$\mathcal{I}_{g^\star}$-NORMAL AND $\mathcal{I}_{g^\star}$-REGULAR SPACES

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Abstract. $\mathcal{I}_{g^\star}$-normal and $\mathcal{I}_{g^\star}$-regular spaces are introduced and various characterizations and properties are given. Characterizations of normal, mildly normal, $g^\star$-normal and regular spaces are also given.

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1. Introduction and preliminaries

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If A$\subseteq$X, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ). A subset A of a space (X, τ) is said to be regular open [18] if A=int(cl(A)) and A is said to be regular closed [18] if A=cl(int(A)). A subset A of a space (X, τ) is said to be an α-open [11] (resp. preopen [8]) if A$\subseteq$int(cl(int(A))) (resp. A$\subseteq$int(cl(A))). The complement of α-open set is α-closed [9]. The α-closure [9] of a subset A of X, denoted by αcl(A), is defined to be the intersection of all α-closed sets containing A. The α-interior [9] of a subset A of X, denoted by αint(A), is defined to be the union of all α-open sets contained in A. The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The interior of a subset A in (X, τ^α) is denoted by int_α(A). The closure of a subset A in (X, τ^α) is denoted by cl_α(A). A subset A of a space (X, τ) is said to be g-closed [6] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is open. The complement of g-closed set is g-open. A subset A of a space (X, τ) is said to be $g^\#\alpha$-closed [13] (resp. rαg-closed [12]) if cl_α(A)$\subseteq$U whenever A$\subseteq$U and U is g-open (resp. regular open). A is said to be $g^\#\alpha$-open (resp. rαg-open) if X$-$A is $g^\#\alpha$-closed (resp. rαg-closed). A subset A of a space (X, τ) is said to be $g^\star$-closed [20] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is g-open. A space (X, τ) is said to be $g^\star$-normal, if for every disjoint $g^\star$-closed sets A and B, there exist disjoint open sets U and V such that A$\subseteq$U and B$\subseteq$V.

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) A$\in\mathcal{I}$ and B$\subseteq$A imply B$\in\mathcal{I}$ and (ii) A$\in\mathcal{I}$ and B$\in\mathcal{I}$ imply A$\cup$B$\in\mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^*$: $\wp$(X) $\rightarrow$ $\wp$(X), called a local function [5] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A$\subseteq$X, A$^*(\mathcal{I}, \tau)$={x$\in$X : U$\cap$A$\notin\mathcal{I}$ for every U$\in\tau$(x)} where τ(x)={U$\in\tau$: x$\in$U}. We will make use of the basic facts about the local functions [[4], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator cl*(.) for a topology $\tau^*(\mathcal{I}, \tau)$, called the $\star$-topology, finer than τ is defined by cl*(A)=A$\cup$A$^*(\mathcal{I}, \tau)$ [4]. When there is no chance for confusion, we will simply write A* for A$^*(\mathcal{I}, \tau)$ and τ^* for $\tau^*(\mathcal{I}, \tau)$. int*(A) will denote the interior of A in (X, τ^*). If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space. N is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is τ^*-closed [4] or $\star$-closed (resp. $*$-dense in itself [3]) if A$^*\subseteq$A (resp. A$\subseteq$A*). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is $\mathcal{I}_{g^\star}$-closed [15] if A$^*\subseteq$U whenever U is g-open and A$\subseteq$U. By Theorem 2.3 of [15], every $\star$-closed and hence every closed set is $\mathcal{I}_{g^\star}$-closed. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}_{g^\star}$-open [15] if X$-$A is $\mathcal{I}_{g^\star}$-closed. In this paper, we define $\mathcal{I}_{g^\star}$-normal, ${}_{g^\star}\mathcal{I}$-normal and $\mathcal{I}_{g^\star}$-regular spaces using $\mathcal{I}_{g^\star}$-open sets and give characterizations and properties of such spaces. Also, characterizations of normal, mildly normal, $g^\star$-normal and regular spaces are given.

An ideal $\mathcal{I}$ is said to be codense [2] if $\tau\cap\mathcal{I}=\{\emptyset\}$. $\mathcal{I}$ is said to be completely codense [2] if PO(X)$\cap\mathcal{I}=\{\emptyset\}$, where PO(X) is the family of all preopen sets in (X, τ). Every completely codense ideal is codense but not conversely [2]. The following lemmas will be useful in the sequel.

