VERTEX-TO-EDGE CENTERS W.R.T. D-DISTANCE

D. Reddy Babu

Department of Mathematics
K.L. University
Vaddeswaram
Guntur 522 502
India
e-mail: reddybabu@kluniversity.in, reddybabu17@gmail.com

P.L.N. Varma
Department of Science & Humanities
V.F.S.T.R. University
Vadlamudi
Guntur 522 237
India
e-mail: varma_sh@vignanuniversity.org

Abstract. Between any two vertices of a graph we can define many distances, the usual distance, detour distance, superior distance, signal distance, degree distance etc. In some of these distances only the lengths of various paths were considered. By considering the degrees of various vertices present in a path, in addition to the length of the path, in an earlier article we introduced the concept of D-distance, $d^D(u,v)$, in graphs. In this article we study D-distance between a vertex and an edge of a graph and determine the eccentrics, radius and diameters of some classes of graphs. We also obtain relations between their eccentrics and also determine centers of graphs. Further, we prove that for any graph G either $C^D(G) \subseteq C_1^D(G)$ or $C_1^D(G) \subseteq C^D(G)$.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops and multiple edges. The concept of distance is one of the important concepts in study of graphs. It is used in isomorphism testing, graph operations, hamiltonicity problems, extremal problems on connectivity and diameter, convexity in graphs etc. Distance is the basis of many concepts of symmetry in graphs.

In addition to the usual distance, $d(u,v)$, between two vertices $u, v \in V$ we have detour distance, superior distance, signal distance, degree distance etc.

In an earlier article [3], the authors have introduced the concept of D-distance by considering not only path length between vertices, but also the degrees of all the vertices present in a path.

The concept of distance between vertex and edge was introduced by Slater [4]. Later the concept of centers in graphs with respect to edges etc. was studied by A.P. Santhakumaran [2]. In this article we study vertex-to-edge centers with respect to D-distance.

In this article, we prove for any graph G either $C^D(G) \subseteq C_1^D(G)$ or $C_1^D(G) \subseteq C^D(G)$. We also prove vertex-to-edge center of a graph G is a subgraph of some block of G and the vertex set of every graph G is the vertex-to-edge center of some connected graph.

2. Preliminaries

For reader convenience we recall the definition of D-distance which the authors have introduced in [3].

Definition 2.1. If u, v are vertices of a connected graph G, the D-length of a connected $u - v$ path s is defined as $l^D(s) = l(s) + deg(v) + deg(u) + \sum deg(w)$ where sum runs over all intermediate vertices w of s and $l(s)$ is the length of the path.

Definition 2.2 (D-distance)**.** The D-*distance*, $d^D(u, v)$, between two vertices u, v of a connected graph G is defined as $d^D(u, v) = \min\{l(s)\}$ if u, v are distinct and $d^D(u, v) = 0$ if $u = v$, where the minimum is taken over all $u - v$ paths s in G. In other words,

$$d^D(u,v) = \begin{cases} \min\limits_{s} \left\{ d(u,v) + deg(u) + deg(v) + \sum deg(w) \right\} & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

where the sum runs over all intermediate vertices w in s and minimum is taken over all $u - v$ paths s in G.

Some Properties of D-distance:

1. The D-distance is a metric on the set of all vertices of a connected graph G.
2. In a connected graph, two distinct vertices u, v are *adjacent* if and only if $d^D(u, v) = deg(u) + deg(v) + 1$.
3. For any two vertices u, v we have $d^D(u, v) \leq d^D(u, w) + d^D(w, v) - deg(w)$ for all $w \in G$.

We can define D-eccentricity, D-radius and D-diameter etc. in a natural way.

Definition 2.3. The *D-eccentricity* of any vertex v, $e^D(v)$, is defined as the maximum distance from v to any other vertex, i.e., $e^D(v) = \max\left\{d^D(u,v) : u \in V\right\}$. Any vertex u for which $e^D(v) = d^D(u,v)$ is called the D-eccentric vertex of v. Further, a vertex u is said to be D-eccentric vertex of G if it is the D-eccentric vertex of some vertex.

