T, S, 0, 1) \) defined in Example 4.3, and max-min IGLFA \( \tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4) \) as in Figure 4,

![Figure 4: The IGLFA of Example 4.10](image)

where \( Q = \{q_0, q_1, q_2, q_3\}, X = \{a_0 = u, a_1 = v\}, \tilde{R} = \{(q_0, 1, 0)\}, Z = \{o\} \) and \( \delta : Q \times X \times Q \rightarrow L \times L \) is defined as follows:

\[
\begin{align*}
\delta(q_0, u, q_1) = (a, 0) & & \delta(q_1, v, q_3) = (a, d) \\
\delta(q_2, u, q_1) = (1, 0) & & \delta(q_2, v, q_0) = (c, d)
\end{align*}
\]

\( \delta(q, x, q') = (0, 1) \) for all other \((q, x, q') \in Q \times X \times Q\) and \( \omega : Q \times Z \rightarrow L \times L \) is defined by: \( \omega(q_1, o) = (1, 0) \) and \( \omega(q, e) = (0, 1) \) for all other \((q, e) \in Q \times Z\).

Now, considering the complete algorithm in the proof of Theorem 4.9, and \( \alpha = a, \beta = d \), we have

**Stage 1** Let \( i = 0, j = 0 \). Then \( T = \emptyset \) and \( \delta_1^*(q_0, u, t) = c, \delta_2^*(q_0, u, t) = 0 \).

**Stage 2** Let \( i = 0, j = 1 \). Then \( T = \emptyset \) and \( \delta_1^*(q_0, v, t) = c, \delta_2^*(q_0, v, t) = 0 \).

**Stage 3** Let \( i = 1, j = 0 \). Then \( T = \emptyset \) and \( \delta_1^*(q_1, u, t) = c, \delta_2^*(q_1, u, t) = 0 \).

**Stage 4** Let \( i = 1, j = 1 \). Then \( T = \emptyset \) and \( \delta_1^*(q_1, v, t) = c, \delta_2^*(q_1, u, t) = 0 \).

**Stage 5** Let \( i = 2, j = 0 \). Then \( T = \{q_1\} \) and we have \( \delta_2(q_2, u, q_1) = 0 <_L d \).

**Stage 6** Let \( i = 2, j = 1 \). Then \( T = \{q_0\} \). Since \( \delta_2(q_2, v, q_0) = d \),

thus \( T = T - \{q_0\} = \emptyset \). Then \( \delta_1^*(q_2, v, t) = c, \delta_2^*(q_2, v, t) = 0 \).

**Stage 7** Let \( i = 3, j = 0 \). Then \( T = \emptyset \) and \( \delta_1^*(q_3, u, t) = c, \delta_2^*(q_3, u, t) = 0 \).

**Stage 8** Let \( i = 3, j = 1 \). Then \( T = \emptyset \) and \( \delta_1^*(q_3, v, t) = c, \delta_2^*(q_3, v, t) = 0 \).
Therefore, we have an \((a, d)\)-complete \(\tilde{F}^{sc} = (Q^c, X, \tilde{R}, Z, \tilde{\delta}^{sc}, \tilde{\omega}^c, F_1, F_2, F_3, F_4)\) as in Figure 5.

**Theorem 4.11** Let \(\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)\) be a max-min IGLFA and \(\tilde{R} \neq \emptyset\). Then there exists a deterministic max-min IGLFA \(\tilde{F}^*_d\) such that \(L^{\alpha, \beta}(\tilde{F}^*) = L^{\alpha, \beta}(\tilde{F}^*_d)\), where \(\alpha, \beta \in L\) and \(\alpha \leq_L N(\beta)\).

**Proof.** Let

\[
I_x = \{q' \in Q | \exists q \in \tilde{R} \text{ such that } \tilde{\delta}_1^*(((p, \mu^0(p), \nu^0(p)), x, q') >_L \alpha, \\
\tilde{\delta}_2^*((p, \mu^0(p), \nu^0(p)), x, q') <_L \beta\},
\]

for all \(x \in X^*\). Then \(I_A = \{q' \in Q | q' \in \tilde{R}\}\).

Let \(Q_d = \{I_x | x \in X^*\}\). Define \(\delta_d : Q_d \times X \times Q_d \rightarrow L \times L\), where

\[
\delta_{d1}(I_y, a, I_x) = \begin{cases} 
\gamma_1 & \text{if } I_x = I_ya, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\delta_{d2}(I_y, a, I_x) = \begin{cases} 
\eta_1 & \text{if } I_x = I_ya, \\
1 & \text{otherwise},
\end{cases}
\]

and \(\omega_d : Q_d \times Z_d \rightarrow L \times L\), where

\[
\omega_{d1}(I_x, e) = \begin{cases} 
\gamma_2 & \text{if } x \in L^{\alpha, \beta}(\tilde{F}^*), \\
0 & \text{otherwise},
\end{cases}
\]

\[
\omega_{d2}(I_x, e) = \begin{cases} 
\eta_2 & \text{if } x \in L^{\alpha, \beta}(\tilde{F}^*), \\
1 & \text{otherwise},
\end{cases}
\]
where \( \gamma_1, \eta_1, \gamma_2, \eta_2 \in L, \gamma_1 \land \gamma_2 > L \alpha, \eta_1 \lor \eta_2 < L \beta, \gamma_1 \leq L \forall \eta_1 \) and \( \gamma_2 \leq L \forall \eta_2 \).

Consider \( \mu_0(I_\lambda) = \vee \{ \mu_0(q) \mid q \in I_\lambda \}, \nu_0(I_\lambda) = \wedge \{ \nu_0(q) \mid q \in I_\lambda \} \) and \( Z_d = \{ \epsilon \} \).

Now, suppose that \( \tilde{F}_d^* = (Q_d, X, I_d, Z_d, \tilde{\delta}_d, \omega_d, F_1, F_2, F_3, F_4) \). It is clear that \( \delta_d \) is well defined. Now, we show that \( \omega_d \) is well defined. Let \( I_x = I_y \) and \( e \in Z_d \). If \( x \in \mathcal{L}^{\alpha, \beta}(\tilde{F}^*) \), then there exist \( q \in \tilde{R}, p \in Q, b, b' \in Z \) such that

\[
\tilde{\delta}_1^*((q, \mu_0(q), \nu_0(q)), x, p) \land \tilde{\omega}_1((p, \mu_0+(x)(p), \nu_0+(x)(p)), b) > L \alpha,
\]

and

\[
\tilde{\delta}_2^*((q, \mu_0(q), \nu_0(q)), x, p) \lor \tilde{\omega}_2((p, \mu_0+(x)(p), \nu_0+(x)(p)), b') < L \beta.
\]

Thus \( p \in I_x = I_y \). Therefore

\[
\tilde{\delta}_1^*((q, \mu_0(q), \nu_0(q)), y, p) > L \alpha \quad \text{and} \quad \tilde{\delta}_2^*((q, \mu_0(q), \nu_0(q)), y, p) < L \beta
\]

for some \( q \in \tilde{R} \). Also \( \omega_1(p, b) > L \alpha \) and \( \omega_2(p, b) < L \beta \). Hence \( y \in \mathcal{L}^{\alpha, \beta}(\tilde{F}^*) \). In a similar way, if \( y \in \mathcal{L}^{\alpha, \beta}(\tilde{F}^*) \), then \( x \in \mathcal{L}^{\alpha, \beta}(\tilde{F}^*) \). Hence \( \omega_d(I_x, e) = \omega_d(I_y, e) \).