Lemma 1.1 ([16], Theorem 6) *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I}$ is completely codense, then $\tau^* \subseteq \tau^\alpha$.*

Lemma 1.2 ([15], Theorem 2.16) *Let $(X,\tau,\mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

1. *X is normal.*
2. *For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that $A\subseteq U$, $B\subseteq V$.*
3. *For any closed set A and open set V containing A, there exists an $\mathcal{I}_{g^\star}$-open set U such that $A\subseteq U\subseteq cl^\star(U)\subseteq V$.*

Lemma 1.3 [15] *If $(X,\tau,\mathcal{I})$ is an ideal topological space and $A\subseteq X$, then the following hold.*

1. *If $\mathcal{I}=\{\emptyset\}$, then A is $\mathcal{I}_{g^\star}$-closed if and only if A is $g^\star$-closed.*
2. *If $\mathcal{I}=N$, then A is $\mathcal{I}_{g^\star}$-closed if and only if A is $g^{\#}\alpha$-closed.*

Lemma 1.4 ([15], Theorem 2.2) *If $(X,\tau,\mathcal{I})$ is an ideal topological space and $A\subseteq X$, then the following are equivalent.*

1. *A is $\mathcal{I}_{g^\star}$-closed.*
2. *$cl^\star(A)\subseteq U$ whenever $A\subseteq U$ and U is g-open in X.*

Lemma 1.5 ([15], Theorem 2.12) *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and $A\subseteq X$. Then A is $\mathcal{I}_{g^\star}$-open if and only if $F\subseteq int^\star(A)$ whenever F is g-closed and $F\subseteq A$.*

Lemma 1.6 ([15], Theorem 2.15) *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then every subset of X is $\mathcal{I}_{g^\star}$-closed if and only if every g-open set is $\star$-closed.*

Proposition 1.7 [6] *Every open set is g-open but not conversely.*

2. $\mathcal{I}_{g^\star}$-normal and ${}_{g^\star}\mathcal{I}$-normal spaces

An ideal topological space $(X,\tau,\mathcal{I})$ is said to be an $\mathcal{I}_{g^\star}$-normal space if for every pair of disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that $A\subseteq U$ and $B\subseteq V$. Since every open set is an $\mathcal{I}_{g^\star}$-open set, every normal space is $\mathcal{I}_{g^\star}$-normal. The following Example 2.1 shows that an $\mathcal{I}_{g^\star}$-normal space is not necessarily a normal space. Theorem 2.2 below gives characterizations of $\mathcal{I}_{g^\star}$-normal spaces. Theorem 2.3 below shows that the two concepts coincide for completely codense ideal topological spaces.

Example 2.1 Let $X=\{a,b,c\}$, $\tau=\{\emptyset,\{b\},\{a,b\},\{b,c\},X\}$ and $\mathcal{I}=\{\emptyset,\{b\}\}$. Then $\emptyset^*=\emptyset$, $(\{a,b\})^*=\{a\}$, $(\{b,c\})^*=\{c\}$, $(\{b\})^*=\emptyset$ and $X^*=\{a,c\}$. Here every g-open set is $\star$-closed and so, by Lemma 1.6, every subset of X is $\mathcal{I}_{g^\star}$-closed and hence every subset of X is $\mathcal{I}_{g^\star}$-open. This implies that $(X,\tau,\mathcal{I})$ is $\mathcal{I}_{g^\star}$-normal. Now $\{a\}$ and $\{c\}$ are disjoint closed subsets of X which are not separated by disjoint open sets and so (X,τ) is not normal.

Theorem 2.2 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following are equivalent.*

1. *X is $\mathcal{I}_{g^\star}$-normal.*
2. *For every closed set A and an open set V containing A, there exists an $\mathcal{I}_{g^\star}$-open set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.*

Proof. (1)⇒(2). Let A be a closed set and V be an open set containing A. Since A and $X - V$ are disjoint closed sets, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and W such that $A \subseteq U$ and $X - V \subseteq W$. Again, $U \cap W = \emptyset$ implies that $U \cap \text{int}^\star(W) = \emptyset$ and so $\text{cl}^\star(U) \subseteq X - \text{int}^\star(W)$. Since $X - V$ is g-closed and W is $\mathcal{I}_{g^\star}$-open, $X - V \subseteq W$ implies that $X - V \subseteq \text{int}^\star(W)$ and so $X - \text{int}^\star(W) \subseteq V$. Thus, we have $A \subseteq U \subseteq \text{cl}^\star(U) \subseteq X - \text{int}^\star(W) \subseteq V$ which proves (2).

