\(\alpha A^*_T\)-SETS, \(\alpha C^*_T\)-SETS, \(\alpha C^*_T\)-SETS AND DECOMPOSITIONS OF \(\alpha\)-\(T\)-CONTINUITY

O. Ravi

Department of Mathematics
P.M. Thevar College
Usilampatti, Madurai District, Tamil Nadu
India
e-mail: siingam@yahoo.com

V. Rajendran

Department of Mathematics
KSG College
Coimbatore, Tamil Nadu
India
e-mail: mathsrj05@yahoo.co.in

K. Indirani

Department of Mathematics
Nirmala College for Women
Coimbatore, Tamil Nadu
India
e-mail: indirani009@gmail.com

S. Vijaya

Department of Mathematics
Sethu Institute of Technology
Kariyapatti
Virudhunagar District, Tamil Nadu
India
e-mail: viviphd.11@gmail.com

Abstract. The aim of this paper is to introduce and study the notions of \(\alpha A^*_T\)-sets, \(\alpha C^*_T\)-sets and \(\alpha C^*_T\)-sets in ideal topological spaces. Properties of \(\alpha A^*_T\)-sets, \(\alpha C^*_T\)-sets and \(\alpha C^*_T\)-sets are investigated. Moreover, decompositions of \(\alpha\)-\(T\)-continuous functions and decompositions of \(\alpha A^*_T\)-continuous functions via \(\alpha A^*_T\)-sets, \(\alpha C^*_T\)-sets and \(\alpha C^*_T\)-sets in ideal topological spaces are established.

Keywords: \(\alpha A^*_T\)-set, \(\alpha C^*_T\)-set, \(\alpha C^*_T\)-set, pre-\(T\)-regular set, ideal topological space, decomposition, \(*\)-extremally disconnected ideal space, \(*\)-hyperconnected ideal space, \(T\)-submaximal ideal space.

2010 Mathematics Subject Classification: 54A05, 54A10, 54C08, 54C10.
1. Introduction and preliminaries

In this paper, $\alpha A^*_I$-sets, $\alpha C_I$-sets and $\alpha C^*_I$-sets in ideal topological spaces are introduced and studied. The relationships and properties of $\alpha A^*_I$-sets, $\alpha C_I$-sets and $\alpha C^*_I$-sets are investigated. Furthermore, decompositions of $\alpha$-continuous functions and decompositions of $\alpha A^*_I$-continuous functions via $\alpha A^*_I$-sets, $\alpha C_I$-sets and $\alpha C^*_I$-sets in ideal topological spaces are provided.

Throughout this paper $(X, \tau)$, $(Y, \sigma)$ (or simply $X$, $Y$) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset $A$ of a space $X$, the closure and interior of $A$ with respect to $\tau$ are denoted by $\text{cl}(A)$ and $\text{int}(A)$ respectively.

An ideal $I$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies

1. $A \in I$ and $B \subseteq A \Rightarrow B \in I$ and
2. $A \in I$ and $B \in I \Rightarrow A \cup B \in I$.

If $I$ is an ideal on $X$ and $X \notin \mathcal{I}$, then $F = \{X \setminus G : G \in \mathcal{I}\}$ is a filter [14]. Given a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$ and if $\varphi(X)$ is the set of all subsets of $X$, a set operator $(.)^*: \varphi(X) \rightarrow \varphi(X)$, called a local function [16] of $A$ with respect to $\tau$ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$, $A^*((\mathcal{I}, \tau)) = \{x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $\text{cl}^*(.)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the $\star$-topology, finer than $\tau$ is defined by $\text{cl}^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [14]. When there is no chance for confusion, we will simply write $A^*$ for $A^*(\mathcal{I}, \tau)$ and $\tau^*$ for $\tau^*(\mathcal{I}, \tau)$. $\text{int}^*(A)$ will denote the interior of $A$ in $(X, \tau^*, \mathcal{I})$.

Remark 1.1 [14] The $\star$-topology is generated by $\tau$ and by the filter $F$. Also the family $\{H \cap G : H \in \tau, G \in F\}$ is a basis for this topology.

Lemma 1.2 [13] Let $A$ be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. If $N$ is open, then $N \cap \text{cl}^*(A) \subseteq \text{cl}^*(N \cap A)$.

Definition 1.3 A subset $A$ of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be

1. pre-$\mathcal{I}$-open [4] if $A \subseteq \text{int}(\text{cl}^*(A))$.
2. semi-$\mathcal{I}$-open [11] if $A \subseteq \text{cl}(\text{int}(A))$.
3. $\alpha$-$\mathcal{I}$-open [11] if $A \subseteq \text{int}(\text{cl}^*(\text{int}(A)))$.
4. strongly $\beta$-$\mathcal{I}$-open [12] if $A \subseteq \text{cl}^*(\text{int}^*(A))$.
5. $\star$-dense [5] if $\text{cl}^*(A) = X$.
6. $t$-$\mathcal{I}$-set [11] if $\text{int}(A) = \text{int}(\text{cl}^*(A))$.
7. semi$^*$-$\mathcal{I}$-open [8, 9] if $A \subseteq \text{cl}(\text{int}^*(A))$. 


The family of all $\alpha$-$I$-open (resp. pre-$I$-open) sets in an ideal topological space $(X, \tau, I)$ is denoted by $\alpha I O(X)$ (resp. $P I O(X)$).

**Remark 1.4** [8] For several subsets defined above, we have the following implications.

$$\text{pre-$I$-open set} \rightarrow \text{strongly $\beta$-$I$-open set} \uparrow \uparrow \text{open set} \rightarrow \alpha$-$I$-open set $\rightarrow$ semi-$I$-open set

The reverse implications are not true.