Since there exists \( q \in \tilde{R} \) such that \( \mu_0(q) > L 0 \), then \( \mu_0(I_\lambda) > L 0 \). It is easy to see that the max-min IGLFA \( \tilde{F}_d^* \) is deterministic. We show that \( \mathcal{L}^{\alpha, \beta}(\tilde{F}^*) = \mathcal{L}^{\alpha, \beta}(\tilde{F}_d^*) \). Let \( x = u_1 u_2 \ldots u_{k+1} \in \mathcal{L}^{\alpha, \beta}(\tilde{F}^*) \). Then there exist \( q \in \tilde{R}, p \in Q, b, b' \in Z \) such that

\[
\tilde{\delta}_1^*((q, \mu_0(q), \nu_0(q)), u_1 \ldots u_{k+1}, p) > L \alpha \quad \text{and} \quad \tilde{\delta}_2^*((q, \mu_0(q), \nu_0(q)), u_1 \ldots u_{k+1}, p) < L \beta,
\]

then \( \mu_0(q) > L \alpha \) and \( \nu_0(q) < L \beta \). Thus \( \mu_0(I_\lambda) > L \alpha \) and \( \nu_0(I_\lambda) < L \beta \). Also

\[
\tilde{\delta}_d^*((I_\lambda, \mu_0(I_\lambda), \nu_0(I_\lambda)), u_1, u_1, u_1) = \tilde{F}_1^T(\mu_0(I_\lambda), \nu_0(I_\lambda), \delta_d(I_\lambda, u_1, u_1)) \geq L \alpha,
\]

thus \( \mu^1(I_{u_1}) \geq L \alpha \). Also we have

\[
\tilde{\delta}_d^*((I_{u_1}, \mu^1(I_{u_1}), \nu^1(I_{u_1})), u_2, u_1 u_2) = \tilde{F}_1^T(\mu^1(I_{u_1}), \nu^1(I_{u_1}), \delta_d(I_{u_1}, u_2, u_1 u_2)) \geq L \alpha.
\]

So if we continue this process, then by some manipulation we get that

\[
\tilde{\delta}_d^*((I_\lambda, \mu_0(I_\lambda), \nu_0(I_\lambda)), x, I_x) \geq L \alpha.
\]

Also, we have

\[
\tilde{\delta}_d^*((I_\lambda, \mu_0(I_\lambda), \nu_0(I_\lambda)), u_1, u_1) = \tilde{F}_1^T(\nu_0(I_\lambda), \delta_d(I_\lambda, u_1, u_1)) \leq L \beta,
\]

then \( \nu^1(I_{u_1}) \leq L \beta \). Therefore

\[
\tilde{\delta}_d^*((I_{u_1}, \mu^1(I_{u_1}), \nu^1(I_{u_1})), u_2, u_1 u_2) \leq L \beta.
\]
By continuing this process we obtain that

$$\tilde{\delta}_{d_2}((I_A, \mu^{lo}(I_A), \nu^{lo}(I_A)), x, I_x) \leq_L \beta.$$ 

Also we have $\omega_{d_1}(I_x, e) = \gamma_2$ and $\omega_{d_2}(I_x, e) = \eta_2$. Hence $x \in \mathcal{L}^{\alpha, \beta}(\hat{F}^*_a)$. Now, suppose that $x \in \mathcal{L}^{\alpha, \beta}(\hat{F}^*_d)$. Then

$$\tilde{\delta}_{d_1}((I_A, \mu^{lo}(I_A), \nu^{lo}(I_A)), x, I_x) \land \tilde{\omega}_{d_1}((I_x, \mu^{lo+\epsilon x}(I_x), \nu^{lo+\epsilon x}(I_x)), e) >_L \alpha,$$

and

$$\tilde{\delta}_{d_2}((I_A, \mu^{lo}(I_A), \nu^{lo}(I_A)), x, I_x) \lor \tilde{\omega}_{d_2}((I_x, \mu^{lo+\epsilon x}(I_x), \nu^{lo+\epsilon x}(I_x)), e) <_L \beta.$$ 

Since $\omega_{d_1}(I_x, e) >_L \alpha$ and $\omega_{d_2}(I_x, e) <_L \beta$, then by (4.5) and (4.6) $x \in \mathcal{L}^{\alpha, \beta}(\hat{F}^*_a)$.

**Theorem 4.12** Let $\hat{F}^* = (Q, X, \hat{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA and $\hat{R} \neq \emptyset$. Then there exists an $(\alpha, \beta)$-accessible max-min IGLFA $\tilde{F}^*_a$ such that $\mathcal{L}^{\alpha, \beta}(\hat{F}^*) = \mathcal{L}^{\alpha, \beta}(\tilde{F}^*_a)$, where $\alpha, \beta \in L$ and $\alpha \leq L N(\beta)$.

**Proof.** By Theorem 4.11, without loss of generality we assume that $\tilde{F}^*$ is deterministic. Let $S = \{q \in Q \mid q$ be an $(\alpha, \beta)$-accessible state$\}, \hat{R}_a = \{(q, \mu^{lo}(q), \nu^{lo}(q)) \mid q \in S|_R\}, Z_a = \hat{Z}, \delta_a = \delta|_{S \times X \times S}$, and $\omega_a = \omega|_{S \times Z}$, i.e., $\delta_a$ is the restriction of $\delta$ to $S \times X \times S$ and $\omega_a$ is the restriction of $\omega$ to $S \times Z$. Then the max-min IGLFA $\tilde{F}^*_a$ is $(\alpha, \beta)$-accessible. It is clear that $\mathcal{L}^{\alpha, \beta}(\tilde{F}^*_a) \subseteq \mathcal{L}^{\alpha, \beta}(\tilde{F}^*)$. Now, let $x = u_1 u_2 ... u_{k+1} \in \mathcal{L}^{\alpha, \beta}(\tilde{F}^*)$. Then there exist $q \in \hat{R}, p \in Q, b \in Z$ such that

$$\tilde{\delta}_1^*((q, \mu^{lo}(q), \nu^{lo}(q)), x, p) \land \tilde{\omega}_1((p, \mu^{lo+\epsilon x}(p), \nu^{lo+\epsilon x}(p)), b) >_L \alpha,$$

and

$$\tilde{\delta}_2^*((q, \mu^{lo}(q), \nu^{lo}(q)), x, p) \lor \tilde{\omega}_2((p, \mu^{lo+\epsilon x}(p), \nu^{lo+\epsilon x}(p)), b') <_L \beta.$$ 

Therefore, there exist $p_1, p_2, ..., p_k, p'_2, ..., p'_k \in Q$ such that

$$\tilde{\delta}_1^*((q, \mu^{lo}(q), \nu^{lo}(q)), u_1, p_1) \land ... \land \tilde{\delta}_1^*((p_k, \mu^{lo+\epsilon x}(p_k), \nu^{lo+\epsilon x}(p_k)), u_{k+1}, p) >_L \alpha,$$

and

$$\tilde{\delta}_2^*((q, \mu^{lo}(q), \nu^{lo}(q)), u_1, p'_1) \lor ... \lor \tilde{\delta}_2^*((p'_k, \mu^{lo+\epsilon x}(p'_k), \nu^{lo+\epsilon x}(p'_k)), u_{k+1}, p) <_L \beta.$$ 

These imply that $q \in \hat{R}_a$ and, since $\tilde{F}^*$ is a deterministic, then

$$p_1 = p'_1, p_2 = p'_2, ..., p_k = p'_k.$$

Thus $p_1, p_2, ..., p_k, p \in S$. Hence

$$\tilde{\delta}_1^*((q, \mu^{lo}(q), \nu^{lo}(q)), x, p) \land \tilde{\omega}_1((p, \mu^{lo+\epsilon x}(p), \nu^{lo+\epsilon x}(p)), b) >_L \alpha,$$

and

$$\tilde{\delta}_2^*((q, \mu^{lo}(q), \nu^{lo}(q)), x, p) \lor \tilde{\omega}_2((p, \mu^{lo+\epsilon x}(p), \nu^{lo+\epsilon x}(p)), b') <_L \beta.$$ 

Then $x \in \mathcal{L}^{\alpha, \beta}(\tilde{F}^*_a)$. Hence $\mathcal{L}^{\alpha, \beta}(\tilde{F}^*_a) \subseteq \mathcal{L}^{\alpha, \beta}(\hat{F}^*_a)$. Thus the claim is hold.
Example 4.13 Consider the complete lattice $L = (L, \leq_L, T, S, 0, 1)$ defined in Example 4.3, and max-min IGLFA $\tilde{F}^* = (Q, X, \tilde{R}, Z, \delta^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ as in Figure 6,

![Figure 6: The IGLFA of Example 4.13](image)

where $Q = \{q_0, q_1, q_2, q_3, q_4\}$, $X = \{u, v\}$, $\tilde{R} = \{(q_0, 1, 0)\}$, $Z = \{o\}$. It is clear that $L_{a,b}(\tilde{F}^*) = \{u, v\}^*uv\{u, v\} \cup \{u, v\}^*vu\{u, v\}$. Considering the proof of Theorems 4.9, 4.11, 4.12, we obtain an $(a, b)$-complete, deterministic and $(a, b)$-accessible max-min IGLFA $\tilde{F}^*_{cda}$ as in Figure 7, such that $L_{a,b}(\tilde{F}^*_{cda}) = L_{a,b}(\tilde{F}^*)$.

