THE NONLOCAL BOUNDARY VALUE PROBLEMS FOR STRONGLY SINGULAR HIGHER-ORDER NONLINEAR FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. A priori boundedness principle is proven for the nonlocal boundary value problems for strongly singular higher-order nonlinear functional-differential equations. Several sufficient conditions of solvability of the Dirichlet problem under consideration are derived from the a priori boundedness principle. The proof of the a priori boundedness principle is based on the Agarwal–Kiguradze type theorems, which guarantee the existence of the Fredholm property for strongly singular higher-order linear differential equations with argument deviations under the nonlocal boundary conditions.

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1. Statement of the main results

1.1. Statement of the problem and a survey of the literature. Consider the functional differential equation

$$u^{(2m+1)}(t) = F(u)(t) \tag{1.1}$$

with the boundary conditions

$$\int_a^b u(s)d\varphi(s) = 0 \tag{1.2}$$

where $\varphi(b) - \varphi(a) \neq 0,\ u^{(i)}(a) = 0,\ u^{(i)}(b) = 0\ (i = 1, ..., m)$.

Here, $-\infty < a < b < +\infty$, $\varphi : [a,b] \to R$ is a function of bounded variation, and the operator F acting from the set of $(m-1)$-th time continuously differentiable on $]a,b[$ functions, to the set $L_{loc}(]a,b[)$. By $u^{(i)}(a)$ $(u^{(i)}(b))$, we denote the right (the left) limit of the function $u^{(i)}$ at the point $a\,(b)$.

The problem is singular in the sense that for an arbitrary x the right-hand side of equation (1.1) may have nonintegrable singularities at the points a and b.

Throughout the paper we use the following notations:

$R^+ = [0, +\infty[$;

$[x]_+$ the positive part of number x, that is $[x]_+ = \frac{x+|x|}{2}$;

$L_{loc}(]a,b[)$ $(L_{loc}(]a,b]))$ is the space of functions $y :]a,\, b[\to R$, which are integrable on $[a+\varepsilon,\, b-\varepsilon]$ for arbitrary small $\varepsilon > 0$;

$L_{\alpha,\beta}(]a,b[)$ $(L^2_{\alpha,\beta}(]a,b[))$ is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y :]a,b[\to R$, with the norm

$$||y||_{L_{\alpha,\beta}} = \int_a^b (s-a)^\alpha (b-s)^\beta |y(s)|ds \ \Big(||y||_{L^2_{\alpha,\beta}} = \Big(\int_a^b (s-a)^\alpha (b-s)^\beta y^2(s)ds\Big)^{1/2}\Big);$$

$L([a,b]) = L_{0,0}(]a,b[)$, $L^2([a,b]) = L^2_{0,0}(]a,b[)$;

$M(]a,b[)$ is the set of the measurable functions $\tau :]a,b[\to]a,b[$;

$\widetilde{L}^2_{\alpha,\beta}(]a,b[)$ $(\widetilde{L}^2_{\alpha}(]a,b]))$ is the Banach space of $y \in L_{loc}(]a,b[)$ $(L_{loc}(]a,b]))$ functions, with the norm

$$||y||_{\widetilde{L}^2_{\alpha,\beta}} \equiv \max\Big\{\Big[\int_a^t (s-a)^\alpha \Big(\int_s^t y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : a \le t \le \frac{a+b}{2}\Big\}$$
$$+ \max\Big\{\Big[\int_t^b (b-s)^\beta \Big(\int_t^s y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : \frac{a+b}{2} \le t \le b\Big\} < +\infty.$$

$L_m(]a,b[)$ is the Banach space of $y \in L_{loc}(]a,b[)$ functions, with the norm

$$||y||_{L_m} = \sup\Big\{[(s-a)(b-t)]^{m-1/2} \int_s^t |y(\xi)|d\xi : \ \ a < s \le t < b\Big\} < +\infty.$$

$C^n_{loc}(]a,b[)$, $(\widetilde{C}^{n-1}_{loc}(]a,b[))$ is the space of the functions $y :]a,b[\to R$, which are continuous (absolutely continuous) together with $y', y'', ..., y^{(n-1)}$ on $[a+\varepsilon,$ $b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$.

$\widetilde{C}^{n,m}(]a,b[)$ $(m \le n)$ is the space of the functions $y \in \widetilde{C}^n_{loc}(]a,b[)$, such that

$$\int_a^b |x^{(m)}(s)|^2 ds < +\infty. \tag{1.3}$$

$C_2^m(]a,b[)$ is the Banach space of the functions $y \in C_{loc}^m(]a,b[)$, such that

$$\limsup_{t\to a} \frac{|x^{(i)}(t)|}{(t-a)^{m-i+1/2}} < +\infty \ (i=1,...,m), \qquad (1.4)$$
$$\limsup_{t\to b} \frac{|x^{(i)}(t)|}{(b-t)^{m-i+1/2}} < +\infty \ (i=1,...,m),$$

with the norm:

$$||x||_{C_2^m} = ||x||_C + \sum_{i=1}^m \sup\Big\{\frac{|x^{(i)}(t)|}{\alpha_i(t)} : a<t<b\Big\},$$

where $\alpha_i(t) = (t-a)^{m-i+1/2}(b-t)^{m-i+1/2}$.