**Remark 1.6** The reverse implication of the above Lemma is not true in general as shown in [9].

**Definition 1.7** The complement of a pre-$I$-open (resp. semi-$I$-open, $\alpha$-$I$-open, semi-$\ast$-$I$-open) set is called pre-$I$-closed [4](resp. semi-$I$-closed [11], $\alpha$-$I$-closed [11], semi-$\ast$-$I$-closed [8, 9]).

**Definition 1.8** [9] The pre-$I$-closure of a subset $A$ of an ideal topological space $(X, \tau, I)$, denoted by $p I cl(A)$, is defined as the intersection of all pre-$I$-closed sets of $X$ containing $A$.

**Lemma 1.9** [9] For a subset $A$ of an ideal topological space $(X, \tau, I)$, $p I cl(A) = A \cup cl(int^\ast(A))$.

**Definition 1.10** A function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is said to be pre-$I$-continuous [4] (resp. semi-$I$-continuous [11], $\alpha$-$I$-continuous [11]) if $f^{-1}(V)$ is pre-$I$-open (resp. semi-$I$-open, $\alpha$-$I$-open) in $X$ for each open set $V$ in $Y$.

**Definition 1.11** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be

1. an $\eta\zeta$-set [17] if $A = L \cap M$, where $L$ is open and $M$ is clopen in $X$.
2. locally closed [3] if $A = L \cap M$, where $L$ is open and $M$ is closed in $X$.
3. $\alpha I N_3$-set [2] if $A = U \cap V$, where $U \in \alpha I O(X)$ and $int(cl^\ast(V)) = int(V)$.
5. $B_I$-set [11] if $A = U \cap V$, where $U$ is open and $V$ is a $t$-$I$-set.

The family of all $B_I$-sets (resp. $\alpha I N_3$-sets) of $X$ is denoted by $B_I(X)$ (resp. $\alpha I N_3(X)$).
Proposition 1.12 [1] Let \((X, \tau, I)\) be an ideal topological space. If \(V \in P\text{IO}(X)\) and \(A \in \alpha\text{IO}(X)\), then \(V \cap A \in P\text{IO}(X)\).

Definition 1.13 An ideal topological space \((X, \tau, I)\) is called

1. \(I\)-submaximal if every \(*\)-dense subset of \(X\) is open in \(X\); [10]
2. \(*\)-extremally disconnected if \(*\)-closure of every open subset of \(X\) is open. [8]

Lemma 1.14 [8] A subset \(A\) of an ideal topological space \((X, \tau, I)\) is semi-\(*\)-open in \((X, \tau, I)\) if and only if \(\text{cl}(A) = \text{cl}(\text{int}^*(A))\).

Lemma 1.15 [7] For a subset \(A\) of an ideal topological space, \(p_I\text{int}(A) = A \cap \text{int}(\text{cl}^*(A))\).

Theorem 1.16 [2] For an ideal topological space \((X, \tau, I)\), we have \(\alpha\text{IO}(X) = \text{P}\text{IO}(X) \cap \alpha\text{IN}_3(X)\).

Definition 1.17 [7] A subset \(A\) of an ideal topological space \((X, \tau, I)\) is called pre-\(I\)-regular if \(A\) is pre-\(I\)-open and pre-\(I\)-closed in \((X, \tau, I)\).

Definition 1.18 [6], [7] A subset \(A\) of an ideal topological space \((X, \tau, I)\) is called \(A^*_I\)-set if \(A = L \cap M\), where \(L\) is an open and \(M = \text{cl}(\text{int}^*(M))\).

Proposition 1.19 [11] Let \((X, \tau, I)\) be an ideal topological space. For a subset \(A\) of \((X, \tau, I)\), the following conditions are equivalent:

1. \(A\) is open.
2. \(A\) is pre-\(I\)-open and a \(B_I\)-set.

Lemma 1.20 [1] Let \((X, \tau, I)\) be an ideal topological space. A subset \(A\) of \(X\) is \(\alpha\)-\(I\)-open if and only if it is semi-\(I\)-open and pre-\(I\)-open.

Theorem 1.21 [10] For an ideal topological space \((X, \tau, I)\), then the following properties are equivalent.

1. \(X\) is \(I\)-submaximal.
2. Every pre-\(I\)-open set is open.
3. Every pre-\(I\)-open set is semi-\(I\)-open and every \(\alpha\)-\(I\)-open set is open.

Theorem 1.22 [7] Let \((X, \tau, I)\) be an ideal topological space and \(K \subseteq X\). The following properties are equivalent.

1. \(K\) is an open set.
2. \(K\) is an \(\alpha\)-\(I\)-open set and an \(A^*_I\)-set.
3. \(K\) is a pre-\(I\)-open set and an \(A^*_I\)-set.
2. \( \alpha I \)-sets, \( \alpha C I \)-sets and \( \alpha C^*_I \)-sets

**Definition 2.1** Let \((X, \tau, I)\) be an ideal topological space and \(A \subseteq X\). A is said to be

1. an \( \alpha C^*_I \)-set if \( A = L \cap M \), where \( L \) is an \( \alpha I \)-open and \( M \) is a pre-\( I \)-regular set in \( X \).

2. an \( \alpha A^*_I \)-set if \( A = L \cap M \), where \( L \) is \( \alpha I \)-open and \( M \) is semi-\( I \)-regular set in \( X \).

**Theorem 2.2** Let \((X, \tau, I)\) be an ideal topological space. Then each \( \alpha C^*_I \)-set in \( X \) is a pre-\( I \)-open set.