(2)⇒(1). Let A and B be two disjoint closed subsets of X. By hypothesis, there exists an $\mathcal{I}_{g^\star}$-open set U such that $A \subseteq U \subseteq \text{cl}^\star(U) \subseteq X - B$. If $W = X - \text{cl}^\star(U)$, then U and W are the required disjoint $\mathcal{I}_{g^\star}$-open sets containing A and B respectively. So, $(X, \tau, \mathcal{I})$ is $\mathcal{I}_{g^\star}$-normal.

Theorem 2.3 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely codense. If $(X, \tau, \mathcal{I})$ is $\mathcal{I}_{g^\star}$-normal, then it is a normal space.*

Proof. It is obvious from Theorem 2.2 and Lemma 1.2.

Theorem 2.4 *Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}_{g^\star}$-normal space. If F is closed and A is a $g^\star$-closed set such that $A \cap F = \emptyset$, then there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.*

Proof. Since $A \cap F = \emptyset$, $A \subseteq X - F$ where $X - F$ is g-open. Therefore, by hypothesis, $\text{cl}(A) \subseteq X - F$. Since $\text{cl}(A) \cap F = \emptyset$ and X is $\mathcal{I}_{g^\star}$-normal, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that $\text{cl}(A) \subseteq U$ and $F \subseteq V$. Thus $A \subseteq U$ and $F \subseteq V$.

The following Corollaries 2.5 and 2.6 give properties of normal spaces. If $\mathcal{I}=\{\emptyset\}$ in Theorem 2.4, then we have the following Corollary 2.5, the proof of which follows from Theorem 2.3 and Lemma 1.3, since $\{\emptyset\}$ is a completely codense ideal. If $\mathcal{I}$=N in Theorem 2.4, then we have the Corollary 2.6 below, since $\tau^*(N) = \tau^\alpha$ and $\mathcal{I}_{g^\star}$-open sets coincide with $g^\#\alpha$-open sets.

Corollary 2.5 *Let (X, τ) be a normal space with $\mathcal{I} = \{\emptyset\}$. If F is a closed set and A is a $g^\star$-closed set disjoint from F, then there exist disjoint $g^\star$-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.*

Corollary 2.6 *Let $(X, \tau, \mathcal{I})$ be a normal ideal topological space where $\mathcal{I}=N$. If F is a closed set and A is a $g^\star$-closed set disjoint from F, then there exist disjoint $g^\#\alpha$-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.*

Theorem 2.7 *Let* $(X,\tau,\mathcal{I})$ *be an ideal topological space which is* $\mathcal{I}_{g^\star}$*-normal. Then the following hold.*

1. *For every closed set* A *and every* $g^\star$*-open set* B *containing* A*, there exists an* $\mathcal{I}_{g^\star}$*-open set* U *such that* $A \subseteq int^\star(U) \subseteq U \subseteq B$.
2. *For every* $g^\star$*-closed set* A *and every open set* B *containing* A*, there exists an* $\mathcal{I}_{g^\star}$*-closed set* U *such that* $A \subseteq U \subseteq cl^\star(U) \subseteq B$.

Proof. (1) Let A be a closed set and B be a $g^\star$-open set containing A. Then $A \cap (X - B) = \emptyset$, where A is closed and $X - B$ is $g^\star$-closed. By Theorem 2.4, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that $A \subseteq U$ and $X - B \subseteq V$. Since $U \cap V = \emptyset$, we have $U \subseteq X - V$. By Lemma 1.5, $A \subseteq \text{int}^\star(U)$. Therefore, $A \subseteq \text{int}^\star(U) \subseteq U \subseteq X - V \subseteq B$. This proves (1).

(2) Let A be a $g^\star$-closed set and B be an open set containing A. Then $X-B$ is a closed set contained in the $g^\star$-open set $X-A$. By (1), there exists an $\mathcal{I}_{g^\star}$-open set V such that $X-B \subseteq \text{int}^\star(V) \subseteq V \subseteq X-A$. Therefore, $A \subseteq X-V \subseteq \text{cl}^\star(X-V) \subseteq B$. If $U = X - V$, then $A \subseteq U \subseteq \text{cl}^\star(U) \subseteq B$ and so U is the required $\mathcal{I}_{g^\star}$-closed set.