![Figure 7: The $(a, b)$-complete, deterministic and $(a, b)$-accessible $\tilde{F}^*_{cda}$ of Example 4.13](image)
Definition 4.14 Let $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be an $(\alpha, \beta)$-accessible, $(\alpha, \beta)$-complete, deterministic max-min IGLFA, where $\alpha, \beta \in L$ and $\alpha \leq \beta \leq N(\beta)$. We define a relation on $Q$ by $q_1 \rho^{\alpha, \beta} q_2$, if and only if the following two sets are equal

(i) \[ \{ w \in X^* \mid \tilde{\delta}_1^*((q_1, \mu^{\alpha}(q_1), \nu^{\beta}(q_1)), w, q) \land \tilde{\omega}_1((q, \mu^{\alpha+1}[w](q), \nu^{\beta+1}[w](q)), b) >_L \alpha, \] \[ \tilde{\delta}_2^*((q_1, \mu^{\alpha}(q_1), \nu^{\beta}(q_1)), w, q) \lor \tilde{\omega}_2((q, \mu^{\alpha+1}[w](q), \nu^{\beta+1}[w](q)), b') <_L \beta, \] for some $b, b' \in Z, q \in Q \} \]

and

(ii) \[ \{ w \in X^* \mid \tilde{\delta}_1^*((q_2, \mu^{\alpha}(q_2), \nu^{\beta}(q_2)), w, q) \land \tilde{\omega}_1((q, \mu^{\alpha+1}[w](q), \nu^{\beta+1}[w](q)), b) >_L \alpha, \] \[ \tilde{\delta}_2^*((q_2, \mu^{\alpha}(q_2), \nu^{\beta}(q_2)), w, q) \lor \tilde{\omega}_2((q, \mu^{\alpha+1}[w](q), \nu^{\beta+1}[w](q)), b') <_L \beta, \] for some $b, b' \in Z, q \in Q \}, \]

where $q_1 \in Q_{act}(t_i)$, $q_2 \in Q_{act}(t_j)$. It is clear that $\rho^{\alpha, \beta}$ is an equivalence relation.

Definition 4.15 We say that the $(\alpha, \beta)$-accessible, $(\alpha, \beta)$-complete, deterministic max-min IGLFA $\tilde{F}^*$, where $\alpha, \beta \in L$ and $\alpha \leq \beta \leq N(\beta)$, is $(\alpha, \beta)$-reduced if $q_1 \rho^{\alpha, \beta} q_2$ implies that $q_1 = q_2$, for any $q_1, q_2 \in Q$.

Let $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be an $(\alpha, \beta)$-accessible, $(\alpha, \beta)$-complete, deterministic max-min IGLFA, where $\alpha, \beta \in L$ and $\alpha \leq \beta \leq N(\beta)$. Now, suppose that $\rho^{\alpha, \beta}$ be the equivalence relation defined in Definition 4.14. Consider $Q/\rho^{\alpha, \beta} = \{ q \rho^{\alpha, \beta} \mid q \in Q \}$ and $R/\rho^{\alpha, \beta} = q_0 \rho^{\alpha, \beta}, \mu^{\alpha}(q_0) = \mu^{\alpha}(q_0), \nu^{\alpha}(q_0) = \nu^{\alpha}(q_0)$. We define $\delta_\rho: Q/\rho^{\alpha, \beta} \times X \times Q/\rho^{\alpha, \beta} \rightarrow L \times L$ by:

\[
\delta_{\rho_1}(q_1 \rho^{\alpha, \beta}, a, q_2 \rho^{\alpha, \beta}) = \begin{cases} 
\gamma_1 & \text{if } \delta_1(q_1, a, q_2) >_L \alpha \land \delta_2(q_1, a, q_2') <_L \beta, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\delta_{\rho_2}(q_1 \rho^{\alpha, \beta}, a, q_2 \rho^{\alpha, \beta}) = \begin{cases} 
\eta_1 & \text{if } \delta_1(q_1, a, q_2') >_L \alpha \land \delta_2(q_1, a, q_2') <_L \beta, \\
1 & \text{otherwise},
\end{cases}
\]

where $\gamma_1, \eta_1 \in L, \gamma_1 >_L \alpha, \eta_1 <_L \beta$ and $\gamma_1 \leq_L N(\eta_1)$.

Define $\omega_\rho: Q/\rho^{\alpha, \beta} \times Z \rightarrow L \times L$ by:

\[
\omega_{\rho_1}(q_1 \rho^{\alpha, \beta}, b) = \begin{cases} 
\gamma_2 & \text{if } \omega_1(q_1, b') >_L \alpha \land \delta_2(q_1, b'') <_L \beta, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\omega_{\rho_2}(q_1 \rho^{\alpha, \beta}, b) = \begin{cases} 
\eta_2 & \text{if } \omega_1(q_1, b') >_L \alpha \land \delta_2(q_1, b'') <_L \beta, \\
1 & \text{otherwise},
\end{cases}
\]

for some $b, b'' \in Z$ and $\gamma_2, \eta_2 \in L, \gamma_2 >_L \alpha, \eta_2 <_L \beta$ and $\gamma_2 \leq_L N(\eta_2)$.

Theorem 4.16 Suppose that $\alpha, \beta \in L, \alpha \leq_L N(\beta)$. Then the following properties hold:
1. \( \delta_\rho \) and \( \omega_\rho \) are well defined,

2. \( \hat{F}^*/\rho^{\alpha,\beta} = (Q/\rho^{\alpha,\beta}, X, q_0 \rho^{\alpha,\beta}, Z, \bar{\delta}_\rho, \bar{\omega}_\rho, F_1, F_2, F_3, F_4) \) is an \((\alpha, \beta)\)-reduced maximin IGLFA,

3. \( \mathcal{L}^{\alpha,\beta}(\hat{F}^*/\rho^{\alpha,\beta}) = \mathcal{L}^{\alpha,\beta}(\hat{F}^*) \).

**Proof.** Let \( q_1 \rho^{\alpha,\beta}, q_2 \rho^{\alpha,\beta}, p_1 \rho^{\alpha,\beta}, p_2 \rho^{\alpha,\beta} \in Q/\rho^{\alpha,\beta}, q_1 \rho^{\alpha,\beta} = p_1 \rho^{\alpha,\beta} \) and \( q_2 \rho^{\alpha,\beta} = p_2 \rho^{\alpha,\beta} \). Then \( q_1 \rho^{\alpha,\beta} p_1 \) and \( q_2 \rho^{\alpha,\beta} p_2 \). Therefore

\[
A = \{ w \in X^* \mid \tilde{\delta}_1((q_1, \mu^{i_1}(q_1), \nu^{i_1}(q_1)), w, q) \land \tilde{\omega}_1((q, \mu^{i_1+|w|}(q), \nu^{i_1+|w|}(q)), b) >_L \alpha, \\
\tilde{\delta}_2((q_1, \mu^{i_1}(q_1), \nu^{i_1}(q_1)), w, q) \lor \tilde{\omega}_2((q, \mu^{i_1+|w|}(q), \nu^{i_1+|w|}(q)), b') <_L \beta, \\
\text{for some } b, b' \in Z, q \in Q \} = \\
\{ w \in X^* \mid \tilde{\delta}_1((q_1, \mu^{i_1}(q_1), \nu^{i_1}(q_1)), w, q) \land \tilde{\omega}_1((q_2, \mu^{i_1+|w|}(q_2), \nu^{i_1+|w|}(q_2)), w, q) \land \tilde{\omega}_2((q, \mu^{i_1+|w|}(q), \nu^{i_1+|w|}(q)), b') <_L \beta, \\
\text{for some } b, b' \in Z, q_2 \in Q \}.
\]