**Proof.** Let \( A \) be an \( \alpha C^*_I \)-set in \( X \). It follows that \( A = L \cap M \), where \( L \) is an \( \alpha I \)-open set and \( M \) is a pre-\( I \)-regular set in \( X \). Since \( M \) is a pre-\( I \)-open set, then by Proposition 1.12, \( A = L \cap M \) is a pre-\( I \)-open set in \( X \).

**Remark 2.3** The converse of the Theorem 2.2 need not be true in general as shown in the following Example.

**Example 2.4** Let \( X = \{a, b, c\} \), \( \tau = \emptyset, \{a\}, X \) \( I = \emptyset, \{a\} \). Then \( A = \{a, b\} \) is a pre-\( I \)-open set but not an \( \alpha C^*_I \)-set.

**Remark 2.5** In an ideal topological space, every \( \alpha I \)-open set and every pre-\( I \)-regular set is an \( \alpha C^*_I \)-set. The converses are not true in general as shown in the following Examples.

**Example 2.6** In Example 2.4, \( A = \{a\} \) is an \( \alpha C^*_I \)-set but not a pre-\( I \)-regular set.

**Example 2.7** Let \( X = \{a, b, c\} \), \( \tau = \emptyset, \{a, b\}, X \) \( I = \emptyset \). Then \( A = \{a\} \) is an \( \alpha C^*_I \)-set but not an \( \alpha I \)-open set.

**Remark 2.8** By Remark 2.5 and Theorem 2.2, the following diagram holds for a subset \( A \) of an ideal topological space \((X, \tau, I)\).

\[
\begin{array}{ccc}
\text{pre-} I \text{-open set} & \uparrow & \text{\( \alpha C^*_I \)-set} \\
\text{pre-} I \text{-regular set} & \longrightarrow \\
\end{array}
\]

**Definition 2.9** A subset \( A \) of an ideal topological space \((X, \tau, I)\) is said to be

1. an \( \alpha C_I \)-set if \( A = L \cap M \), where \( L \) is an \( \alpha I \)-open set and \( M \) is a pre-\( I \)-closed set in \( X \).

2. an \( \alpha I \)-set if \( A = L \cap M \), where \( L \) is an \( \alpha I \)-open set and \( M \) is an \( \alpha I \)-closed set in \( X \).

3. an \( \alpha A^*_I \)-set if \( A = L \cap M \), where \( L \) is an \( \alpha I \)-open set and \( M = \text{cl}( \text{int}^*(M) ) \).
**Remark 2.10** Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \(A \subseteq X\). The following diagram holds for \(A\).

\[
\begin{array}{ccc}
\alpha C^*_I\text{-set} & \rightarrow & \alpha C_I\text{-set} \\
A^*_I\text{-set} & \rightarrow & \alpha A^*_I\text{-set} & \rightarrow & \alpha \eta_I\text{-set}
\end{array}
\]

The following Examples show that these implications are not reversible in general.

**Example 2.11** Let \(X = \{a, b, c, d\}\), \(\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}\) and \(\mathcal{I} = \{\emptyset\}\). Then \(A = \{a, b, d\}\) is \(\alpha A^*_I\)-set but not an \(A^*_I\)-set.

**Example 2.12** In Example 2.11, \(A = \{c\}\) is \(\alpha \eta_I\)-set but not an \(\alpha A^*_I\)-set.

**Example 2.13** In Example 2.11, \(A = \{c\}\) is \(\alpha C_I\)-set but not an \(\alpha C^*_I\)-set.

**Example 2.14** In Example 2.4, \(A = \{c\}\) is \(\alpha C_I\)-set but not an \(\alpha \eta_I\)-set.

**Theorem 2.15** For a subset \(A\) of an ideal topological space \((X, \tau, \mathcal{I})\), the following properties are equivalent.

1. \(A\) is an \(\alpha C_I\)-set and a semi\(^*\)-\(\mathcal{I}\)-open set in \(X\).
2. \(A = L \cap \text{cl}(\text{int}^*(A))\) for an \(\alpha\)-\(\mathcal{I}\)-open set \(L\).

**Proof.** (1) \(\Rightarrow\) (2): Suppose that \(A\) is an \(\alpha C_I\)-set and a semi\(^*\)-\(\mathcal{I}\)-open set in \(X\). Since \(A\) is an \(\alpha C_I\)-set, then we have \(A = L \cap M\), where \(L\) is an \(\alpha\)-\(\mathcal{I}\)-open set and \(M\) is a pre-\(\mathcal{I}\)-closed set in \(X\). We have \(A \subseteq M\), so \(\text{cl}(\text{int}^*(A)) \subseteq \text{cl}(\text{int}^*(M))\). Since \(M\) is a pre-\(\mathcal{I}\)-closed set in \(X\), we have \(\text{cl}(\text{int}^*(M)) \subseteq M\). Since \(A\) is a semi\(^*\)-\(\mathcal{I}\)-open set in \(X\), We have \(A \subseteq \text{cl}(\text{int}^*(A))\). It follows that \(A = A \cap \text{cl}(\text{int}^*(A)) = L \cap M \cap \text{cl}(\text{int}^*(A)) = L \cap \text{cl}(\text{int}^*(A))\).

(2) \(\Rightarrow\) (1): Let \(A = L \cap \text{cl}(\text{int}^*(A))\) for an \(\alpha\)-\(\mathcal{I}\)-open set \(L\). We have \(A \subseteq \text{cl}(\text{int}^*(A))\). It follows that \(A\) is a semi\(^*\)-\(\mathcal{I}\)-open set in \(X\). Since \(\text{cl}(\text{int}^*(A))\) is a closed set, then \(\text{cl}(\text{int}^*(A))\) is a pre-\(\mathcal{I}\)-closed set in \(X\). Hence, \(A\) is an \(\alpha C_I\)-set in \(X\).