The following Corollaries 2.8 and 2.9 give some properties of normal spaces. If $\mathcal{I} = \{\emptyset\}$ in Theorem 2.7, then we have the following Corollary 2.8. If $\mathcal{I} = N$ in Theorem 2.7, then we have the Corollary 2.9 below.

Corollary 2.8 *Let* (X,τ) *be a normal space with* $\mathcal{I} = \{\emptyset\}$*. Then the following hold.*

1. *For every closed set* A *and every* $g^\star$*-open set* B *containing* A*, there exists a* $g^\star$*-open set* U *such that* $A \subseteq \text{int}(U) \subseteq U \subseteq B$.
2. *For every* $g^\star$*-closed set* A *and every open set* B *containing* A*, there exists a* $g^\star$*-closed set* U *such that* $A \subseteq U \subseteq \text{cl}(U) \subseteq B$.

Corollary 2.9 *Let* (X,τ) *be a normal space with* $\mathcal{I} = N$*. Then the following hold.*

1. *For every closed set* A *and every* $g^\star$*-open set* B *containing* A*, there exists an* $g^\#\alpha$*-open set* U *such that* $A \subseteq \text{int}_\alpha(U) \subseteq U \subseteq B$.
2. *For every* $g^\star$*-closed set* A *and every open set* B *containing* A*, there exists an* $g^\#\alpha$*-closed set* U *such that* $A \subseteq U \subseteq \text{cl}_\alpha(U) \subseteq B$.

An ideal topological space $(X,\tau,\mathcal{I})$ is said to be ${}_{g^\star}\mathcal{I}$-normal if for each pair of disjoint $\mathcal{I}_{g^\star}$-closed sets A and B, there exist disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$. Since every closed set is $\mathcal{I}_{g^\star}$-closed, every ${}_{g^\star}\mathcal{I}$-normal space is normal. But a normal space need not be ${}_{g^\star}\mathcal{I}$-normal as the following Example 2.10 shows. Theorems 2.11 and 2.13 below give characterizations of ${}_{g^\star}\mathcal{I}$-normal spaces.

Example 2.10 Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b,c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$. Every g-open set is $\star$-closed and so every subset of X is $\mathcal{I}_{g^\star}$-closed. Now $A = \{a,b\}$ and $B = \{c\}$ are disjoint $\mathcal{I}_{g^\star}$-closed sets, but they are not separated by disjoint open sets. So $(X,\tau,\mathcal{I})$ is not ${}_{g^\star}\mathcal{I}$-normal. But $(X,\tau,\mathcal{I})$ is normal.

Theorem 2.11 *In an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent.*

1. *X is ${}_{g^\star}\mathcal{I}$-normal.*
2. *For every $\mathcal{I}_{g^\star}$-closed set A and every $\mathcal{I}_{g^\star}$-open set B containing A, there exists an open set U of X such that $A \subseteq U \subseteq \mathrm{cl}(U) \subseteq B$.*

Proof. It is similar to the proof of Theorem 2.2.

If $\mathcal{I} = \{\emptyset\}$, then ${}_{g^\star}\mathcal{I}$-normal spaces coincide with $g^\star$-normal spaces and so if we take $\mathcal{I} = \{\emptyset\}$, in Theorem 2.11, then we have the following characterization for $g^\star$-normal spaces.

Corollary 2.12 *In a space (X, τ), the following are equivalent.*

1. *X is $g^\star$-normal.*
2. *For every $g^\star$-closed set A and every $g^\star$-open set B containing A, there exists an open set U of X such that $A \subseteq U \subseteq \mathrm{cl}(U) \subseteq B$.*

Theorem 2.13 *In an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent.*

1. *X is ${}_{g^\star}\mathcal{I}$-normal.*
2. *For each pair of disjoint $\mathcal{I}_{g^\star}$-closed subsets A and B of X, there exists an open set U of X containing A such that $\mathrm{cl}(U) \cap B = \emptyset$.*
3. *For each pair of disjoint $\mathcal{I}_{g^\star}$-closed subsets A and B of X, there exist an open set U containing A and an open set V containing B such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$.*

Proof. (1)⇒(2). Suppose that A and B are disjoint $\mathcal{I}_{g^\star}$-closed subsets of X. Then the $\mathcal{I}_{g^\star}$-closed set A is contained in the $\mathcal{I}_{g^\star}$-open set $X - B$. By Theorem 2.11, there exists an open set U such that $A \subseteq U \subseteq \mathrm{cl}(U) \subseteq X - B$. Therefore, U is the required open set containing A such that $\mathrm{cl}(U) \cap B = \emptyset$.