$\widetilde{C}_2^m(]a,b[)$ is the Banach space of the functions $y \in \widetilde{C}_{loc}^m(]a,b[)$, such that conditions $\Big(\int\limits_a^b |x^{(m+1)}(s)|^2 ds\Big)^{1/2} < +\infty$ and (1.4) hold, with the norm:

$$||x||_{\widetilde{C}_2^m} = ||x||_{C_1^m} + \Big(\int_a^b |x^{(m+1)}(s)|^2 ds\Big)^{1/2}.$$

$D_n(]a,b[\times R^+)$ is the set of such functions $\delta :]a,b[\times R^+ \to L_n(]a,b[)$ that $\delta(t,\cdot) : R^+ \to R^+$ is nondecreasing for every $t \in]a,b[$, and $\delta(\cdot,\rho) \in L_n(]a,b[)$ for any $\rho \in R^+$.

$D_{2m-2,2m-2}(]a,b[\times R^+)$ is the set of such functions $\delta :]a,b[\times R^+ \to \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$ that $\delta(t,\cdot) : R^+ \to R^+$ is nondecreasing for every $t \in]a,b[$, and $\delta(\cdot,\rho) \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$ for any $\rho \in R^+$.

A solution of problem (1.1), (1.2) is sought in the space $\widetilde{C}^{2m,m+1}(]a,b[)$.

The singular ordinary differential and functional-differential equations, have been studied with sufficient completeness under different boundary conditions, see for example [1], [2], [4] – [14], [17], [25]-[29] and the references cited therein. But the equation (1.1), under the boundary condition (1.2), is not studied even in the case when equation (1.1) has the form

$$x^{(2m+1)}(t) = \sum_{j=0}^m p_j(t) x^{(j)}(\tau_j(t)) + f(x)(t), \qquad (1.5)$$

where the singularity of the functions $p_j : L_{loc}([a,\, b])$ be such that the inequalities

$$\int_a^b (s-a)^{2m-1}(b-s)^{2m-1}[(-1)^{n-m}p_1(s)]_+ ds < +\infty, \qquad (1.6)$$
$$\int_a^b (s-a)^{2m-j}(b-s)^{2m-j}|p_j(s)| ds < +\infty \quad (j=2,...,m),$$

are not fulfilled (in this case we sad that the linear part of the operator F is a strongly singular), the operator f continuously acting from $C_2^m(]a,b[)$ to $L_{\widetilde{L}^2_{2m-2,2m-2}}(]a,b[)$, and the inclusion

$$\sup\{f(x)(t) : ||x||_{C_2^m} \le \rho\} \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[). \tag{1.7}$$

holds. The first step in studying of the differential equations with strong singularities was made by R.P. Agarwal and I. Kiguradze in the article [3], where the linear ordinary differential equations under conditions (1.2), in the case when the functions p_j have strong singularities at the points a and b, are studied. Also the ordinary differential equations with strong singularities under two-point boundary conditions are studied in the articles of I. Kiguradze [15], [16], and N. Partsvania [24]. In the papers [20], [21] these results are generalized for linear differential equation with deviating arguments i.e., are proven the Agarwal-Kiguradze type theorems, which guarantee Fredholm's property for linear differential equation with deviating arguments.

In this paper, on the basis of articles [3] and [19], we prove a priori boundedness principle for the problem (1.1), (1.2) in the case where equation (1.1) is in form (1.5).

Now, we introduce some results from articles [20], [21], which we need for this work. Consider the equation

$$u^{(2m+1)}(t) = \sum_{j=1}^{m} p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \qquad \text{for} \qquad a < t < b. \tag{1.8}$$

By $h_j :]a,b[\times]a,b[\to R_+$ and $f_j : [a,b]\times M(]a,b[) \to C_{loc}(]a,b[\times]a,b[)$ $(j = 1,...,m)$ we denote the functions and operator, respectively defined by the equalities

$$\begin{aligned} h_1(t,s) &= \Big|\int_s^t (\xi-a)^{n-2m}[(-1)^{n-m}p_1(\xi)]_+ d\xi\Big|, \\ h_j(t,s) &= \Big|\int_s^t (\xi-a)^{n-2m}p_j(\xi) d\xi\Big| \quad (j=2,...,m), \end{aligned} \tag{1.9}$$

and

$$f_j(c,\tau_j)(t,s) = \Big|\int_s^t (\xi-a)^{n-2m}|p_j(\xi)|\Big|\int_\xi^{\tau_j(\xi)} (\xi_1-c)^{2(m-j)}d\xi_1\Big|^{1/2} d\xi\Big|, \tag{1.10}$$

and also we put that

$$f_0(t,s) = \Big|\int_s^t |p_0(\xi)|d\xi\Big|.$$

Let $k = 2k_1 + 1$ ($k_1 \in N$), then

$$k!! = \begin{cases} 1 & \text{for } k \le 0, \\ 1 \cdot 3 \cdot 5 \cdots k & \text{for } k \ge 1. \end{cases}$$

Now, we can introduce the main theorem of the paper [20].