Let \( \delta_1(q_1 \rho^{\alpha,\beta}, a, q_2 \rho^{\alpha,\beta}) = \gamma \) and \( \delta_2(q_1 \rho^{\alpha,\beta}, a, q_2 \rho^{\alpha,\beta}) = \eta \), where \( \gamma, \eta \in L \) and \( \gamma \leq_L N(\eta) \). Then there exists \( q_2' \in Q \) such that \( q_2' \rho^{\alpha,\beta} q_2 \) and \( \delta_1(q_1, a, q_2') >_L \alpha \) and \( \delta_2(q_1, a, q_2') <_L \beta \). We have \( w \in A \) if and only \( w \in B \), for all \( w \in X^* \). This holds in particular for all \( w \in aX^* \). Then

\[
\{ aw \in aX^* \mid \tilde{\delta}_1((q_1, \mu^{i_1}(q_1), \nu^{i_1}(q_1)), a, q_2'), w, q) \land \tilde{\omega}_1((q_2, \mu^{i_1+1}(q_2), \nu^{i_1+1}(q_2)), w, q) \land \tilde{\omega}_2((q, \mu^{i_1+|w|}(q), \nu^{i_1+|w|}(q)), b') <_L \beta, \\
\text{for some } b, b' \in Z, q_2' \in Q \} = \\
\{ aw \in aX^* \mid \tilde{\delta}_1((p_1, \mu^{i_1}(p_1), \nu^{i_1}(p_1)), a, p') \land \tilde{\delta}_1((p', \mu^{i_1+1}(p'), \nu^{i_1+1}(p')), w, q) \land \tilde{\omega}_1((q, \mu^{i_1+|w|}(q), \nu^{i_1+|w|}(q)), b') <_L \beta, \\
\text{for some } b, b' \in Z, q, p' \in Q \}.
\]

These imply that

\[
\{ w \in X^* \mid \tilde{\delta}_1((q_2, \mu^{i_1+1}(q_2), \nu^{i_1+1}(q_2)), w, q) \land \tilde{\omega}_1((q, \mu^{i_1+|w|}(q), \nu^{i_1+|w|}(q)), b') <_L \beta, \\
\text{for some } b, b' \in Z, q \in Q \} = \\
\{ w \in X^* \mid \tilde{\delta}_1((p', \mu^{i_1+1}(p'), \nu^{i_1+1}(p')), w, q) \land \tilde{\omega}_1((q, \mu^{i_1+|w|}(q), \nu^{i_1+|w|}(q)), b') <_L \beta, \\
\text{for some } b, b' \in Z, q \in Q \},
\]
Thus \( p'_1 \rho^{\alpha, \beta} q'_1 \rho^{\alpha, \beta} q_2 \rho^{\alpha, \beta} p_2 \) i.e., \( p'_1 \rho^{\alpha, \beta} p_2 \). Also, \( \tilde{F}^* \) is \((\alpha, \beta)\)-complete and deterministic, therefore \( \delta_1 (p_1, a, p') >_L \alpha \) and \( \delta_2 (p_1, a, p') <_L \beta \). Hence

\[
\delta_{p_1} (p_1 \rho^{\alpha, \beta}, a, p_2 \rho^{\alpha, \beta}) = \gamma \quad \text{and} \quad \delta_{p_2} (p_1 \rho^{\alpha, \beta}, a, p_2 \rho^{\alpha, \beta}) = \eta.
\]

So \( \delta_\rho \) is well defined.

Now, let \( q_1 \rho^{\alpha, \beta} = p_1 \rho^{\alpha, \beta} \). If \( \omega_{p_1} (q_1 \rho^{\alpha, \beta}, b) = \gamma' \) and \( \omega_{p_2} (q_1 \rho^{\alpha, \beta}, b) = \eta' \), where \( \gamma', \eta' \in L \) and \( \gamma' \leq_L N(\eta') \), then \( \omega_{q_1} (q_1, b') >_L \alpha \) and \( \omega_{q_2} (q_1, b'') <_L \beta \), for some \( b \in Z_p, b', b'' \in Z \). Since \( \tilde{F}^* \) is \((\alpha, \beta)\)-accessible, then there exists a positive integer \( j \) such that \( \mu^l_j (q_1) >_L \alpha \) and \( \nu^l_j (q_1) <_L \beta \). These imply that \( \Lambda \in A \). Therefore \( \Lambda \in B \). Then \( \omega_{q_1} (q_2, b') >_L \alpha \) and \( \omega_{q_2} (q_2, b'') <_L \beta \), for some \( b', b'' \in Z \). Hence

\[
\omega_{p_1} (q_2 \rho^{\alpha, \beta}, b) = \gamma' \quad \text{and} \quad \omega_{p_2} (q_2 \rho^{\alpha, \beta}, b) = \eta'.
\]

Then \( \omega_\rho \) is well defined.

2. Let \((q_1 \rho^{\alpha, \beta}) \rho^{\alpha, \beta} (q_2 \rho^{\alpha, \beta}) \). Now, we have to show that \( q_1 \rho^{\alpha, \beta} = q_2 \rho^{\alpha, \beta} \). It suffices to show that \( q_1 \rho^{\alpha, \beta} q_2 \).

Let for \( w \in X^* \)

\[
\tilde{\delta}_{p_1} ((q_1 \rho^{\alpha, \beta}, \mu^l_k (q_1 \rho^{\alpha, \beta}), \nu^l_k (q_1 \rho^{\alpha, \beta})), w, p_1) \land \tilde{\omega}_{p_1} ((p_1 \rho^{\alpha, \beta}, \mu^{l_1 + |w|} (p_1 \rho^{\alpha, \beta}), \nu^{l_1 + |w|} (p_1 \rho^{\alpha, \beta})), b) >_L \alpha,
\]

\[
\tilde{\delta}_{p_2} ((q_1 \rho^{\alpha, \beta}, \mu^l_k (q_1 \rho^{\alpha, \beta}), \nu^l_k (q_1 \rho^{\alpha, \beta})), w, p_1) \lor \tilde{\omega}_{p_2} ((p_1 \rho^{\alpha, \beta}, \mu^{l_1 + |w|} (p_1 \rho^{\alpha, \beta}), \nu^{l_1 + |w|} (p_1 \rho^{\alpha, \beta})), b') <_L \beta,
\]

for some \( b, b' \in Z \). Then

\[
\tilde{\delta}_{p_1} ((q_1 \rho^{\alpha, \beta}, \mu^l_j (q_2 \rho^{\alpha, \beta}), \nu^l_j (q_2 \rho^{\alpha, \beta})), w, p'_1 \rho^{\alpha, \beta}) \land \tilde{\omega}_{p_1} ((p'_1 \rho^{\alpha, \beta}, \mu^{l_1 + |w|} (p'_1 \rho^{\alpha, \beta}), \nu^{l_1 + |w|} (p'_1 \rho^{\alpha, \beta})), b) >_L \alpha,
\]

\[
\tilde{\delta}_{p_2} ((q_2 \rho^{\alpha, \beta}, \mu^l_j (q_2 \rho^{\alpha, \beta}), \nu^l_j (q_2 \rho^{\alpha, \beta})), w, p'_1 \rho^{\alpha, \beta}) \lor \tilde{\omega}_{p_2} ((p'_1 \rho^{\alpha, \beta}, \mu^{l_1 + |w|} (p'_1 \rho^{\alpha, \beta}), \nu^{l_1 + |w|} (p'_1 \rho^{\alpha, \beta})), b') <_L \beta,
\]