Definition 2.4. The *D-radius*, $r^D(G)$, is the minimum D-eccentricity among all vertices of G and *D-diameter*, $d^D(G)$, is the maximum D-eccentricity i.e., $r^D(G) = \min\left\{e^D(u) : u \in V\right\}$ and $d^D(G) = \max\left\{e^D(u) : u \in V\right\}$. We have

$$r^D(G) \leq d^D(G) \leq 2r^D(G).$$

Futher,

Definition 2.5. The *D-centre* of G, $C^D(G)$, is the subgraph induced by the set of all vertices of minimum D-eccentricity. A graph is called *D-self-centered* if $C^D(G) = V$ or equivalently $d^D(G) = r^D(G)$. Similarly, the set of all vertices of maximum D-eccentricity is the *D-periphery* of G .

Next, we define the *vertex-to-edge D-distance.*

Definition 2.6. If v is a vertex and $e = xy$ is any edge (joining the vertices x and y) we define the distance between them as $d^D(v,e) = min\left\{d^D(v,x), d^D(v,y)\right\}$. Similarly, for any two subsets S, T of V we have $d^D(v,S) = min\left\{d^D(v,s) : s \in S\right\}$ and $d^D(S,T) = min\left\{d^D(s,t) : s \in S, t \in T\right\}$.

One more,

Definition 2.7. For any vertex v of G the *vertex-to-edge eccentricity*, e_1^D, of v is given by $e_1^D(v) = \max\left\{d^D(v,e) : e \in E\right\}$. A vertex v for which $e_1^D(v)$ is minimum is called a *vertex-to-edge central vertex* and the set of all vertex-to-edge central vertices of G is the *vertex-to-edge center* of G. This is denoted by $C_1^D(G)$.

3. Results on centers

We begin this section with an important result.

Theorem 3.1. *For any vertex v in a graph G we have $e^D(v) - deg(v) - 1 \leq e_1^D(v) \leq e^D(v)$. Further $e^D(v) = e_1^D(v)$ if and only if both the vertices of the eccentric edge of v are eccentric vertices of v.*

Proof. Let v be any vertex of G and let u be an eccentric vertex of v. Let $p = u = v_0, v_1, v_2, \cdots, v_k = v$ be a $u - v$ minimum D-path. Then $d^D(u,v) = k + deg(v_0) + deg(v_1) + deg(v_2) + \cdots + deg(v_k) = e^D(v)$. Let $e = v_{k-1}v_k$, since $d^D(v,e) = k - 1 + deg(v_0) + deg(v_1) + deg(v_2) + \cdots + deg(v_{k-1}) = e^D(v) - deg(v) - 1$. It follows that $e_1^D(v) \geq e^D(v) - deg(v) - 1$. Also since $e_1^D(v) = \min\left\{d^D(v,e) : e \in E\right\}$ it follows that that $e_1^D(v) \leq d^D(v,u) = e^D(v)$. Thus $e^D(v) - deg(v) - 1 \leq e_1^D(v) \leq e^D(v)$.

Now suppose that $e_1^D(v) = e^D(v)$, for some vertex v of G. Then for any eccentric edge $e = xy$ of v we have $d^D(v, e) = e_1^D = e^D(v) \geq d^D(v, x)$. Thus $d^D(v, x) \leq d^D(v, e) \leq d^D(v, y)$.

Similarly we can prove that $d^D(v, y) \leq d^D(v, e)$ it follows that $d^D(v, y) = d^D(v, x)$ and both are equal to $e_1^D(v)$ so x and y are eccentric vertices of v.

Conversely, if both x and y are eccentric vertices of v, then $e^D(v) = d^D(v, x) = d^D(v, y)$. Hence $e_1^D(v) = e^D(v)$. ■

Theorem 3.2. *For any graph either* $C^D(G) \subseteq C_1^D(G)$ *or* $C_1^D(G) \subseteq C^D(G)$.