Theorem 1.1. *Let there exist numbers* $t^* \in]a, b[$, $l_{k0} > 0$, $l_{kj} > 0$, $\bar{l}_{kj} \ge 0$, *and* $\gamma_{kj} > 0$ $(k = 0, 1;\ j = 1, ..., m)$ *such that along with*

$$(1.11)\quad \begin{aligned} B_0 \equiv & \bar{l}_{00}\Big(\frac{2^{m-1}}{(2m-3)!!}\Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{(t^*-a)^{\gamma_{00}}}{\sqrt{2\gamma_{00}}} \int\limits_a^b \frac{|\varphi(\xi)-\varphi(a)|+|\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|} d\xi \\ & + \sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1}\, l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\, \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \Big) < \frac{1}{2}, \end{aligned}$$

$$(1.12)\quad \begin{aligned} B_1 \equiv & \bar{l}_{10}\Big(\frac{2^{m-1}}{(2m-3)!!}\Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{(b-t^*)^{\gamma_{10}}}{\sqrt{2\gamma_{10}}} \int\limits_a^b \frac{|\varphi(\xi)-\varphi(a)|+|\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|} d\xi \\ & + \sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1}\, l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\, \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \Big) < \frac{1}{2}, \end{aligned}$$

the conditions

$$(1.13)\quad \begin{aligned} & (t-a)^{m-\gamma_{00}-1/2} f_0(t,s) \le \bar{l}_{00}, \qquad (t-a)^{2m-j} h_{j_1}(t,s) \le l_{0j_1}, \\ & (t-a)^{m-\gamma_{0j}-1/2} f_j(a,\tau_j)(t,s) \le \bar{l}_{0j} \qquad \text{for } a<t\le s\le t^*, \end{aligned}$$

$$(1.14)\quad \begin{aligned} & (b-t)^{m-\gamma_{10}-1/2} f_0(t,s) \le \bar{l}_{10}, \qquad (b-t)^{2m-j} h_{j_1}(t,s) \le l_{1j_1}, \\ & (b-t)^{m-\gamma_{1j}-1/2} f_j(b,\tau_j)(t,s) \le \bar{l}_{1j} \qquad \text{for } t^*\le s\le t<b \end{aligned}$$

$j = 1, \cdots, m$ *hold. Then for every* $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a, b[)$ *problem* (1.8), (1.2) *is uniquely solvable in the space* $\widetilde{C}^{2m,\,m+1}(]a,\, b[)$.

Also, in [21], the following theorem is proven:

Theorem 1.2. *Let all the conditions of Theorem* 1.5 *are satisfied. Then the unique solution* u *of problem* (1.8), (1.2) *for every* $q \in \widetilde{L}^2_{2m-2,2m-2}(]a, b[)$ *admits the estimate*

$$(1.15)\qquad ||u^{(m+1)}||_{L^2} \le r||q||_{\widetilde{L}^2_{2m-2,\,2m-2}},$$

with

$$r = \frac{2^m}{(1-2\max\{B_0, B_1\})(2m-1)!!},$$

and thus constant $r > 0$ *depends only on the numbers* l_{kj}, $\bar{l}_{kj}, \gamma_{kj}$ $(k = 0, 1;\ j = 0, ..., m)$, *and* a, b, t^*.

Remark 1.1. Under conditions of Theorem 1.2, for every $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,b[)$ the unique solution u of problem (1.8), (1.2) admits the estimate

$$||u^{(m+1)}||_{\widetilde{C}_2^{2m}} \leq r_{m,\varphi}||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}, \tag{1.16}$$

with

$$r_{m,\varphi} = \Big(1+\int\limits_a^b \frac{|\varphi(s)-\varphi(a)|+|\varphi(s)-\varphi(b)|}{|\varphi(b)-\varphi(a)|}ds\frac{(b-a)^{m-1/2}}{(m-1)!(2m-1)^{1/2}2^{m-1/2}}$$
$$+\sum_{j=1}^m \frac{1}{(m-i)!(2m-2i+1)^{1/2}}\Big(\frac{2}{b-a}\Big)^{m-i+1/2}\Big)\frac{2^m}{(1-2\max\{B_0,B_1\})(2m-1)!!}.$$

1.2. Theorems on a solvability of problem (1.1)**,** (1.2)**.** Define the operator $P : C_2^m(]a,b[) \times C_2^m(]a,b[) \to L_{loc}(]a,b[)$, by the equality

$$P(x,y)(t) = \sum_{j=0}^m p_j(x)(t)y^{(j)}(\tau_j(t)) \qquad \text{for} \quad a < t < b \tag{1.17}$$

where $p_j : C_2^m(]a,\, b[) \to L_{loc}(]a,\, b[)$, and $\tau_j \in M(]a,b[)$. Also, for any $\gamma > 0$, define the set A_γ by the relation

$$A_\gamma = \{x \in \widetilde{C}_2^m(]a,\, b[) : ||x||_{\widetilde{C}_2^m} \leq \gamma\}. \tag{1.18}$$

To formulate this a priori boundedness principle, we have to introduce

Definition 1.1. Let γ_0 and γ be the positive numbers. We said that the continuous operator $P : C_2^m(]a,b[) \times C_2^m(]a,b[) \to L_m(]a,b[)$ is γ_0, γ consistent with boundary condition (1.2) if:

(i) for any $x \in A_{\gamma_0}$ and almost all $t \in]a,b[$, the inequality

$$\sum_{j=0}^m |p_j(x)(t)x^{(j)}(\tau_j(t))| \leq \delta(t,\, ||x||_{\widetilde{C}_2^m})||x||_{\widetilde{C}_2^m} \tag{1.19}$$

holds, where $\delta \in D_{2m}(]a,b[\times R^+)$.