for some \( b, b' \in Z \). Therefore

\[
\tilde{\delta}_{p_1} ((q_2 \rho^{\alpha, \beta}, \mu^l_j (q_2 \rho^{\alpha, \beta}), \nu^l_j (q_2 \rho^{\alpha, \beta})), w, p'_1 \rho^{\alpha, \beta}) \land \tilde{\omega}_{p_1} ((p'_1 \rho^{\alpha, \beta}, \mu^{l_1 + |w|} (p'_1 \rho^{\alpha, \beta}), \nu^{l_1 + |w|} (p'_1 \rho^{\alpha, \beta})), b) >_L \alpha,
\]

\[
\tilde{\delta}_{p_2} ((q_2 \rho^{\alpha, \beta}, \mu^l_j (q_2 \rho^{\alpha, \beta}), \nu^l_j (q_2 \rho^{\alpha, \beta})), w, p'_1 \rho^{\alpha, \beta}) \lor \tilde{\omega}_{p_2} ((p'_1 \rho^{\alpha, \beta}, \mu^{l_1 + |w|} (p'_1 \rho^{\alpha, \beta}), \nu^{l_1 + |w|} (p'_1 \rho^{\alpha, \beta})), b') <_L \beta,
\]

\( p'_1 \rho^{\alpha, \beta} \in Q/\rho^{\alpha, \beta} \) for some \( b, b' \in Z \). Since \( \tilde{F}^* \) is \((\alpha, \beta)\)-accessible, then there exists a positive integer \( j' \) such that \( \mu^{l_1} (q_2) >_L \alpha \) and \( \nu^{l_1} (q_2) <_L \beta \). Thus

\[
\tilde{\delta}_1 ((q_2, \mu^{l'_1} (q_2), \nu^{l'_1} (q_2)), w, p) \land \tilde{\omega}_1 ((p, \mu^{l_1 + |w|} (p), \nu^{l_1 + |w|} (p)), b) >_L \alpha,
\]

\[
\tilde{\delta}_2 ((q_2, \mu^{l'_1} (q_2), \nu^{l'_1} (q_2)), w, p) \lor \tilde{\omega}_2 ((p, \mu^{l_1 + |w|} (p), \nu^{l_1 + |w|} (p)), b') <_L \beta,
\]
where $p p^{\alpha, b} p'_1$ for some $b, b' \in Z$. The converse follows in a similar manner. Hence

$$\{ w \in X^* | \tilde{\delta}^*_1((q_1, \mu^{t_1}(q_1), \nu^{t_1}(q_1)), w, q) \land \tilde{\omega}_1((q, \mu^{t_1+|w|}(q), \nu^{t_1+|w|}(q)), b) >_L \alpha, $$

$$\tilde{\delta}^*_2((q_1, \mu^{t_1}(q_1), \nu^{t_1}(q_1)), w, q) \lor \tilde{\omega}_2((q, \mu^{t_1+|w|}(q), \nu^{t_1+|w|}(q)), b') <_L \beta, $$

for some $b, b' \in Z, q \in Q \} = \{ w \in X^* | \delta^*_1((q_2, \mu^{t_2}(q_2), \nu^{t_2}(q_2)), w, q) \land \tilde{\omega}_1((q, \mu^{t_2+|w|}(q), \nu^{t_2+|w|}(q)), b) >_L \alpha, $$

$$\tilde{\delta}^*_2((q_2, \mu^{t_2}(q_2), \nu^{t_2}(q_2)), w, q) \lor \tilde{\omega}_2((q, \mu^{t_2+|w|}(q), \nu^{t_2+|w|}(q)), b') <_L \beta, $$

for some $b, b' \in Z, q \in Q \}$. Hence the claim is hold.

3. Considering the proof of 2, it is trivial.

**Example 4.17** Consider the complete lattice $L = (L, \leq, T, S, 0, 1)$ defined in Example 4.3, and the max-min IGLFA $\tilde{F}^*_{oda} = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ as in Figure 7. By the Definition 4.14, $I_{uv} \rho^{a,b} I_{uv} \rho^{a,b} I_{uv} \rho^{a,b} I_{uv}$. Then the max-min IGLFA $\tilde{F}^*/\rho^{a,b}$ has the state diagram as in Figure 8.

![Figure 8: The reduced \( \tilde{F}^*/\rho^{a,b} \) of Example 4.17](image)

**Definition 4.18** Let

$$\tilde{F}^*_1 = (Q_1, X, \tilde{R}_1, Z_1, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$$

and

$$\tilde{F}^*_2 = (Q_2, X, \tilde{R}_2, Z_2, \tilde{\delta}'^*, \tilde{\omega}', F_1, F_2, F_3, F_4)$$

be two max-min IGLFA. A homomorphism from $\tilde{F}^*_1$ onto $\tilde{F}^*_2$ with threshold $(\alpha, \beta)$, where $\alpha, \beta \in L$ and $\alpha \leq L N(\beta)$, is a function $\xi$ from $Q_1$ onto $Q_2$ such that for every $q', q'' \in Q_1$, $u \in X$ and $b_1, b_2 \in Z$ the following conditions hold:
1. \((\mu_{Q_1}^0(q') \succ_L \alpha \& \nu_{Q_1}^0(q') \prec_L \beta)\)
   if and only if \((\mu_{Q_2}^0(\xi(q')) \succ_L \alpha \& \nu_{Q_2}^0(\xi(q')) \prec_L \beta)\).
2. \((\delta_1(q', u, q''') \succ_L \alpha \& \delta_2(q', u, q''') \prec_L \beta)\)
   if and only if \((\delta'_1(\xi(q'''), u, \xi(q''')) \succ_L \alpha \& \delta'_2(\xi(q'''), u, \xi(q''')) \prec_L \beta)\).
3. \((\omega_1(q'), b_1) \succ_L \alpha \& \omega_2(q', b_2) \prec_L \beta)\)
   implies \((\omega'_1(\xi(q'''), b) \succ_L \alpha \& \omega_2(\xi(q'''), b_2) \prec_L \beta)\) for some \(b, b' \in Z'\).

We say that \(\xi\) is an isomorphism with threshold \((\alpha, \beta)\) if and only if \(\xi\) is a homomorphism with threshold \((\alpha, \beta)\) that is one-one and \((\omega_1(q', b_1) \succ_L \alpha \& \omega_2(q', b_2) \prec_L \beta)\) if and only if \((\omega'_1(\xi(q'''), b) \succ_L \alpha \& \omega_2(\xi(q'''), b_2) \prec_L \beta)\), for some \(b, b' \in Z'\).

**Definition 4.19** Let \(L\) be an \((\alpha, \beta)\)-language, where \(\alpha, \beta \in L\) and \(\alpha \leq_L N(\beta)\).
A relation \(R_L\) on \(X^*\) is defined by:

for any two strings \(x\) and \(y\) in \(X^*\), \(xR_Ly\) if, for all \(z \in X^*\),
either \(xz, yz \in L\) or \(xz, yz \notin L\).

**Definition 4.20** Let \(\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)\) be a deterministic max-min IGLFA. Then for any \(\alpha, \beta \in L\), where \(\alpha \leq_L N(\beta)\), define a relation \(R_{F, \alpha, \beta}^*\) on \(X^*\). For strings \(x\) and \(y\) in \(X^*\), \(xR_{F, \alpha, \beta}^*y\) if and only if there exists \(q \in Q\) such that

\[
\tilde{\delta}_1^*(q_0, \mu^0(q_0), \nu^0(q_0), x, q) \succ_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*(q_0, \mu^0(q_0), \nu^0(q_0), x, q) \prec_L \beta,
\]

iff

\[
\tilde{\delta}_1^*(q_0, \mu^0(q_0), \nu^0(q_0), y, q) \succ_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*(q_0, \mu^0(q_0), \nu^0(q_0), y, q) \prec_L \beta.
\]

Now, we show that \(R_{F, \alpha, \beta}^*\) is an equivalence relation. It is clear that \(xR_{F, \alpha, \beta}^*x\) and if \(xR_{F, \alpha, \beta}^*y\), then \(yR_{F, \alpha, \beta}^*x\).