(2)⇒(3). Let A and B be two disjoint $\mathcal{I}_{g^\star}$-closed subsets of X. By hypothesis, there exists an open set U of X containing A such that $\mathrm{cl}(U) \cap B = \emptyset$. Also, $\mathrm{cl}(U)$ and B are disjoint $\mathcal{I}_{g^\star}$-closed sets of X. By hypothesis, there exists an open set V of X containing B such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$.

(3)⇒(1). The proof is clear.

If $\mathcal{I} = \{\emptyset\}$, in Theorem 2.13, then we have the following characterizations for $g^\star$-normal spaces.

Corollary 2.14 *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is $g^\star$-normal.*
2. *For each pair of disjoint $g^\star$-closed subsets A and B of X, there exists an open set U of X containing A such that $\mathrm{cl}(U) \cap B = \emptyset$.*

3. *For each pair of disjoint $g^\star$-closed subsets A and B of X, there exist an open set U containing A and an open set V containing B such that* $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$.

Theorem 2.15 *Let $(X, \tau, \mathcal{I})$ be an ${}_{g^\star}\mathcal{I}$-normal space. If A and B are disjoint $\mathcal{I}_{g^\star}$-closed subsets of X, then there exist disjoint open sets U and V such that* $\mathrm{cl}^\star(A) \subseteq U$ *and* $\mathrm{cl}^\star(B) \subseteq V$.

Proof. Suppose that A and B are disjoint $\mathcal{I}_{g^\star}$-closed sets. By Theorem 2.13(3), there exist an open set U containing A and an open set V containing B such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$. Since A is $\mathcal{I}_{g^\star}$-closed, $A \subseteq U$ implies that $\mathrm{cl}^\star(A) \subseteq U$. Similarly, $\mathrm{cl}^\star(B) \subseteq V$.

If $\mathcal{I} = \{\emptyset\}$, in Theorem 2.15, then we have the following property of disjoint $g^\star$-closed sets in $g^\star$-normal spaces.

Corollary 2.16 *Let (X, τ) be a $g^\star$-normal space. If A and B are disjoint $g^\star$-closed subsets of X, then there exist disjoint open sets U and V such that* $\mathrm{cl}(A) \subseteq U$ *and* $\mathrm{cl}(B) \subseteq V$.

Theorem 2.17 *Let $(X, \tau, \mathcal{I})$ be an ${}_{g^\star}\mathcal{I}$-normal space. If A is an $\mathcal{I}_{g^\star}$-closed set and B is an $\mathcal{I}_{g^\star}$-open set containing A, then there exists an open set U such that* $A \subseteq \mathrm{cl}^\star(A) \subseteq U \subseteq \mathrm{int}^\star(B) \subseteq B$.

Proof. Suppose A is an $\mathcal{I}_{g^\star}$-closed set and B is an $\mathcal{I}_{g^\star}$-open set containing A. Since A and $X - B$ are disjoint $\mathcal{I}_{g^\star}$-closed sets, by Theorem 2.15, there exist disjoint open sets U and V such that $\mathrm{cl}^\star(A) \subseteq U$ and $\mathrm{cl}^\star(X - B) \subseteq V$. Now, $X - \mathrm{int}^\star(B) = \mathrm{cl}^\star(X - B) \subseteq V$ implies that $X - V \subseteq \mathrm{int}^\star(B)$. Again, $U \cap V = \emptyset$ implies $U \subseteq X - V$ and so $A \subseteq \mathrm{cl}^\star(A) \subseteq U \subseteq X - V \subseteq \mathrm{int}^\star(B) \subseteq B$.

If $\mathcal{I} = \{\emptyset\}$, in Theorem 2.17, then we have the following Corollary 2.18.

Corollary 2.18 *Let (X, τ) be a $g^\star$-normal space. If A is a $g^\star$-closed set and B is a $g^\star$-open set containing A, then there exists an open set U such that* $A \subseteq \mathrm{cl}(A) \subseteq U \subseteq \mathrm{int}(B) \subseteq B$.

The following Theorem 2.19 gives a characterization of normal spaces in terms of $g^\star$-open sets which follows from Lemma 1.2 if $\mathcal{I} = \{\emptyset\}$.