(ii) for any $x \in A_{\gamma_0}$ and $q \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$, the equation

$$y^{(2m+1)}(t) = \sum_{j=0}^m p_j(x)(t)y^{(j)}(\tau_j(t)) + q(t) \tag{1.20}$$

under boundary conditions (1.2), has the unique solution y in the space $\widetilde{C}^{2m,\,m+1}(]a,\, b[)$ and

$$||y||_{\widetilde{C}_2^m} \leq \gamma||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}. \tag{1.21}$$

Definition 1.2. We said that the operator P is γ consistent with boundary condition (1.2), if the operator P is γ_0, γ consistent with boundary condition (1.2) for any $\gamma_0 > 0$.

In the sequel, it will always be assumed that the operator F_p, defined by the equality

$$F_p(x)(t) = |F(x)(t) - \sum_{j=0}^{m} p_j(x)(t)x^{(j)}(\tau_j(t))(t)|,$$

continuously acting from $C_2^m(]a,b[)$ to $L_{\widetilde{L}^2_{2m-2,2m-2}}(]a,b[)$, and

$$\widetilde{F}_p(t,\rho) \equiv \sup\{F_p(x)(t) : ||x||_{C_2^m} \leq \rho\} \in \widetilde{L}^2_{2m-2,2m-2}(]a,\ b[) \tag{1.22}$$

for each $\rho \in [0, +\infty[$.

Then, the following theorem is valid.

Theorem 1.3. *Let the operator P be γ_0, γ consistent with boundary condition* (1.2)*, and there exist a positive number $\rho_0 \leq \gamma_0$, such that*

$$||\widetilde{F}_p(\cdot, \min\{2\rho_0, \gamma_0\})||_{\widetilde{L}^2_{2m-2,2m-2}} \leq \frac{\gamma_0}{\gamma}. \tag{1.23}$$

Moreover, for any $\lambda \in]0,1[$, let an arbitrary solution $x \in A_{\gamma_0}$ of the equation

$$x^{(2m+1)}(t) = (1-\lambda)P(x,x)(t) + \lambda F(x)(t) \tag{1.24}$$

under conditions (1.2)*, admits the estimate*

$$||x||_{\widetilde{C}_2^m} \leq \rho_0. \tag{1.25}$$

Then problem (1.1)*,* (1.2) *is solvable in the space $\widetilde{C}^{2m,m+1}(]a,b[)$.*

From Theorem 1.3 with $\rho_0 = \gamma_0$ immediately follows

Corollary 1.1. *Let the operator P be γ_0, γ consistent with the boundary condition* (1.2)*, and*

$$|F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}(\tau_j(t))(t)| \leq \eta(t,\ ||x||_{\widetilde{C}_1^{m-1}}) \tag{1.26}$$

for $x \in A_{\gamma_0}$ and almost all $t \in]a,\ b[$, and

$$||\eta(\cdot,\ \gamma_0)||_{\widetilde{L}^2_{2m-2,\ 2m-2}} \leq \frac{\gamma_0}{\gamma}, \tag{1.27}$$

where $\eta \in D_{2m-2,\ 2m-2}(]a,\ b[\times R^+)$. Then problem (1.1)*,* (1.2) *is solvable in the space $\widetilde{C}^{2m,m+1}(]a,\ b[)$.*

Corollary 1.2. *Let the operator P be γ consistent with the boundary condition* (1.2)*, inequality* (1.26) *holds for $x \in \widetilde{C}_2^m(]a,\, b[)$ and almost all $t \in]a, b[$, where $\eta(\cdot,\, \rho) \in \widetilde{L}^2_{2m-2,\, 2m-2}(]a,\, b[)$ for any $\rho \in R^+$, and*

$$(1.28) \qquad \limsup_{\rho\to+\infty} \frac{1}{\rho}||\eta(\cdot,\ \rho)||_{\widetilde{L}^2_{2m-2,\,2m-2}} < \frac{1}{\gamma}.$$

Then, problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{2m,m+1}(]a, b[)$.*

Now define the operators

$$h_j : C_1^{m-1}(]a, b[)\times]a, b[\times]a, b[\to R_+,$$
$$f_j : C_1^{m-1}(]a, b[) \times [a, b] \times M(]a, b[) \to R,\ (j = 1, ..., m)$$

by the equalities

$$(1.29) \qquad \begin{aligned} h_1(x, t, s) &= \Big|\int_s^t (\xi - a)^{n-2m}[(-1)^{n-m}p_1(x)(\xi)]_+ d\xi\Big|,\\ h_j(x, t, s) &= \Big|\int_s^t (\xi - a)^{n-2m}p_j(x)(\xi)d\xi\Big| \quad (j = 2, ..., m), \end{aligned}$$

and

$$(1.30) \qquad f_j(x, c, \tau_j)(t, s) = \Big|\int_s^t (\xi - a)^{n-2m}|p_j(x)(\xi)|\Big|\int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)}d\xi_1\Big|^{1/2} d\xi\Big|.$$