**Theorem 2.16** For a subset \(A\) of an ideal topological space \((X, \tau, \mathcal{I})\), the following properties are equivalent.

1. \(A\) is an \(\alpha A^*_I\)-set in \(X\).
2. \(A\) is an \(\alpha \eta_I\)-set and a semi\(^*\)-\(\mathcal{I}\)-open set in \(X\).
3. \(A\) is an \(\alpha C_I\)-set and a semi\(^*\)-\(\mathcal{I}\)-open set in \(X\).
αA∗I-sets, αC∗I-sets, αC∗I-sets and decompositions of αI-continuity

Proof. (1) ⇒ (2): Suppose that A is an αA∗I-set in X. It follows that $A = L \cap M$, where $L$ is an αI-open set and $M = \text{cl}(\text{int}^*(M))$. This implies $A = L \cap M = \text{cl}(\text{int}^*(L)) \cap \text{int}(M) \subseteq \text{cl}(\text{int}^*(L)) \cap \text{int}(M) \subseteq \text{cl}(\text{int}^*(L)) \cap \text{int}(M) = \text{cl}(\text{int}^*(L \cap M))$. Thus $A \subseteq \text{cl}(\text{int}^*(A))$ and hence $A$ is a semi∗I-open set in $X$. Moreover, by Remark 2.10, $A$ is an αηI-set in $X$.

(2) ⇒ (3): It follows from the fact that every αηI-set is an αC∗I-set in $X$ by Remark 2.10.

(3) ⇒ (1): Suppose that $A$ is an αC∗I-set and a semi∗I-open set in $X$. By Theorem 2.15, $A = L \cap \text{cl}(\text{int}^*(L))$ for an αI-open set $L$. We have $\text{cl}(\text{int}^*(\text{cl}(\text{int}^*(A)))) = \text{cl}(\text{int}^*(A))$. It follows that $A$ is an αA∗I-set in $X$.

Remark 2.17

1. The notions of αηI-set and semi∗I-open set are independent of each other.
2. The notions of αC∗I-set and semi∗I-open set are independent of each other.

Example 2.18

1. In Example 2.11, $A = \{c, d\}$ is αC∗I-set as well as αηI-set but not semi∗I-open set.
2. In Example 2.4, $A = \{a, c\}$ is a semi∗I-open set but it is neither αC∗I-set nor αηI-set.

Theorem 2.19 [7] A subset $A$ of an ideal topological space $(X, \tau, \mathcal{I})$ is semi∗I-closed if and only if $A$ is a tI-set.

Theorem 2.20 Let $(X, \tau, \mathcal{I})$ be an I-submaximal and ∗-extremally disconnected ideal topological space. Then $B_\mathcal{I}(X) = \alpha_{\mathcal{I}}N_3(X)$.

Proof. It follows from Lemma 1.14 and Theorem 1.21.

Theorem 2.21 Let $(X, \tau, \mathcal{I})$ be an I-submaximal and ∗-extremally disconnected ideal topological space and $A \subseteq X$. The following properties are equivalent.

1. $A$ is an open set in $X$.
2. $A$ is an αI-open set and an $A_\mathcal{I}^*$-set.
3. $A$ is a preI-open and an αA∗I-set.

Proof. (1) ⇔ (2): It follows from Theorem 1.22.

(2) ⇒ (3): It follows from the fact that every αI-open set is preI-open and every $A_\mathcal{I}^*$-set is an αA∗I-set.

(3) ⇒ (1): Suppose that $A$ is a preI-open set and an $A_\mathcal{I}^*$-set. Since $A$ is an αA∗I-set, then we have $A = L \cap M$, where $L$ is an αI-open set and $M = \text{cl}(\text{int}^*(M))$. It follows that $\text{int}(\text{cl}^*(M)) \subseteq \text{cl}^*(M) \subseteq \text{cl}(M) = \text{cl}(\text{int}^*(M)) = M$. 
Since $\text{int}(\text{cl}^*(M)) \subseteq M$, then $M$ is a semi*-I-closed set. By Theorem 2.19, $M$ is a t-I-set. Hence, $A$ is an $\alpha_\mathcal{I}N_3$-set. Since $A$ is an $\alpha_\mathcal{I}N_3$-set and a pre-I-open set, then by Theorem 2.20, $A$ is a $B_\mathcal{I}$-set and a pre-I-open set. By Proposition 1.19, $A$ is an open set in $X$.

**Theorem 2.22** Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. The following properties are equivalent.

1. $A$ is an $\alpha$-I-open set in $X$.
2. $A$ is a pre-I-open and an $\alpha A^*_\mathcal{I}$-set.

**Proof.** (1) $\Rightarrow$ (2): It follows from the fact that every $\alpha$-I-open set is pre-I-open and every $\alpha$-I-open set is an $\alpha A^*_\mathcal{I}$-set.

(2) $\Rightarrow$ (1): Suppose that $A$ is a pre-I-open set and an $\alpha A^*_\mathcal{I}$-set. Since $A$ is an $\alpha A^*_\mathcal{I}$-set, then we have $A = L \cap M$, where $L$ is an $\alpha$-I-open set and $M = \text{cl}(\text{int}^*(M))$. It follows that $\text{int}(\text{cl}^*(M)) \subseteq \text{cl}^*(M) \subseteq \text{cl}(M) = \text{cl}(\text{int}^*(M)) = M$. Since int$(\text{cl}^*(M)) \subseteq M$, then $M$ is a semi*-I-closed set. By Theorem 2.19, $M$ is a t-I-set. Hence, $A$ is an $\alpha_\mathcal{I}N_3$-set. Since $A$ is an $\alpha_\mathcal{I}N_3$-set and a pre-I-open set, then by Theorem 1.16, $A$ is an $\alpha$-I-open set in $X$.