Theorem 2.19 *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is normal.*
2. *For any disjoint closed sets A and B, there exist disjoint $g^\star$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
3. *For any closed set A and open set V containing A, there exists a $g^\star$-open set U such that* $A \subseteq U \subseteq \mathrm{cl}(U) \subseteq V$.

The rest of the section is devoted to the study of mildly normal spaces in terms of $\mathcal{I}_{g^\star}$-open sets, $\mathcal{I}_g$-open sets and $\mathcal{I}_{rg}$-open sets. A space (X,τ) is said to be a mildly normal space [17] if disjoint regular closed sets are separated by disjoint open sets. A subset A of a space (X,τ) is said to be αg-closed [7] if $\mathrm{cl}_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is open. A subset A of a space (X,τ) is said to be rg-closed [14] if cl(A)⊆U whenever A⊆U and U is regular open in X. The complements of the above said closed sets are called their respective open sets.

A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}_g$-closed [1] if A*⊆U whenever A⊆U and U is open. A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is said to be a regular generalized closed set with respect to an ideal $\mathcal{I}$ ($\mathcal{I}_{rg}$-closed) [10] if A*⊆U whenever A⊆U and U is regular open. A is called $\mathcal{I}_g$-open (resp. $\mathcal{I}_{rg}$-open) if X−A is $\mathcal{I}_g$-closed (resp. $\mathcal{I}_{rg}$-closed). Clearly, every $\mathcal{I}_{g^\star}$-closed set is $\mathcal{I}_g$-closed and every $\mathcal{I}_g$-closed set is $\mathcal{I}_{rg}$-closed but the separate converses are not true. Theorem 2.21 below gives characterizations of mildly normal spaces. Corollary 2.22 below gives characterizations of mildly normal spaces in terms of $g^\#\alpha$-open, αg-open and rαg-open sets. Corollary 2.23 below gives characterizations of mildly normal spaces in terms of $g^\star$-open, g-open and rg-open sets. The following Lemma 2.20 is essential to prove Theorem 2.21.

Lemma 2.20 [10] *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. A subset $A \subseteq X$ is $\mathcal{I}_{rg}$-open if and only if $F \subseteq \mathrm{int}^\star(A)$ whenever F is regular closed and $F \subseteq A$.*

Theorem 2.21 *Let $(X,\tau,\mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

1. *X is mildly normal.*
2. *For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
3. *For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_g$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
4. *For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_{rg}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
5. *For a regular closed set A and a regular open set V containing A, there exists an $\mathcal{I}_{rg}$-open set U of X such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.*
6. *For a regular closed set A and a regular open set V containing A, there exists an $\star$-open set U of X such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.*
7. *For disjoint regular closed sets A and B, there exist disjoint $\star$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

Proof. (1)⇒(2). Suppose that A and B are disjoint regular closed sets. Since X is mildly normal, there exist disjoint open sets U and V such that A⊆U and B⊆V. But every open set is an $\mathcal{I}_{g^\star}$-open set. This proves (2).

(2)⇒(3). The proof follows from the fact that every $\mathcal{I}_{g^\star}$-open set is an $\mathcal{I}_g$-open set.

(3)⇒(4). The proof follows from the fact that every $\mathcal{I}_g$-open set is an $\mathcal{I}_{rg}$-open set.

(4)⇒(5). Suppose A is a regular closed and B is a regular open set containing A. Then A and X−B are disjoint regular closed sets. By hypothesis, there exist disjoint $\mathcal{I}_{rg}$-open sets U and V such that A⊆U and X−B⊆V. Since X−B is regular closed and V is $\mathcal{I}_{rg}$-open, by Lemma 2.20, X−B⊆int*(V) and so X−int*(V)⊆B. Again, U∩V=∅ implies that U∩int*(V)=∅ and so cl*(U)⊆X−int*(V)⊆B. Hence U is the required $\mathcal{I}_{rg}$-open set such that A⊆U⊆cl*(U)⊆B.

(5)⇒(6). Let A be a regular closed set and V be a regular open set containing A. Then there exists an $\mathcal{I}_{rg}$-open set G of X such that $A \subseteq G \subseteq \mathrm{cl}^\star(G) \subseteq V$. By Lemma 2.20, $A \subseteq \mathrm{int}^\star(G)$. If $U = \mathrm{int}^\star(G)$, then U is an $\star$-open set and $A \subseteq U \subseteq \mathrm{cl}^\star(U) \subseteq \mathrm{cl}^\star(G) \subseteq V$. Therefore, $A \subseteq U \subseteq \mathrm{cl}^\star(U) \subseteq V$.