Theorem 1.4. *Let the continuous operator $P : C_1^{m-1}(]a, b[) \times C_1^{m-1}(]a, b[) \to L_n(]a,\, b[)$ admit to condition* (1.19)*, where $\delta \in D_n(]a, b[\times R^+)$, $\tau_j \in M(]a,\, b[)$ and the numbers $\gamma_0, t^* \in]a, b[$, $l_{kj} > 0$, $\bar{l}_{kj} > 0$, $\gamma_{kj} > 0$ $(k = 1, 2;\ j = 1, ..., m)$, be such that the inequalities*

$$(1.31) \qquad (t - a)^{2m-j}h_j(x, t, s) \le l_{0j}, \quad \limsup_{t\to a}(t - a)^{m-\frac{1}{2}-\gamma_{0j}} f_j(x, a, \tau_j)(t, s) \le \bar{l}_{0j}$$

for $a < t \le s \le t^*$, $||x||_{\widetilde{C}_1^{m-1}} \le \gamma_0$,

$$(1.32) \qquad (b - t)^{2m-j}h_j(x, t, s) \le l_{1j}, \quad \limsup_{t\to b}(b - t)^{m-\frac{1}{2}-\gamma_{1j}} f_j(x, b, \tau_j)(t, s) \le \bar{l}_{1j}$$

for $t^* \le s \le t < b$, $||x||_{\widetilde{C}_1^{m-1}} \le \gamma_0$, and conditions (1.11), (1.12) *hold. Moreover, let the operator F and the function $\eta \in D_{2n-2m-2,2m-2}(]a, b[\times R^+)$ be such that condition* (1.26) *and the inequality*

$$(1.33) \qquad ||\eta(\cdot, \gamma_0)||_{\widetilde{L}^2_{2n-2m-2,2m-2}} < \frac{\gamma_0}{r_n},$$

be fulfilled, where

$$r_n=\Big(1+\sum_{j=1}^{m}\frac{2^{m-j+1/2}}{(m-j)!(2m-2j+1)^{1/2}(b-a)^{m-j+1/2}}\Big)\frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n-2\max\{B_0,B_1\})(2m-1)!!}.$$

Then, problem (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,b[)$.

Theorem 1.5. *Let the operator* F *and function* η *are such that condition* (1.26), (1.28) *hold and the continuous operator* $P : C_1^{m-1}(]a,b[)\times C_1^{m-1}(]a,b[) \to L_n(]a,b[)$ *admits condition* (1.19) *where* $\delta \in D_n(]a,b[\times R^+)$. *Let moreover the measurable functions* $\tau_j \in M(]a,\ b[)$ *and the numbers* $t^* \in]a,b[,\ l_{kj} > 0,\ \bar{l}_{kj} > 0,\ \gamma_{kj} > 0,$ $(k = 1,2;\ j = 1,...,m)$ *be such that the inequalities*

$$(1.34)\qquad (t-a)^{2m-j}h_j(x,t,s) \le l_{0j},\qquad \limsup_{t\to a}(t-a)^{m-\frac{1}{2}-\gamma_{0j}}f_j(x,a,\tau_j)(t,s) \le \bar{l}_{0j}$$

for $a < t \le s \le t^*$, $x \in \widetilde{C}_1^{m-1}(]a,b[)$,

$$(1.35)\qquad (b-t)^{2m-j}h_j(x,t,s) \le l_{1j},\qquad \limsup_{t\to b}(b-t)^{m-\frac{1}{2}-\gamma_{1j}}f_j(x,b,\tau_j)(t,s) \le \bar{l}_{1j}$$

for $t^* \le s \le t < b,\ x \in \widetilde{C}_1^{m-1}(]a,\ b[)$, *and conditions* (1.11), (1.12) *hold. Then, problem* (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,b[)$.

Remark 1.2. Let $\gamma_0 > 0$, operators $\alpha_j(t)p_j(x)(t)$ $(j{=}1,...,m)$ continuously acting from the space $C_1^{m-1}(]a,b[)$ to the space $L_n(]a,b[)$, exist the function $\delta_j \in D_n(]a,b[)$ such that for any $x \in A_{\gamma_0}$

$$(1.36)\qquad |p_j(x)(t)|\alpha_j(t) \le \delta_j(t,\ ||x||_{\widetilde{C}_1^{m-1}})\quad \text{for } a<t<b,$$

and exists constants $\kappa > 0,\ \varepsilon > 0$ such that

$$(1.37)\qquad \begin{array}{ll} |\tau_j(t)-t| \le \kappa(t-a)\ \ (j=1,\cdots,m) & \text{for } a<t<a+\varepsilon,\\ |\tau_j(t)-t| \le \kappa(b-t)\ \ (j=1,\cdots,m) & \text{for } b-\varepsilon<t<b,\end{array}$$

Then, the operator P defined by equality (1.17), continuously acting from A_{γ_0} to the space $L_n(]a,b[)$, and there exists the function $\delta \in D_n(]a,b[)$ such that item (ii) of Definition 1.1 holds.

Now, consider the equation with deviating arguments

$$(1.38)\qquad u^{(n)}(t) = f(t,u(\tau_1(t)),u'(\tau_2(t)),\cdots,u^{(m-1)}(\tau_m(t)))\quad\text{for}\quad a<t<b,$$

where $-\infty < a < b < +\infty,\ f :]a,\ b[\times R^m \to R$ is a function, satisfying the local Carathéodory conditions and $\tau_j \in M(]a,b[)$ $(j = 0,...,n-1)$ are measurable functions.