**Remark 2.23** The notions of pre-I-open set and $\alpha A^*_\mathcal{I}$-set are independent of each other.

**Example 2.24**

1. In Example 2.4, $A = \{b, c\}$ is $\alpha A^*_\mathcal{I}$-set but not a pre-I-open set.
2. In Example 2.4, $A = \{a, c\}$ is pre-I-open set but not an $\alpha A^*_\mathcal{I}$-set.

**Theorem 2.25** Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. The following properties are equivalent.

1. $A$ is an $\alpha$-I-open set.
2. $A$ is an $\alpha C^*_\mathcal{I}$-set and a semi*-I-open set.

**Proof.** (1) $\Rightarrow$ (2): It follows from the fact that every $\alpha$-I-open set is an $\alpha C^*_\mathcal{I}$-set and a semi*-I-open set by Remark 1.4 and Lemma 1.5.

(2) $\Rightarrow$ (1): Let $A$ be an $\alpha C^*_\mathcal{I}$-set and a semi*-I-open set. Since $A$ is an $\alpha C^*_\mathcal{I}$-set, then $A$ is an $\alpha C_\mathcal{I}$-set. Since $A$ is an $\alpha C_\mathcal{I}$-set and a semi*-I-open set in $X$, then by Theorem 2.16, $A$ is an $\alpha A^*_\mathcal{I}$-set. Moreover, since $A$ is an $\alpha C^*_\mathcal{I}$-set, then $A$ is a pre-I-open by Theorem 2.2. Hence, by Theorem 2.22, $A$ is an $\alpha$-I-open set in $X$.

**Remark 2.26** The notions of $\alpha C^*_\mathcal{I}$-set and semi*-I-open set are independent of each other.
Example 2.27
1. Let \( X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) and \( I = \emptyset \). Then \( A = \{a, c\} \) is semi*-\( I \)-open set but not an \( \alpha C^*_I \)-set.

2. In Example 2.7, \( A = \{a, c\} \) is \( \alpha C^*_I \)-set but not semi*-\( I \)-open set.

Theorem 2.28 Let \((X, \tau, I)\) be an ideal topological space and \( A \subseteq X \). The following properties are equivalent.

1. \( A \) is an \( \alpha I \)-open set.
2. \( A \) is a semi-\( I \)-open set and an \( \alpha C^*_I \)-set.
3. \( A \) is a semi-\( I \)-open set and a pre-\( I \)-open set.

Proof. (1) \( \Rightarrow \) (2): It is obvious.
(2) \( \Rightarrow \) (3): It follows from the fact that every \( \alpha C^*_I \)-set is a pre-\( I \)-open set by Theorem 2.2.
(3) \( \Rightarrow \) (1): It follows from Lemma 1.20.

Remark 2.29 The notions of semi-\( I \)-open set and \( \alpha C^*_I \)-set are independent of each other.

Example 2.30
1. In Example 2.27(1), \( A = \{a, c\} \) is semi-\( I \)-open set but not an \( \alpha C^*_I \)-set.
2. In Example 2.7, \( A = \{a, c\} \) is \( \alpha C^*_I \)-set but not a semi-\( I \)-open set.

Definition 2.31 A subset \( A \) of an ideal topological space \((X, \tau, I)\) is said to be \( \alpha gp I \)-open if \( N \subseteq p_I \text{int}(A) \) whenever \( N \subseteq A \) and \( N \) is an \( \alpha I \)-closed set in \( X \).

Definition 2.32 A subset \( A \) of an ideal topological space \((X, \tau, I)\) is said to be \( \alpha \)-generalized pre-\( I \)-closed (\( \alpha gp I \)-closed) in \( X \) if \( X \setminus A \) is \( \alpha gp I \)-open.

Theorem 2.33 For a subset \( A \) of an ideal topological space \((X, \tau, I)\), \( A \) is \( \alpha gp I \)-closed if and only if \( p_I \text{cl}(A) \subseteq N \) whenever \( A \subseteq N \) and \( N \) is an \( \alpha I \)-open set in \((X, \tau, I)\).

Proof. Let \( A \) be an \( \alpha gp I \)-closed set in \( X \). Suppose that \( A \subseteq N \) and \( N \) is an \( \alpha I \)-open set in \((X, \tau, I)\). Then \( X \setminus A \) is \( \alpha gp I \)-open and \( X \setminus N \subseteq X \setminus A \) where \( X \setminus N \) is an \( \alpha I \)-closed. Since \( X \setminus A \) is \( \alpha gp I \)-open, then we have \( X \setminus N \subseteq p_I \text{int}(X \setminus A) \), where \( p_I \text{int}(X \setminus A) = (X \setminus A) \cap \text{int}(\text{cl}(X \setminus A)) \). Since \((X \setminus A) \cap \text{int}(\text{cl}(X \setminus A)) = (X \setminus A) \cap (X \setminus \text{cl}(\text{int}(X \setminus A))) = X \setminus (A \cup \text{cl}(\text{int}(A))) \), then by Lemma 1.9, \((X \setminus A) \cap \text{int}(\text{cl}(X \setminus A)) = X \setminus (A \cup \text{cl}(\text{int}(A))) = X \setminus p_I \text{cl}(A) \). It follows that \( p_I \text{int}(X \setminus A) = X \setminus p_I \text{cl}(A) \). Thus \( p_I \text{cl}(A) = X \setminus p_I \text{int}(X \setminus A) \subseteq N \) and hence \( p_I \text{cl}(A) \subseteq N \). The converse is similar.
Theorem 2.34 Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \(V \subseteq X\). Then \(V\) is an \(\alpha C_I\)-set in \(X\) if and only if \(V = G \cap p_I \text{cl}(V)\) for an \(\alpha I\)-open set \(G\) in \(X\).