Proof. Suppose that the result is false. Then there exit vertices u and v such that $u \in C_1^D(G) - C^D(G)$ and $v \in C^D(G) - C_1^D(G)$. Thus

$$e^D(v) < e^D(u) \tag{3.1}$$

and

$$e_1^D(u) < e_1^D(v) \tag{3.2}$$

If $e_1^D(u) = e^D(u)$ it follows from (3.1) and (3.2) that $e^D(v) < e_1^D(v)$ which is contradiction to Theorem(3.1). Similarly if $e_1^D(v) = e^D(v) - deg(v) - 1$ it follows from (3.1) and (3.2) that $e_1^D(u) < e^D(u) - deg(u) - 1$ which is again a contradiction to theorem(3.1). Hence it follows from theorem that

$$e_1^D(u) = e^D(u) - deg(u) - 1 \tag{3.3}$$

and

$$e_1^D(v) = e^D(v) \tag{3.4}$$

Now, it follows from (3.2), (3.3) and (3.4) that $e^D(u) - deg(u) - 1 < e^D(v)$. This gives $e^D(u) \leq e^D(v)$, which is a contradiction to (3.1). Hence the proof. ■

Now, let us look at some examples.

Example 3.3. Consider the $(5, 5)$ graph kite (K), shown in Figure 1. For this, we calculate the D-eccentricities of the vertices.

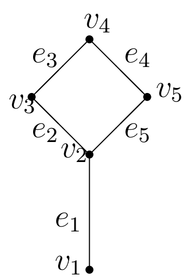

Figure 1: Kite

The following table gives the D-eccentricities of vertices of kite, K.

	v_1	v_2	v_3	v_4	v_5
$e^D(v)$	11	9	8	11	8
$e_1^D(v)$	8	6	6	9	6

Example 3.4. Consider the $(5, 6)$ graph hut H, shown in Figure 2. For this, we calculate the D-eccentricities of the vertices.

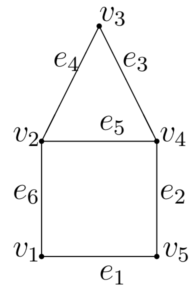

Figure 2: Hut

The following table gives the D-eccentricities of vertices of hut, H.

	v_1	v_2	v_3	v_4	v_5
$e^D(v)$	9	9	9	9	9
$e_1^D(v)$	9	7	9	7	9

Example 3.5. Consider the $(6, 8)$ graph tent (T), shown in Figure 3. For this, we calculate the D-eccentricities of the vertices.

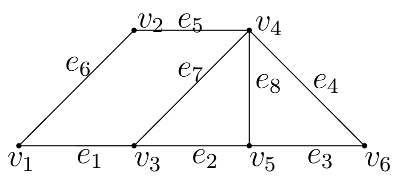

Figure 3: Tent

The following table gives the D-eccentricities of vertices in tent.

	v_1	v_2	v_3	v_4	v_5	v_6
$e^D(v)$	13	11	10	10	11	13
$e_1^D(v)$	10	10	8	8	10	10

After these examples, now we have

Remark 1. From these examples, it is clear that $C^D(K) \subseteq C_1^D(K), C^D(H) \supseteq C_1^D(H)$ and $C^D(T) = C_1^D(T)$.

Now, we have one of the result mentioned in the introduction.

Theorem 3.6. *The vertex set of every graph G, with at least two vertices, is the vertex-to-edge center of some connected graph.*

Proof. Let G be a given graph with n vertices. We construct a connected graph H such that vertex-to-edge center of H is the vertex set of G. First, if G is not complete make G a complete graph, by considering all the remaining edges. Then introduce two new vertices u and v to $V(G)$ and join them to every vertex of G but not each other as shown below. Next introduce two more vertices p and q and join p to u and q to v. The resulting graph H is given in Figure 4.

Then it is clear that

$$e_1^D(p) = e_1^D(q) = 3n + 7, e_1^D(u) = e_1^D(v) = 3n + 5 \text{ and } e_1^D(w) = 2n + 3$$

for every vertex w in G. Therefore, every w of G has minimum D-eccentricity. Hence the vertex set of G is the vertex-to-edge center of H.

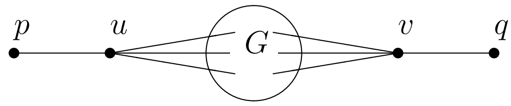

Figure 4: Construction of H

After this result, we prove a result on bipartite graphs.