(6)⇒(7). Let A and B be disjoint regular closed subsets of X. Then X−B is a regular open set containing A. By hypothesis, there exists an $\star$-open set U of X such that A⊆U⊆cl*(U)⊆X−B. If V=X−cl*(U), then U and V are disjoint $\star$-open sets of X such that A⊆U and B⊆V.

(7)⇒(1). Let A and B be disjoint regular closed sets of X. Then there exist disjoint $\star$-open sets U and V such that A⊆U and B⊆V. Since $\mathcal{I}$ is completely codense, by Lemma 1.1, $\tau^* \subseteq \tau^\alpha$ and so U, V $\in \tau^\alpha$. Hence A⊆U⊆int(cl(int(U)))=G and B⊆V⊆int(cl(int(V)))=H. G and H are the required disjoint open sets containing A and B respectively. This proves (1).

If $\mathcal{I}$=N, in the above Theorem 2.21, then $\mathcal{I}_{rg}$-closed sets coincide with rαg-closed sets and so we have the following Corollary 2.22.

Corollary 2.22 *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is mildly normal.*
2. *For disjoint regular closed sets A and B, there exist disjoint $g^\#\alpha$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
3. *For disjoint regular closed sets A and B, there exist disjoint αg-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
4. *For disjoint regular closed sets A and B, there exist disjoint $r\alpha g$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
5. *For a regular closed set A and a regular open set V containing A, there exists an $r\alpha g$-open set U of X such that $A \subseteq U \subseteq cl_\alpha(U) \subseteq V$.*
6. *For a regular closed set A and a regular open set V containing A, there exists an α-open set U of X such that $A \subseteq U \subseteq cl_\alpha(U) \subseteq V$.*
7. *For disjoint regular closed sets A and B, there exist disjoint α-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

If $\mathcal{I} = \{\emptyset\}$ in the above Theorem 2.21, we get the following Corollary 2.23.

Corollary 2.23 *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is mildly normal.*
2. *For disjoint regular closed sets A and B, there exist disjoint $g^\star$-open sets U and V such that A⊆U and B⊆V.*
3. *For disjoint regular closed sets A and B, there exist disjoint g-open sets U and V such that A⊆U and B⊆V.*
4. *For disjoint regular closed sets A and B, there exist disjoint rg-open sets U and V such that A⊆U and B⊆V.*
5. *For a regular closed set A and a regular open set V containing A, there exists an rg-open set U of X such that A⊆U⊆cl(U)⊆V.*
6. *For a regular closed set A and a regular open set V containing A, there exists an open set U of X such that A⊆U⊆cl(U)⊆V.*
7. *For disjoint regular closed sets A and B, there exist disjoint open sets U and V such that A⊆U and B⊆V.*

3. $\mathcal{I}_{g^\star}$-regular spaces

An ideal topological space $(X, \tau, \mathcal{I})$ is said to be an $\mathcal{I}_{g^\star}$-regular space if for each pair consisting of a point x and a closed set B not containing x, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that x∈U and B⊆V. Every regular space is $\mathcal{I}_{g^\star}$-regular, since every open set is $\mathcal{I}_{g^\star}$-open. The following Example 3.1 shows that an $\mathcal{I}_{g^\star}$-regular space need not be regular. Theorem 3.2 gives a characterization of $\mathcal{I}_{g^\star}$-regular spaces.

Example 3.1 Consider the ideal topological space $(X, \tau, \mathcal{I})$ of Example 2.1. Then ∅*=∅, ({b})*=∅, ({a, b})*={a}, ({b, c})*={c} and X*={a, c}. Since every *g*-open set is ⋆-closed, every subset of X is $\mathcal{I}_{g^\star}$-closed and so every subset of X is $\mathcal{I}_{g^\star}$-open. This implies that $(X, \tau, \mathcal{I})$ is $\mathcal{I}_{g^\star}$-regular. Now, {c} is a closed set not containing a∈X, {c} and a are not separated by disjoint open sets. So $(X, \tau, \mathcal{I})$ is not regular.