Corollary 1.3. *Let the functions* $\tau_j \in M(]a,\, b[)$ *and the numbers* $t^* \in]a, b[$, $\kappa \geq 0$, $\varepsilon > 0$, $l_{kj} > 0$, $\bar{l}_{kj} > 0$, $\gamma_{kj} > 0$, $(k = 1, 2;\ j = 1, ..., m)$ *be such that the conditions* (1.11)-(1.14), (1.37) *and the inclusions*

$$\alpha_j p_j \in L_n(]a, b[)\ (j = 1, ..., m) \tag{1.39}$$

are fulfilled. Moreover, let

$$\Big|f(t, x(\tau_1(t)), x'(\tau_2(t)), ..., x^{(m-1)}(\tau_m(t))) - \sum_{j=1}^{m} p_j(t) x^{(j-1)}(\tau_j(t))(t)\Big| \leq \eta(t,\, ||x||_{\widetilde{C}_1^{m-1}}) \tag{1.40}$$

for $x \in \widetilde{C}_1^{m-1}(]a,\, b[)$ *and almost all* $t \in]a, b[$, *where* $\eta(\cdot, \rho) \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a, b[)$ *for any* $\rho \in R^+$, *and condition* (1.28) *holds. Then problem* (1.38), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a, b[)$.

Remark 1.3. From conditions (1.39), conditions (1.6) do not follow.

Now, to illustrate our results, consider on $]a,\, b[$ the second order functional-differential equations

$$u''(t) = -\frac{\lambda |u(t)|^k}{[(t-a)(b-t)]^{2+k/2}} u(\tau(t)) + q(x)(t), \tag{1.41}$$

$$u''(t) = -\frac{\lambda |\sin u^k(t)|}{[(t-a)(b-t)]^2} u(\tau(t)) + q(x)(t), \tag{1.42}$$

where $\lambda, k \in R^+$, the function $\tau \in M(]a, b[)$, the operator $q : C_1^{m-1}(]a, b[) \to \widetilde{L}^2_{0,0}(]a, b[)$ is continuous and

$$\eta(t, \rho) \equiv \sup\{|q(x)(t)| : ||x||_{\widetilde{C}_1^{m-1}} \leq \rho\} \in \widetilde{L}^2_{0,0}(]a, b[).$$

Then, from Theorems 1.4 and 1.5, it follows

Corollary 1.4. *Let the function* $\tau \in M(]a, b[)$, *the continuous operator* $q : C_1^{m-1}(]a, b[) \to \widetilde{L}^2_{0,0}(]a, b[)$, *and the numbers* $\gamma_0 > 0,\ \lambda \geq 0,\ k > 0$, *be such that*

$$|\tau(t) - t| \leq \begin{cases} (t-a)^{3/2} & \textit{for}\ \ a < t \leq (a+b)/2 \\ (b-t)^{3/2} & \textit{for}\ \ (a+b)/2 \leq t < b \end{cases}, \tag{1.43}$$

$$||\eta(t,\, \gamma_0)||_{\widetilde{L}^2_{0,\,0}} \leq \Big(1 + \sqrt{\frac{2}{b-a}}\Big)^{-1} \frac{(b-a)^2 - 16\lambda\gamma_0^k(1 + [2(b-a)]^{1/4})}{2(1+b-a)(b-a)^2}, \tag{1.44}$$

and

$$\lambda < \frac{(b-a)^2}{32\gamma_0^k(1 + [2(b-a)]^{1/4})}. \tag{1.45}$$

Then, problem (1.41), (1.2) *is solvable.*

Corollary 1.5. *Let the function* $\tau \in M(]a,\, b[)$, *continuous operator* $q : C_1^{m-1}(]a, b[) \to \widetilde{L}_{0,0}^2(]a, b[)$, *and the number* $\lambda \geq 0$ *by such, that inequalities* (1.28) *with* $n = 2$, (1.43) *and*

$$\lambda < \frac{(b-a)^2}{32(1 + [2(b-a)]^{1/4})}, \tag{1.46}$$

hold. Then, problem (1.42), (1.2) *is solvable.*

2. Auxiliary propositions

2.1. Lemmas on some properties of the equation $x^{(2m)}(t) = \lambda(t)$. First, we introduce two lemmas without proofs. The first Lemma is proved in [3].

Lemma 2.1. *Let* $i \in 1, 2,\ \ x \in \widetilde{C}_{loc}^{m-1}(]t_0,\, t_1[)$ *and*

$$x^{(j-1)}(t_i) = 0 \quad (j = 1, \ldots, m), \qquad \int\limits_{t_0}^{t_1} |x^{(m)}(s)|^2 ds < +\infty. \tag{2.1}$$

Then

$$\Big| \int\limits_{t_i}^{t} \frac{(x^{(j-1)}(s))^2}{(s-t_i)^{2m-2j+2}} ds \Big|^{1/2} \leq \frac{2^{m-j+1}}{(2m-2j+1)!!} \Big| \int\limits_{t_i}^{t} |x^{(m)}(s)|^2 ds \Big|^{1/2} \tag{2.2}$$

for $t_0 \leq t \leq t_1$.

This second lemma is a particular case of Lemma 4.1 in [9]

Lemma 2.2. *If* $x \in C_{loc}^{2m-1}(]a, a_1])$, *then for any* $s, t \in]a, a_1]$, *we have the equality*

$$(-1)^m \int\limits_{s}^{t} x^{(n)}(\xi) x(\xi) d\xi = w_{2m}(x)(t) - w_{2m}(x)(s) + \int\limits_{s}^{t} |x^{(m)}(\xi)|^2 d\xi$$

$w_{2m}(x)(t) = \sum\limits_{j=1}^{m} (-1)^{m+j-1} x^{(2m-j)}(t) x(t)$.