Theorem 3.7. *For any vertex v of a bipartite graph G, $e^D(v) - 3 \leq e_1^D(v)$.*

Proof. Let $e = ab$ be an eccentric edge of v. If both a and b are eccentric vertices of v then $e^D(v) = d^D(v, a) = d^D(v, b)$. Let M and N be $v - a$ and $v - b$ geodesics respectively. If u is the last vertex that is common to both M and N, then the $u - a$ section of M followed by the edge ab and the $b - u$ section of N^{-1} forms an odd cycle in G, which is a contradiction to the fact that G containing even cycles. Hence it follows from Theorem 3.1 that $e^D(v) - 3 \leq e_1^D(v)$. ■

We recall a definition before proving the next result.

Definition 3.8. A *block* of a graph is a maximal connected sub graph having no cut vertices. A graph G with all its blocks complete is called a *block graph.*

Theorem 3.9. *The vertex-to-edge center, $C_1^D(G)$, of every connected graph is a sub graph of some block of G.*

Proof. If possible that assume the center, $C_1^D(G)$, of a connected graph G lies in more than one block. Then G contains a cut vertex v such that $G - v$ has components G_1 and G_2, each of which contain a vertex of $C_1^D(G)$. Let $e = pq$ be an eccentric edge of v then $e_1(v) = d^D(v, e) = d^D(v, p)$ (say). Let M be a $v - p$ geodesic. At least one of G_1 and G_2 contains no vertices of M, say G_2 contains no vertices of M. Let w be a vertex of $C_1^D(G)$ that belongs to G_2 and let N be a $w - p$ geodesic in G. Now N followed by M gives a $w - p$ geodesic whose D-length is greater than that of M. It follows that $e_1^D(w) > e_1^D(v)$ which is contradiction because $w \in C_1^D(G)$. Thus the vertex-to-edge center C_1^D of every connected graph G is a subgraph of some block of G. ■

4. D-radius and D-diameter for vertex-to-edge D-distance

In this section we study some classes of graphs w.r.t. D-distance.

For cycles, we have

$$r_1^D(C_{2n}) = r_1^D(C_{2n-1}) = d_1^D(C_{2n}) = d_1^D(C_{2n-1}) = 3n - 1 \text{ for } n \geq 1.$$

Hence cycles are self centered graphs. Thus we have

Proposition 4.1. *For any cycle, C_n, we have $C^D(C_n) = C_1^D(C_n) = V(C_n)$.*

Proposition 4.2. *In cycle, C_n, the difference between the two eccentricities, $e^D(v)$ and $e_1^D(v)$ is 3 for all vertices V of C_n.*

For complete graphs, we have $r_1^D(K_n) = d_1^D(K_n) = 2n - 1$ for all $n \geq 3$.

Proposition 4.3. *Any complete graph, K_n, is self centered and further $C^D(K_n) = C_1^D(K_n) = V(K_n)$.*

Proposition 4.4. *If G is a complete graph, K_n, then the two eccentricities $e^D(v)$ and $e_1^D(v)$ are equal for all vertices v.*

Proof. This follows from the fact that $d^D(v, e) = 2n-1$ for any vertex v and any edge e of K_n. ■

Proposition 4.5. *For wheel graph, $W_{n,1}$, we have $r_1^D(W_{n,1}) = n + 4 \ \forall\ n \geq 5$, $r_1^D(W_{3,1}) = 7$, $r_1^D(W_{4,1}) = 7$ and $d_1^D(W_{n,1}) = 2n + 1 \ \forall\ n \geq 5$, $d_1^D(W_{3,1}) = 7$, $d_1^D(W_{4,1}) = 8$.*

Proposition 4.6. *For star graph, $St_{n,1}$, we have $r_1^D(St_{n,1}) = 0$ and $d_1^D(St_{n,1}) = n + 2 \ \forall\ n \geq 2$.*

Proposition 4.7. *For any path graph, we have $r_1^D(P_{2n}) = r_1^D(P_{2n+1}) = 3n - 1$ $\forall\ n \geq 2$, $r_1^D(P_2) = r_1^D(P_3) = 0$. Similarly, $d_1^D(P_n) = 3n - 5 \ \forall\ n \geq 3$.*

Proposition 4.8. *For any complete bipartite graph, $K_{m,n}$, we have $r_1^D(K_{m,n}) = d_1^D(K_{m,n}) = m + n + 1 \ \forall\ m, n \geq 2$.*

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