Proof. If \(V\) is an \(\alpha C_I\)-set, then \(V = G \cap M\) for an \(\alpha I\)-open set \(G\) and a pre-\(I\)-closed set \(M\). But then \(V \subseteq M\) and so \(V \subseteq \text{p}_I \text{cl}(V) \subseteq M\). It follows that \(V = V \cap \text{p}_I \text{cl}(V) = G \cap M \cap \text{p}_I \text{cl}(V) = G \cap \text{p}_I \text{cl}(V)\). Conversely, it is enough to prove that \(\text{p}_I \text{cl}(V)\) is a pre-\(I\)-closed set. But \(\text{p}_I \text{cl}(V) \subseteq M\), for any pre-\(I\)-closed set \(M\) containing \(V\). So, \(\text{cl}(\text{int}^*(\text{p}_I \text{cl}(V))) \subseteq \text{cl}(\text{int}^*(M)) \subseteq M\). It follows that \(\text{cl}(\text{int}^*(\text{p}_I \text{cl}(V))) \subseteq \cap V \subseteq M\). \(M\) is pre-\(I\)-closed \(= \text{p}_I \text{cl}(V)\).

Theorem 2.35 Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \(A \subseteq X\). The following properties are equivalent.

1. \(A\) is a pre-\(I\)-closed set in \(X\).
2. \(A\) is an \(\alpha C_I\)-set and an \(\alpha gp_I\)-closed set in \(X\).

Proof. (1) \(\Rightarrow\) (2): It follows from the fact that any pre-\(I\)-closed set in \(X\) is an \(\alpha C_I\)-set and an \(\alpha gp_I\)-closed set in \(X\).

(2) \(\Rightarrow\) (1): Suppose that \(A\) is an \(\alpha C_I\)-set and an \(\alpha gp_I\)-closed set in \(X\). Since \(A\) is an \(\alpha C_I\)-set, then by Theorem 2.34, \(A = G \cap \text{p}_I \text{cl}(A)\) for an \(\alpha I\)-open set \(G\) in \((X, \tau, \mathcal{I})\). Since \(A \subseteq G\) and \(A\) is \(\alpha gp_I\)-closed set in \(X\), then \(\text{p}_I \text{cl}(A) \subseteq G\).

It follows that \(\text{p}_I \text{cl}(A) \subseteq G \cap \text{p}_I \text{cl}(A) = A\). Thus, \(A = \text{p}_I \text{cl}(A)\) and hence \(A\) is pre-\(I\)-closed.

Theorem 2.36 Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \(A \subseteq X\). If \(A\) is an \(\alpha C_I\)-set in \(X\), then \(\text{p}_I \text{cl}(A) \setminus A\) is a pre-\(I\)-closed set and \(A \cup (X \setminus \text{p}_I \text{cl}(A))\) is a pre-\(I\)-open set in \(X\).

Proof. Suppose that \(A\) is an \(\alpha C_I\)-set in \(X\). By Theorem 2.34, we have \(A = L \cap \text{p}_I \text{cl}(A)\) for an \(\alpha I\)-open set \(L\) in \(X\). It follows that \(\text{p}_I \text{cl}(A) \setminus A = \text{p}_I \text{cl}(A) \setminus (L \cap \text{p}_I \text{cl}(A)) = \text{p}_I \text{cl}(A) \cap (X \setminus (L \cap \text{p}_I \text{cl}(A))) = \text{p}_I \text{cl}(A) \cap ((X \setminus L) \cup (X \setminus \text{p}_I \text{cl}(A)))\).

It is \(= \text{p}_I \text{cl}(A) \cap (X \setminus L) \cup (\text{p}_I \text{cl}(A) \cap (X \setminus \text{p}_I \text{cl}(A))) = (\text{p}_I \text{cl}(A) \cap (X \setminus L)) \cup (\text{p}_I \text{cl}(A) \cap (X \setminus \text{p}_I \text{cl}(A))) \cup \phi = \text{p}_I \text{cl}(A) \cap (X \setminus L)\). Thus \(\text{p}_I \text{cl}(A) \setminus A = \text{p}_I \text{cl}(A) \cap (X \setminus L)\) and hence \(\text{p}_I \text{cl}(A) \setminus A\) is pre-\(I\)-closed set. Moreover, since \(\text{p}_I \text{cl}(A) \setminus A\) is a pre-\(I\)-closed set in \(X\), then \(X \setminus (\text{p}_I \text{cl}(A) \setminus A) = (X \setminus \text{p}_I \text{cl}(A) \cap (X \setminus A)) = (X \setminus \text{p}_I \text{cl}(A)) \cup A\) is a pre-\(I\)-open set. Thus, \(X \setminus (\text{p}_I \text{cl}(A) \setminus A) = (X \setminus \text{p}_I \text{cl}(A)) \cup A\) is a pre-\(I\)-open set in \(X\).

3. Further properties

Definition 3.1 [7] Let \((X, \tau, \mathcal{I})\) be an ideal topological space. \((X, \tau, \mathcal{I})\) is said to be pre-\(I\)-connected if \(X\) can not be expressed as the disjoint union of two nonvoid pre-\(I\)-open sets.

Theorem 3.2 Let \((X, \tau, \mathcal{I})\) be an ideal topological space. The following properties are equivalent.
1. \((X, \tau, \mathcal{I})\) is pre-\(\mathcal{I}\)-connected.