Theorem 3.2 *In an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent.*

1. *X is $\mathcal{I}_{g^\star}$-regular.*
2. *For every open set V containing x∈X, there exists an $\mathcal{I}_{g^\star}$-open set U of X such that x∈U⊆cl*(U)⊆V.*

Proof. (1)⇒(2). Let V be an open subset such that $x \in V$. Then $X - V$ is a closed set not containing x. Therefore, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and W such that $x \in U$ and $X - V \subseteq W$. Now, $X - V \subseteq W$ implies that $X - V \subseteq \text{int}^\star(W)$ and so $X - \text{int}^\star(W) \subseteq V$. Again, $U \cap W = \emptyset$ implies that $U \cap \text{int}^\star(W) = \emptyset$ and so $\text{cl}^\star(U) \subseteq X - \text{int}^\star(W)$. Therefore, $x \in U \subseteq \text{cl}^\star(U) \subseteq V$. This proves (2).

(2)⇒(1). Let B be a closed set not containing x. By hypothesis, there exists an $\mathcal{I}_{g^\star}$-open set U such that x∈U⊆cl*(U)⊆X−B. If W=X−cl*(U), then U and W are disjoint $\mathcal{I}_{g^\star}$-open sets such that x∈U and B⊆W. This proves (1).

Theorem 3.3 *If $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_{g^\star}$-regular, T_1-space where $\mathcal{I}$ is completely codense, then X is regular.*

Proof. Let B be a closed set not containing x∈X. By Theorem 3.2, there exists an $\mathcal{I}_{g^\star}$-open set U of X such that $x \in U \subseteq cl^\star(U) \subseteq X - B$. Since X is a T_1-space, {x} is g-closed and so $\{x\} \subseteq int^\star(U)$, by Lemma 1.5. Since $\mathcal{I}$ is completely codense, $\tau^* \subseteq \tau^\alpha$ and so int*(U) and X−cl*(U) are α-open sets. Now, $x \in int^\star(U) \subseteq int(cl(int(int^\star(U)))) = G$ and $B \subseteq X - cl^\star(U) \subseteq int(cl(int(X - cl^\star(U)))) = H$. Then G and H are disjoint open sets containing x and B respectively. Therefore, X is regular.

If $\mathcal{I}$=N in Theorem 3.2, then we have the following Corollary 3.4 which gives characterizations of regular spaces, the proof of which follows from Theorem 3.3.

Corollary 3.4 *If (X, τ) is a T_1-space, then the following are equivalent.*

1. *X is regular.*
2. *For every open set V containing x∈X, there exists an $g^\#\alpha$-open set U of X such that $x \in U \subseteq \mathrm{cl}_\alpha(U) \subseteq V$.*

If $\mathcal{I} = \{\emptyset\}$ in Theorem 3.2, then we have the following Corollary 3.5 which gives characterizations of regular spaces, the proof of which follows from Theorem 3.3.

Corollary 3.5 *If (X, τ) is a T_1-space, then the following are equivalent.*

1. *X is regular.*
2. *For every open set V containing x∈X, there exists a $g^\star$-open set U of X such that $x \in U \subseteq cl(U) \subseteq V$.*

Theorem 3.6 *If every g-open subset of an ideal topological space $(X, \tau, \mathcal{I})$ is $\star$-closed, then $(X, \tau, \mathcal{I})$ is $\mathcal{I}_{g^\star}$-regular.*

Proof. Suppose every g-open subset of X is $\star$-closed. Then by Lemma 1.6, every subset of X is $\mathcal{I}_{g^\star}$-closed and hence every subset of X is $\mathcal{I}_{g^\star}$-open. If B is a closed set not containing x, then {x} and B are the required disjoint $\mathcal{I}_{g^\star}$-open sets containing x and B respectively. Therefore, $(X, \tau, \mathcal{I})$ is $\mathcal{I}_{g^\star}$-regular.

The following Example 3.7 shows that the reverse direction of the above Theorem 3.6 is not true.

Example 3.7 Consider the real line $\mathcal{R}$ with the usual topology with $\mathcal{I} = \{\emptyset\}$. Since $\mathcal{R}$ is regular, $\mathcal{R}$ is $\mathcal{I}_{g^\star}$-regular. Obviously U = (0,1) is g-open being open in $\mathcal{R}$. But U is not $\star$-closed because, when $\mathcal{I} = \{\emptyset\}$, cl*(U) = cl(U) = [0,1] $\neq$ U.

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