Now, let \(xR_{F, \alpha, \beta}^*y\) and \(yR_{F, \alpha, \beta}^*z\). Suppose that there exists \(q \in Q\) such that

\[
\tilde{\delta}_1^*(q_0, \mu^0(q_0), \nu^0(q_0), x, q) \succ_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*(q_0, \mu^0(q_0), \nu^0(q_0), x, q) \prec_L \beta.
\]

Then

\[
\tilde{\delta}_1^*(q_0, \mu^0(q_0), \nu^0(q_0), y, q) \succ_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*(q_0, \mu^0(q_0), \nu^0(q_0), y, q) \prec_L \beta,
\]

and, since \(yR_{F, \alpha, \beta}^*z\), then

\[
\tilde{\delta}_1^*(q_0, \mu^0(q_0), \nu^0(q_0), z, q) \succ_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*(q_0, \mu^0(q_0), \nu^0(q_0), z, q) \prec_L \beta.
\]

Similarity we can obtain the converse, i.e.,

\[
\tilde{\delta}_1^*(q_0, \mu^0(q_0), \nu^0(q_0), z, q) \succ_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*(q_0, \mu^0(q_0), \nu^0(q_0), z, q) \prec_L \beta,
\]

implies that

\[
\tilde{\delta}_1^*(q_0, \mu^0(q_0), \nu^0(q_0), x, q) \succ_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*(q_0, \mu^0(q_0), \nu^0(q_0), x, q) \prec_L \beta.
\]

Then \(xR_{F, \alpha, \beta}^*z\). Hence the claim is hold.
Note 4.21 By Definition 4.20, for any $(\alpha, \beta)$-complete and deterministic max-min IGLFA $\tilde{F}^*$, the number of classes of equivalence relation $R_F^{\alpha, \beta}$ is less than or equal to the number of states of $\tilde{F}^*$.

Theorem 4.22 Let $\alpha, \beta \in L$, where $\alpha \leq L N(\beta)$. Suppose that $L$ be a recognizable $(\alpha, \beta)$-language where $L = \mathcal{L}^{\alpha, \beta}(\tilde{F}^*)$, for some $(\alpha, \beta)$-complete and deterministic max-min IGLFA $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$. Then for a given equivalence class $[w]_{R_F^{\alpha, \beta}}$ of $R_F^{\alpha, \beta}$ there is an equivalence class $[w]_{R_L}$ of $R_L$ such that $[w]_{R_F^{\alpha, \beta}} \subseteq [w]_{R_L}$. Each equivalence class $[w]_{R_L}$ of the relation $R_L$ is a finite union of equivalence classes of $R_F^{\alpha, \beta}$.

Proof. Let $[w]_{R_F^{\alpha, \beta}}$ be an equivalence class of $R_F^{\alpha, \beta}$. Now, suppose that $x \in [w]_{R_F^{\alpha, \beta}}$. Since $\tilde{F}^*$ is an $(\alpha, \beta)$-complete, then there exists $q \in Q$ such that

$$\tilde{\delta}_1^*( (q_0, \mu^0(q_0), \nu^0(q_0), x, q) >_L \alpha \text{ and } \tilde{\delta}_2^*( (q_0, \mu^0(q_0), \nu^0(q_0)), w, q) <_L \beta.$$ 

Therefore

$$\tilde{\delta}_1^*( (q_0, \mu^0(q_0), \nu^0(q_0), w, q) >_L \alpha \text{ and } \tilde{\delta}_2^*( (q_0, \mu^0(q_0), \nu^0(q_0)), w, q) <_L \beta.$$ 

By the $(\alpha, \beta)$-complete property of $\tilde{F}^*$, for any $y \in X^*$, there exists $q' \in Q$ such that

$$\tilde{\delta}_1^*( (q_0, \mu^0(q_0), \nu^0(q_0), xy, q') \land \tilde{\delta}_1^*( (q_0, \mu^0(q_0), \nu^0(q_0)), wy, q') >_L \alpha,$$

and

$$\tilde{\delta}_2^*( (q_0, \mu^0(q_0), \nu^0(q_0), xy, q') \lor \tilde{\delta}_2^*( (q_0, \mu^0(q_0), \nu^0(q_0)), wy, q') <_L \beta.$$ 

If there exist $b, b' \in Z$ such that $\omega_1(q', b) >_L \alpha$ and $\omega_2(q', b) <_L \beta$, then $xy, wy \in L$ otherwise $xy, wy \notin L$. Then $xR_L w$. Thus $x \in [w]_{R_L}$ and $[w]_{R_F^{\alpha, \beta}} \subseteq [w]_{R_L}$. Suppose that $[w]_{R_L}$ is an equivalence class of $R_L$. It is obvious that if $w \notin L$, then for any $x \in [w]_{R_L}$, $x \notin L$ and if $w \in L$, then for any $x \in [w]_{R_L}$, $x \in L$. If $w \notin L$, then for any $q \in Q$, $b, b' \in Z$ we have

$$\tilde{\delta}_1^*( (q_0, \mu^0(q_0), \nu^0(q_0), w, q) \land \omega_1((q, \mu^0 + |w|(q), \nu^0 + |w|(q)), b) \leq_L \alpha,$$

or

$$\tilde{\delta}_2^*( (q_0, \mu^0(q_0), \nu^0(q_0), w, q) \lor \omega_2((q, \mu^0 + |w|(q), \nu^0 + |w|(q)), b') \geq_L \beta.$$ 

If $\omega_1(q, b) >_L \alpha$ and $\tilde{\delta}_2^*( (q_0, \mu^0(q_0), \nu^0(q_0), w, q) <_L \beta$, then

$$\omega_1(q, b) \leq_L \alpha \text{ or } \omega_2(q, b) \geq_L \beta.$$ 

The set

$$(4.9) \quad S = \{ q \in Q \mid \tilde{\delta}_1^*( (q_0, \mu^0(q_0), \nu^0(q_0), x, q) >_L \alpha, \text{ and } \tilde{\delta}_2^*( (q_0, \mu^0(q_0), \nu^0(q_0), x, q) <_L \beta, x \in [\omega]_{R_L} \},$$

is finite. Then we have $\omega_1(q, b) \leq_L \alpha$ or $\omega_2(q, b) \geq_L \beta$ for all $q \in S$ and $b \in Z$, or for all $q \in S$ there exist $b, b' \in Z$ such that $\omega_1(q, b) >_L \alpha$ and $\omega_2(q, b') <_L \beta$. Therefore the equivalence class $[w]_{R_L}$ of $R_L$ is a finite union of the equivalence classes $[w]_{R_F^{\alpha, \beta}}$ of $R_F^{\alpha, \beta}$, where $q \in S$. 

**Theorem 4.23** Let $\mathcal{L}$ be a recognizable $(\alpha, \beta)$-language. Then there is an $(\alpha, \beta)$-complete and deterministic IGLFA $\mathcal{F}^*_m$ such that $\mathcal{L}^{\alpha, \beta}(\mathcal{F}^*_m) = \mathcal{L}$ and $\mathcal{F}^*_m$ is a minimal automaton, where $\alpha, \beta \in L$ and $\alpha \leq L N(\beta)$.