Lemma 2.3. *Let the numbers* $a_1 \in]a, b[,\ t_{0k} \in]a, a_1[$, *and* $\varepsilon_{ik}, \varepsilon_i, \beta_k, \beta \in R^+$, $k \in N$, $i = 1, ..., m$ *be such that*

$$\lim_{k \to +\infty} t_{0\,k} = a, \quad \lim_{k \to +\infty} \beta_k = \beta, \quad \lim_{k \to +\infty} \varepsilon_{i,k} = \varepsilon_i. \tag{2.3}$$

Moreover, let

$$\lambda \in \widetilde{L}_{2m-2,0}^2(]a, a_1]), \tag{2.4}$$

be a nonnegative function, $x_k \in \widetilde{C}^{2m-1,m}(]a,b[)$ *be a solution of the problem*

$$x^{(2m)}(t) = \beta_k \lambda(t), \tag{2.5}$$

$$x^{(i-1)}(t_{0\,k}) = 0 \ \ (i = 1, ..., m), \qquad x^{(i-1)}(a_1) = \varepsilon_{i,k} \ \ (i = 1, ..., m), \tag{2.6}$$

and $x \in \widetilde{C}^{2m-1,m}(]a,b[)$ *be a solution of the problem*

$$x^{(2m)}(t) = \beta \lambda(t), \tag{2.7}$$

$$x^{(i-1)}(a) = 0 \ \ (i = 1, ..., m), \qquad x^{(i-1)}(a_1) = \varepsilon_i \ \ (i = 1, ..., m). \tag{2.8}$$

Then

$$\lim_{k\to+\infty} x_k^{(j-1)}(t) = x^{(j-1)}(t) \quad (j = 1, ..., 2m) \quad \textit{uniformly in} \quad]a, a_1]. \tag{2.9}$$

Analogously one can prove

Lemma 2.4. *Let the numbers* $b_1 \in]a,b[$, $t_{0k} \in]b_1,b[$, *and* $\varepsilon_{ik}, \varepsilon_i$, $\beta_k, \beta \in R^+$, $k \in N$, $i = 1, ..., n-m$ *be such that*

$$\lim_{k\to+\infty} t_{0\,k} = b, \quad \lim_{k\to+\infty} \beta_k = \beta, \quad \lim_{k\to+\infty} \varepsilon_{i,k} = \varepsilon_i.$$

Moreover, let $\lambda \in \widetilde{L}^2_{0,2m-2}(]b_1,b])$ *is a nonnegative function,* $x_k \in \widetilde{C}^{n-1,m}(]a,b[)$ *be a solution of problem* (2.5) *under the conditions*

$$x^{(i-1)}(b_1) = \varepsilon_{i,k} \ \ (i = 1, \cdots, m), \qquad x^{(i-1)}(t_{0\,k}) = 0 \ \ (i = 1, ..., n-m),$$

and $x \in \widetilde{C}^{n-1,\,m}(]a,b[)$ *be a solution of equation* (2.7) *under the conditions*

$$x^{(i-1)}(b_1) = \varepsilon_i \ \ (i = 1, ..., m), \qquad x^{(i-1)}(b) = 0 \ \ (i = 1, ..., n-m). \tag{2.10}$$

Then, equalities (2.9) *hold.*

Lemma 2.5. *Let* $a < a_1 < b_1 < b$, $\varepsilon_i \in R^+$ *and*

$$\lambda \in \widetilde{L}^2_{2n-2m-2,\,0}(]a,\ a_1]) \quad (\lambda \in \widetilde{L}^2_{0,\,2m-2}(]b_1,b]))$$

be nonnegative function. Then for the solution $x \in \widetilde{C}^{n-1,\,m}(]a,\,b[)$ *of problem* (2.7), (2.8) ((2.7), (2.10)) *with* $\beta = 1$, *the estimate*

$$\int_a^{a_1} |x^{(m)}(s)|^2 ds \le \Theta_1(x, a_1, \lambda) \ \Big(\int_{b_1}^{b} |x^{(m)}(s)|^2 ds \le \Theta_2(x, b_1, \lambda) \Big) \ (k \in N) \tag{2.11}$$

is valid, where

$$\begin{gathered} \Theta_1(x, a_1, \lambda) = 2|w_n(x)(a_1)| + \gamma_1 ||\lambda||^2_{\widetilde{L}^2_{2n-2m-2,0}(]a,a_1])}, \\ \Big(\Theta_2(x, b_1, \lambda) = 2|w_n(x)(b_1)| + \gamma_2 ||\lambda||^2_{\widetilde{L}^2_{0,2m-2}(]b_1,b])}\Big), \end{gathered} \tag{2.12}$$

and

$$\gamma_1 = \Big(\frac{2^{m-1}(2m+1)}{(2m-1)!!}\Big)^2, \ \ \gamma_2 = \Big(\frac{2^{m-1}(2m+1)(b-a+1)}{(2m-1)!!}\Big)^2.$$

2.2. Lemmas on the Banach space $\widetilde{C}^{m-1}_{1,\Gamma}(]a,\, b[)$. Let the bounded linear operator $\Gamma : \widetilde{C}^{m-1}_1 \to \widetilde{C}^m_2$ be defined by the equality

$$\Gamma(x)(t) = \int_a^b G(t,s)x(s)ds,$$

and

$$G(t,s) = \frac{1}{\varphi(b)-\varphi(a)} \times \begin{cases} \varphi(s)-\varphi(b) & \text{for } s \geq t \\ \varphi(s)-\varphi(a) & \text{for } s < t \end{cases} \tag{2.13}$$

is the Green's function of the problem:

$$w'(t) = 0, \qquad \int_a^b w(s)d\varphi(s) = 0. \tag{2.14}$$

where $\varphi : [a,b] \to R$ is a function of bounded variation and $\varphi(b)-\varphi(a) \neq 0$.