2. \((X, \tau, \mathcal{I})\) can not be expressed as the disjoint union of two nonvoid \(\alpha C^*_\mathcal{I}\)-sets.

**Proof.** (1) ⇒ (2): Suppose that \((X, \tau, \mathcal{I})\) can be expressed as the disjoint union of two nonvoid \(\alpha C^*_\mathcal{I}\)-sets. Since any \(\alpha C^*_\mathcal{I}\)-set is a pre-\(\mathcal{I}\)-open set, then \((X, \tau, \mathcal{I})\) can be expressed as the disjoint union of two nonvoid pre-\(\mathcal{I}\)-open sets. So, \((X, \tau, \mathcal{I})\) is not pre-\(\mathcal{I}\)-connected. This is a contradiction.

(2) ⇒ (1): Suppose that \((X, \tau, \mathcal{I})\) is not pre-\(\mathcal{I}\)-connected. Then, \(X\) can be expressed as the disjoint union of two nonvoid pre-\(\mathcal{I}\)-open sets. It follows that \(X\) has a nontrivial pre-\(\mathcal{I}\)-regular subset \(A\). Moreover, \(A\) and \(B = X \setminus A\) are pre-\(\mathcal{I}\)-regular. Then \(A\) and \(B\) are \(\alpha C^*_\mathcal{I}\)-sets. Hence \((X, \tau, \mathcal{I})\) can be expressed as the disjoint union of two nonvoid \(\alpha C^*_\mathcal{I}\)-sets. This is a contradiction.

**Theorem 3.3** In an \(\mathcal{I}\)-submaximal ideal space \((X, \tau, \mathcal{I})\), the following properties holds.

1. Any \(\alpha C^*_\mathcal{I}\)-set is an \(\eta\zeta\)-set and an \(\alpha A\beta\mathcal{I}\)-set.

2. Any \(\alpha\eta\mathcal{I}\)-set is a locally closed set.

**Proof.** (1) Suppose that \(A\) is an \(\alpha C^*_\mathcal{I}\)-set in \(X\). It follows that \(A = L \cap M\), where \(L\) is an \(\alpha\mathcal{I}\)-open set and \(M\) is a pre-\(\mathcal{I}\)-regular set in \(X\). By Theorem 1.21, \(M\) is semi-\(\mathcal{I}\)-open and semi-\(\mathcal{I}\)-closed. It follows from Lemma 1.5 that \(M\) is semi-\(\mathcal{I}\)-open and semi\(^*\)\(\mathcal{I}\)-closed. By Theorem 2.19, \(M\) is semi-\(\mathcal{I}\)-open and a t-\(\mathcal{I}\)-set in \(X\). Hence \(M\) is semi-\(\mathcal{I}\)-regular set. Thus, \(A\) is an \(\alpha A\beta\mathcal{I}\)-set in \(X\). Furthermore, by Theorem 1.21, \(A\) is an \(\eta\zeta\)-set.

(2) It follows from Theorem 1.21.

**Definition 3.4** [9] An ideal topological space \((X, \tau, \mathcal{I})\) is said to be \(\star\)-hyperconnected if \(A\) is \(\star\)-dense for every open subset \(A \neq \phi\) of \(X\).

**Theorem 3.5** [9] The following properties are equivalent for an ideal topological space \((X, \tau, \mathcal{I})\).

1. \(X\) is \(\star\)-hyperconnected.

2. \(A\) is \(\star\)-dense for every strongly \(\beta\mathcal{I}\)-open subset \(\phi \neq A \subseteq X\).

**Theorem 3.6** For an ideal topological space \((X, \tau, \mathcal{I})\), the following properties are equivalent.

1. \((X, \tau, \mathcal{I})\) is \(\star\)-hyperconnected.

2. any \(\alpha C^*_\mathcal{I}\)-set in \(X\) is \(\star\)-dense.
Proof. \((1) \Rightarrow (2): \) Let \(A\) be an \(\alpha C^*_\mathcal{I}\)-set in \(X\). By Theorem 2.2, \(A\) is pre-\(\mathcal{I}\)-open. By Remark 1.4, \(A\) is strongly \(\beta\)-\(\mathcal{I}\)-open set. Since \((X, \tau, \mathcal{I})\) is a \(\star\)-hyperconnected ideal topological space, then by Theorem 3.5, \(A\) is \(\star\)-dense.

\((2) \Rightarrow (1): \) Suppose that any \(\alpha C^*_\mathcal{I}\)-set in \((X, \tau, \mathcal{I})\) is \(\star\)-dense in \(X\). Since an open set \(A\) in \(X\) is an \(\alpha\)-\(\mathcal{I}\)-open set and every \(\alpha\)-\(\mathcal{I}\)-open set \(A\) is an \(\alpha C^*_\mathcal{I}\)-set, then \(A\) is \(\star\)-dense. Thus, \((X, \tau, \mathcal{I})\) is \(\star\)-hyperconnected.

4. Decompositions of \(\alpha\)-\(\mathcal{I}\)-continuity and \(\alpha A^*_\mathcal{I}\)-continuity

Definition 4.1 A function \(f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)\) is said to be

1. \(\alpha C^*_\mathcal{I}\)-continuous if \(f^{-1}(A)\) is an \(\alpha C^*_\mathcal{I}\)-set in \(X\) for every open set \(A\) in \(Y\).
2. \(PR^*_\mathcal{I}\)-continuous \([7]\) if \(f^{-1}(A)\) is a pre-\(\mathcal{I}\)-regular set in \(X\) for every open set \(A\) in \(Y\).