**Proof.** Let $\tilde{\mathcal{F}}^*_m = (\tilde{Q}, \tilde{Z}, \tilde{F}, \tilde{T}, F_1, F_2, F_3, F_4)$ be an $(\alpha, \beta)$-complete and deterministic max-min IGLFA such that $\mathcal{L} = \mathcal{L}^{\alpha, \beta}(\tilde{\mathcal{F}}^*_m)$. By Theorem 4.22, we have that the number of equivalence classes of $R^*_m$ i.e., $Q_m = \{[w] | [w]$ is an equivalence class of $R^*_m\}$. Let $\tilde{R}_m = \{([\Lambda], \gamma_1, \gamma_2)\}$, where $\gamma_1, \gamma_2 \in L, \gamma_1 > L \alpha, \gamma_2 < L \beta$ and $\gamma_1 \leq L N(\gamma_2)$. Define $\delta_m : Q_m \times X \times Q_m \rightarrow \tilde{L} \times \tilde{L}$ by

\begin{equation}
\delta_m([z], a, [x]) = \begin{cases} 
\gamma_1 & \text{if } [za] = [x], \\
0 & \text{otherwise}, 
\end{cases}
\end{equation}

and

\begin{equation}
\delta_m([z], a, [x]) = \begin{cases} 
\gamma_2 & \text{if } [za] = [x], \\
1 & \text{otherwise}, 
\end{cases}
\end{equation}

Also define $\omega_m : Q_m \times \{b\} \rightarrow L \times L$ by

\begin{equation}
\omega_m([w], b) = \begin{cases} 
\gamma_1 & \text{if } w \in L, \\
0 & \text{otherwise}, 
\end{cases}
\end{equation}

and

\begin{equation}
\omega_m([w], b) = \begin{cases} 
\gamma_2 & \text{if } w \in L, \\
1 & \text{otherwise}. 
\end{cases}
\end{equation}

It is clear that $\tilde{\mathcal{F}}^*_m = (Q_m, X, \tilde{R}_m, Z = \{b\}, \delta_m^*, \omega_m, F_1, F_2, F_3, F_4)$ is an $(\alpha, \beta)$-complete and deterministic IGLFA. Clearly, $\delta_m$ and $\omega_m$ are well defined.

Let $w \in \tilde{L}$. Then, we have

\begin{equation}
\delta_m^1([\Lambda], \mu^\alpha([\Lambda]), \nu^\alpha([\Lambda])), w, [w]) > L \alpha \quad \text{and} \quad \delta_m^2([\Lambda], \mu^\alpha([\Lambda]), \nu^\alpha([\Lambda])), w, [w]) < L \beta, 
\end{equation}

and

\begin{equation}
\omega_m([w], b) = \gamma_1 > L \alpha, \quad \omega_m([w], b) = \gamma_2 < L \beta.
\end{equation}

These imply that $w \in \mathcal{L}^{\alpha, \beta}(\mathcal{F}^*_m)$.

Now, suppose that $w \in \mathcal{L}^{\alpha, \beta}(\mathcal{F}^*_m)$. Then there exist $[z] \in Q_m, b \in Z$ such that

\begin{equation}
\tilde{\delta}_m^1([\Lambda], \mu^\alpha([\Lambda]), \nu^\alpha([\Lambda])), w, [z]) \wedge \tilde{\omega}_m^1([\Lambda], \mu^\alpha([\Lambda]), \nu^\alpha([\Lambda])), [z], \nu^\alpha([\Lambda]), b) > L \alpha, 
\end{equation}

and

\begin{equation}
\tilde{\delta}_m^2([\Lambda], \mu^\alpha([\Lambda]), \nu^\alpha([\Lambda])), w, [z]) \vee \tilde{\omega}_m^2([\Lambda], \mu^\alpha([\Lambda]), \nu^\alpha([\Lambda])), [z], \nu^\alpha([\Lambda]), b) < L \beta.
\end{equation}
Let the complete lattice $L = (L, \leq_L, T, S, 0, 1)$ defined in Example 4.3, and the max-min IGLFA $\tilde{F}^*$ be equivalence classes of $R_\mathcal{L}$ as in Figure 6. Considering Definition 4.19, we find $[u], [v], [uv] = [vu], [u^2] = [u], [v^2] = [v], [uv^2] = [uv]$ and $[uvu] = [uv]$. Then we have $\tilde{F}^*_m$ as in Figure 9. It is obvious that $\tilde{F}^*/\rho^{\alpha, \beta}$ as in Figure 8, and $\tilde{F}^*_m$ as in Figure 9, have the same number of states.

**Theorem 4.25** For every recognizable $(\alpha, \beta)$-language $\mathcal{L}$, the minimal max-min IGLFA $\tilde{F}^*$ defined in proof of Theorem 4.23, is $(\alpha, \beta)$-reduced, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.

**Proof.** Let $\tilde{F}^*$ be an $(\alpha, \beta)$-accessible, $(\alpha, \beta)$-complete, deterministic max-min IGLFA. Suppose that $q_1 = [u_1], q_2 = [u_2]$ be equivalence classes of $R_\mathcal{L}$. Now, let $q_1 \rho^{\alpha, \beta} q_2$. Then

$$A = \{w \in X^* | \delta^*_1 \left( ([u_1], \mu^+([u_1]), \nu^+([u_1])), w, q \right) \wedge \tilde{\omega}_1((q, \mu^+([w])(q), \nu^+([w])(q)), b) >_L \alpha, \delta^*_2 \left( ([u_1], \mu^+([u_1]), \nu^+([u_1])), w, q \right) \vee \tilde{\omega}_2((q, \mu^+([w])(q), \nu^+([w])(q)), b') <_L \beta, \text{ for some } b, b' \in Z, q \text{ is an equivalence classes of } R_\mathcal{L} \},$$

$$= \{w \in X^* | \delta^*_1 \left( ([u_2], \mu^+([u_2]), \nu^+([u_2])), w, q \right) \wedge \tilde{\omega}_1((q, \mu^+([w])(q), \nu^+([w])(q)), b) >_L \alpha, \delta^*_2 \left( ([u_2], \mu^+([u_2]), \nu^+([u_2])), w, q \right) \vee \tilde{\omega}_2((q, \mu^+([w])(q), \nu^+([w])(q)), b') <_L \beta, \text{ for some } b, b' \in Z, q \text{ is an equivalence classes of } R_\mathcal{L} \}.$$
Therefore $w \in A \iff w \in B$, for all $w \in X^*$, this implies that $u_1w \in \mathcal{L} \iff u_2w \in \mathcal{L}$, for all $w \in X^*$. Then $[u_1] = [u_2]$. Hence $F^*$ is an $(\alpha, \beta)$-reduced.

**Theorem 4.26** Let $\mathcal{L}$ be a recognizable $(\alpha, \beta)$-language, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. Let $\hat{F}^*_m$ be the max-min IGLFA defined in the proof of Theorem 4.23, and $\hat{F}^*$ be an $(\alpha, \beta)$-complete, $(\alpha, \beta)$-accessible, deterministic and $(\alpha, \beta)$-reduced max-min IGLFA. Then $\hat{F}^*_m$ and $\hat{F}^*$ are isomorphism with threshold $(\alpha, \beta)$.

**Proof.** Let $\hat{F}^*_m = \langle Q, X, \{[\Lambda]\}, \{b\}, \delta^*_m, \tilde{\omega}_m, F_1, F_2, F_3, F_4 \rangle$ and $\hat{F}^* = \langle Q, X, \{q_0\}, Z, \delta^*, \tilde{\omega}, F_1, F_2, F_3, F_4 \rangle$. Define $\xi : Q \rightarrow Q_m$ by $\xi(q) = [u]$, where

$$\tilde{\delta}_1^*([q_0, \mu^{io}(q_0), \nu^{io}(q_0)], u, q) >_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*([q_0, \mu^{io}(q_0), \nu^{io}(q_0)], u, q) <_L \beta.$$ 

Since $\hat{F}^*$ is $(\alpha, \beta)$-accessible, then $\mu^{io}_Q([q_0]) >_L \alpha$ and $\nu^{io}_Q([q_0]) <_L \beta$.

Also, $\mu^{io}_{Q_m}([\Lambda]) >_L \alpha$ and $\nu^{io}_{Q_m}([\Lambda]) <_L \beta$.