Remark 4.2 For a function \(f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)\), the following diagram holds. The reverses of these implications are not true in general as shown in the following Examples.

\[
\begin{array}{ccc}
\text{pre-\(\mathcal{I}\)-continuity} & \uparrow & \text{PR^*_\mathcal{I}-continuity} \\
\text{\(\alpha C^*_\mathcal{I}\)-continuity} & \leftarrow & \text{\(PR^*_\mathcal{I}\)-continuity} \\
\end{array}
\]

Example 4.3 Let \(X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}, Y = \{p, q, r\}, \sigma = \{\emptyset, \{p\}, \{p, q\}, \{p, r\}, Y\}, \mathcal{I} = \{\emptyset\}\) and \(\mathcal{J} = \{\emptyset\}\). Define \(f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})\) by \(f(a) = p\); \(f(b) = q\) and \(f(c) = r\). Then \(f\) is pre-\(\mathcal{I}\)-continuous but not \(\alpha C^*_\mathcal{I}\)-continuous.

Example 4.4 Let \(X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}, Y = \{p, q, r\}, \sigma = \{\emptyset, \{p\}, Y\}, \mathcal{I} = \{\emptyset\}\) and \(\mathcal{J} = \{\emptyset\}\). Define \(f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})\) by \(f(a) = p\); \(f(b) = q\) and \(f(c) = r\). Then \(f\) is \(\alpha C^*_\mathcal{I}\)-continuous but not \(PR^*_\mathcal{I}\)-continuous.

Definition 4.5 A function \(f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)\) is said to be

1. \(\alpha C^*_\mathcal{I}\)-continuous if \(f^{-1}(A)\) is an \(\alpha C^*_\mathcal{I}\)-set in \(X\) for every open set \(A\) in \(Y\).
2. \(\alpha A^*_\mathcal{I}\)-continuous if \(f^{-1}(A)\) is an \(\alpha A^*_\mathcal{I}\)-set in \(X\) for every open set \(A\) in \(Y\).
3. \(\alpha \eta^*_\mathcal{I}\)-continuous if \(f^{-1}(A)\) is an \(\alpha \eta^*_\mathcal{I}\)-set in \(X\) for every open set \(A\) in \(Y\).
4. \(A^*_\mathcal{I}\)-continuous \([7]\) if \(f^{-1}(A)\) is an \(A^*_\mathcal{I}\)-set in \(X\) for every open set \(A\) in \(Y\).

Remark 4.6 For a function \(f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)\), the following diagram holds. The reverses of these implications are not true in general as shown in the following Examples.

\[
\begin{array}{ccc}
\text{\(\alpha C^*_\mathcal{I}\)-continuity} & \uparrow & \text{\(\alpha C^*_\mathcal{I}\)-continuity} \\
\text{\(\alpha \eta^*_\mathcal{I}\)-continuity} & \leftarrow & \text{\(\alpha A^*_\mathcal{I}\)-continuity} & \leftarrow & \text{\(A^*_\mathcal{I}\)-continuity} \\
\end{array}
\]
Example 4.7 Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$, $Y = \{p, q, r, s\}$, $\sigma = \{\emptyset, \{p\}, \{q\}, \{p, q, s\}, Y\}$, $\mathcal{I} = \{\emptyset\}$ and $\mathcal{J} = \{\emptyset\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = p$, $f(b) = q$, $f(c) = r$ and $f(d) = s$. Then $f$ is $\alpha A^*_I$-continuous but not $\alpha I^*_R$-continuous.

Example 4.8 Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$, $Y = \{p, q, r, s\}$, $\sigma = \{\emptyset, \{r\}, \{s\}, \{r, s\}, Y\}$, $\mathcal{I} = \{\emptyset\}$ and $\mathcal{J} = \{\emptyset\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = p$, $f(b) = q$, $f(c) = r$ and $f(d) = s$. Then $f$ is $\alpha \eta^*_I$-continuous but not $\alpha \eta^*_R$-continuous.

Example 4.9 In Example 4.8, $f$ is $\alpha C_I$-continuous but not $\alpha C^*_I$-continuous.

Example 4.10 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, $Y = \{p, q, r\}$, $\sigma = \{\emptyset, \{q\}, Y\}$, $\mathcal{I} = \{\emptyset\}$ and $\mathcal{J} = \{\emptyset\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = p$, $f(b) = q$ and $f(c) = r$. Then $f$ is $\alpha C_I$-continuous but not $\alpha \eta^*_I$-continuous.

Definition 4.11 [7] A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be semi*$^*$-$\mathcal{I}$-continuous if $f^{-1}(V)$ is a semi*$^*$-$\mathcal{I}$-open set in $X$ for every open set $V$ in $Y$.

Theorem 4.12 The following properties are equivalent for a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$:

1. $f$ is $\alpha A^*_I$-continuous.
2. $f$ is $\alpha \eta^*_I$-continuous and semi*$^*$-$\mathcal{I}$-continuous.
3. $f$ is $\alpha C^*_I$-continuous and semi*$^*$-$\mathcal{I}$-continuous.

Proof. It follows from Theorem 2.16.

Theorem 4.13 The following properties are equivalent for a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$:

1. $f$ is $\alpha$-$\mathcal{I}$-continuous.
2. $f$ is pre-$\mathcal{I}$-continuous and $\alpha A^*_I$-continuous.
3. $f$ is semi*$^*$-$\mathcal{I}$-continuous and $\alpha C^*_I$-continuous.
4. $f$ is semi-$\mathcal{I}$-continuous and $\alpha C^*_I$-continuous.
5. $f$ is semi-$\mathcal{I}$-continuous and pre-$\mathcal{I}$-continuous.

Proof. It follows from Theorems 2.22, 2.25 and 2.28.

References


Accepted: 11.03.2015