Let $q_1, q_2 \in Q$ and $q_1 \neq q_2$. Then $q_1 \rho^{\alpha, \beta} \sim q_2$. Therefore

$$\{ w \in X^* : \tilde{\delta}_1^*([q_1, \mu^t(q_1), \nu^t(q_1)], w, q) \land \tilde{\omega}_1([q, \mu^{t_i+|w|}(q), \nu^{t_j+|w|}(q)], b) >_L \alpha, \quad \tilde{\delta}_2^*([q_2, \mu^t(q_2), \nu^t(q_2)], w, q) \lor \tilde{\omega}_2([q, \mu^{t_i+|w|}(q), \nu^{t_j+|w|}(q)], b') <_L \beta, \quad \text{for some } b, b' \in Z, q \in Q \} =$$

$$\{ w \in X^* : \tilde{\delta}_1^*([q_2, \mu^t(q_2), \nu^t(q_2)], w, q) \land \tilde{\omega}_1([q, \mu^{t_i+|w|}(q), \nu^{t_j+|w|}(q)], b) >_L \alpha, \quad \tilde{\delta}_2^*([q, \mu^t(q), \nu^t(q)], w, q) \lor \tilde{\omega}_2([q, \mu^{t_i+|w|}(q), \nu^{t_j+|w|}(q)], b') <_L \beta, \quad \text{for some } b, b' \in Z, q \in Q \}.$$ 

Since $\mu^t(q_1) >_L \alpha, \nu^t(q_1) <_L \beta$ and $\mu^t(q_2) >_L \alpha, \nu^t(q_2) <_L \beta$, then there exist $u, v \in X^*$, where $|u| = i, |v| = j$, and

$$\tilde{\delta}_1^*([q_0, \mu^{io}(q_0), \nu^{io}(q_0)], u, q_1) >_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*([q_0, \mu^{io}(q_0), \nu^{io}(q_0)], u, q_1) <_L \beta,$$

and

$$\tilde{\delta}_1^*([q_0, \mu^{io}(q_0), \nu^{io}(q_0)], v, q_2) >_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*([q_0, \mu^{io}(q_0), \nu^{io}(q_0)], v, q_2) <_L \beta.$$ 

Thus $uz \in \mathcal{L}$ if and only if $vz \in \mathcal{L}$ for all $z \in X^*$. Then $[u] = [v]$, i.e., $\xi(q_1) = \xi(q_2)$. Hence $\xi$ is well defined.

Let $[u] \in Q_m$. By the $(\alpha, \beta)$-complete property of $\hat{F}^*$, there exists $q \in Q$ such that

$$\tilde{\delta}_1^*([q_0, \mu^{io}(q_0), \nu^{io}(q_0)], u, q) >_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*([q_0, \mu^{io}(q_0), \nu^{io}(q_0)], u, q) <_L \beta.$$ 

Then $\xi(q) = [u]$. Therefore $\xi$ is surjective.

Now, let $\delta_1(q, a, q') >_L \alpha$ and $\delta_2(q, a, q') <_L \beta$. Since $\hat{F}^*$ is $(\alpha, \beta)$-accessible, then there exists $u \in X^*$ such that

$$\tilde{\delta}_1^*([q_0, \mu^{io}(q_0), \nu^{io}(q_0)], u, q) >_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*([q_0, \mu^{io}(q_0), \nu^{io}(q_0)], u, q) <_L \beta.$$
Therefore
\[ \tilde{\delta}_1^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), ua, q') >_L \alpha \text{ and } \tilde{\delta}_2^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), ua, q') <_L \beta. \]

Hence \( \xi(q) = [u] \) and \( \xi(q') = [ua] \) and let \( \delta_m((\xi(q), a, \xi(q')) >_L \alpha \) and \( \delta_m((\xi(q), a, \xi(q')) <_L \beta. \)

Let \( \delta_m((\xi(q), a, \xi(q')) >_L \alpha \) and \( \delta_m((\xi(q), a, \xi(q')) <_L \beta \), where \( \xi(q) = [u] \) and \( \xi(q') = [v]. \) Then \( [ua] = [v]. \) Thus
\[ \tilde{\delta}_1^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}_2^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), u, q) <_L \beta, \]

and
\[ \tilde{\delta}_1^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), ua, q') >_L \alpha \text{ and } \tilde{\delta}_2^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), ua, q') <_L \beta. \]

Since \( \tilde{F}^* \) is deterministic, then \( \delta_1(q, a, q') >_L \alpha \text{ and } \delta_2(q, a, q') <_L \beta. \)

Let \( q \in Q \) and \( \omega_1(q, b) >_L \alpha \text{ and } \omega_2(q, b') <_L \beta \) for some \( b, b' \in Z. \) By the \((\alpha, \beta)\)-accessibility property
\[ \tilde{\delta}_1^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}_2^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), u, q) <_L \beta. \]

Then \( u \in L \). Therefore \( \omega_m1([u], b) >_L \alpha \) and \( \omega_m2([u], b') <_L \beta \) for some \( b, b' \in Z. \) Hence \( \omega_m1((\xi(q), b) >_L \alpha \text{ and } \omega_m2((\xi(q), b') <_L \beta \text{ for some } b, b' \in Z. \)

Then \( \tilde{F}^m \) and \( \tilde{F}^* \) are homomorphism with threshold \((\alpha, \beta)\). Now, let \( q_1, q_2 \in Q \) and \( \xi(q_1) = \xi(q_2). \) Then there exist \( x, y \in X^* \) such that \( [x] = \xi(q_1) = \xi(q_2) = [y]. \)

Therefore \( q_1, q_2 \in \tilde{F}^* \). Since \( \tilde{F}^* \) is \((\alpha, \beta)\)-reduced, then \( q_1 = q_2. \) Thus \( \xi \) is one-one.

Now, let \( \omega_m1((\xi(q), b) >_L \alpha \text{ and } \omega_m2((\xi(q), b') <_L \beta \text{ for some } b, b' \in Z, \) where \( \xi(q) = [u]. \) On the other hand, \( \omega_m1([u], b) >_L \alpha \text{ and } \omega_m2([u], b') <_L \beta \) for some \( b, b' \in Z. \) These imply that \( u \in L. \) Then
\[ \tilde{\delta}_1^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), u, q') \land \tilde{\delta}_2^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), u, q') >_L \alpha, \]
\[ \delta_2^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), u, q') \lor \delta_2^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), u, q') <_L \beta, \]

for some \( b, b' \in Z. \) Also, we have
\[ \tilde{\delta}_1^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}_2^1((q_0, \mu^{\alpha}(q_0), \nu^{\alpha}(q_0)), u, q) <_L \beta. \]

Since \( \tilde{F}^* \) is deterministic, then \( q = q'. \) Therefore \( \omega_1(q, b) >_L \alpha \) and \( \omega_2(q, b') <_L \beta \) for some \( b, b' \in Z. \)

Hence \( \tilde{F}^m \) and \( \tilde{F}^* \) are isomorphism with threshold \((\alpha, \beta). \)

5. Conclusion

In this paper, we present the notions of intuitionistic general L-fuzzy automata based on lattice valued intuitionistic fuzzy sets, \((\alpha, \beta)\)-language, \((\alpha, \beta)\)-accessible, \((\alpha, \beta)\)-reduced, \((\alpha, \beta)\)-complete, deterministic and isomorphic with threshold \((\alpha, \beta). \)

After that, it is shown that for any recognizable \((\alpha, \beta)\)-language over a bounded lattice \( L, \) there exists a minimal \((\alpha, \beta)\)-complete and deterministic IGLFA, which
preserve \((\alpha, \beta)\)-language. Finally, we show that for any given \((\alpha, \beta)\)-language \(\mathcal{L}\), the minimal \((\alpha, \beta)\)-complete and deterministic IGLFA recognizing \(\mathcal{L}\) is isomorphic with threshold \((\alpha, \beta)\) to any \((\alpha, \beta)\)-complete, \((\alpha, \beta)\)-accessible, deterministic, \((\alpha, \beta)\)-reduced IGLFA recognizing \(\mathcal{L}\).

Now, there is an important question:

Suppose that there is a max-min IGLFA, say \(\tilde{F}^*\). Is there a minimal intuitionistic general \(L\)-fuzzy automata with more that one initial state, in which it has the same \((\alpha, \beta)\)-language as the \((\alpha, \beta)\)-language of \(\tilde{F}^*\)?

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