The problem (2.14) has only the trivial solution, thus $\Gamma(x) = 0$ if and only if $x = 0$, and we can define the Banach spaces:

$C^{m-1}_{1,\Gamma}(]a,\, b[)$ is the Banach space of the functions $x \in C^{m-1}_{loc}(]a,\, b[)$, such that

$$\begin{aligned} &\limsup_{t\to a} \frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}} < +\infty \; (i = 1, \cdots, m), \\ &\limsup_{t\to b} \frac{|x^{(i-1)}(t)|}{(b-t)^{m-i+1/2}} < +\infty \; (i = 1, \cdots, m), \end{aligned} \tag{2.15}$$

with the norm:

$$||x||_{C^{m-1}_{1,\Gamma}} = ||\Gamma(x)||_C + \sum_{i=1}^m \sup\Big\{ \frac{|x^{(i-1)}(t)|}{\alpha_i(t)} \;:\; a<t<b \Big\},$$

where $\alpha_i(t) = (t-a)^{m-i+1/2}(b-t)^{m-i+1/2}$.

$\widetilde{C}^{m-1}_{1,\Gamma}(]a,\, b[)$ is the Banach space of the functions $x \in \widetilde{C}^{m-1}_{loc}(]a,\, b[)$, such that conditions (1.3) and (2.15) hold, with the norm:

$$||x||_{\widetilde{C}^{m-1}_{1,\Gamma}} = ||\Gamma(x)||_C + ||x||_{C^m_1} + \Big(\int_a^b |x^{(m)}(s)|^2 ds \Big)^{1/2}.$$

Definition 2.3. Let $\rho \in R^+$ and the function $\eta \in L_{loc}(]a,b[)$ be nonnegative. Then $S(\rho,\eta)$ is a set of such $z \in C^{n-1}_{loc}(]a,b[)$ that

$$\Big| z^{(i-1)}\Big(\tfrac{a+b}{2}\Big)\Big| \leq \rho \qquad (i = 1, ..., n), \tag{2.16}$$

$$|z^{(n-1)}(t) - z^{(n-1)}(s)| \leq \int_s^t \eta(\xi)d\xi \qquad \text{for } a<s\leq t<b, \tag{2.17}$$

$$z^{(i-1)}(a) = 0 \; (i = 1, ..., m), \qquad z^{(i-1)}(b) = 0 \; (i = 1, ..., m). \tag{2.18}$$

Lemma 2.6. *For the function* $z \in \widetilde{C}^{2m-1,m}(]a,\ b[)$, *let conditions* (2.18) *be satisfied. Then,* $z \in \widetilde{C}^{m-1}_{1,\Gamma}(]a,b[)$ *and the estimates*

$$(2.19) \quad |z^{(i-1)}(t)| \leq \frac{|t-c_k|^{m-i+1/2}}{(m-i)!(2m-2i+1)^{1/2}} \Big| \int\limits_{c_k}^{t} |z^{(m)}(s)|^2 ds \Big|^{1/2} \quad \text{for } a<t<b,$$

$i = 1, ..., m$, *hold for* $k = 1, 2$, *where* $c_1 = a,\ c_2 = b$.

Proof. First, not that in view of inclusion $z \in \widetilde{C}^{2m-1,m}(]a,b[)$, the equality

$$(2.20) \qquad z^{(i-1)}(t) = \sum_{j=i}^{l} \frac{(t-c)^{j-i}}{(j-i)!} z^{(j-1)}(c) + \frac{1}{(l-i)!} \int\limits_{c}^{t} (t-s)^{l-i} z^{(l)}(s) ds$$

for $a<t<b$, $i = 1, ..., l$, $l = 1, ..., 2m$, holds, where

1. $c \in [a,b]$ if $l \leq m$;
2. $c \in]a,b[$ if $l > m$,

and exists $r > 0$ such that

$$(2.21) \qquad \int\limits_{a}^{b} |z^{(m)}(s)|^2 ds \leq r.$$

Equality (2.20), with $l = m,\ c = a$ and with $l = m,\ c = b$ by conditions (2.18), (2.21) and the Schwartz inequality yields (2.19). But, from (2.19) with $i = 1$, we have that $||z||_C < +\infty$, and then, by the inequality

$$(2.22) \qquad |\Gamma(z)(t)| \leq \int\limits_{a}^{b} \frac{|\varphi(s)-\varphi(a)| + |\varphi(s)-\varphi(b)|}{|\varphi(b)-\varphi(a)|} ds ||z||_C,$$

we have

$$(2.23) \qquad ||\Gamma(z)||_C < +\infty.$$

From (2.19), (2.21) and (2.23) it is clear that $z \in \widetilde{C}^{m-1}_{1,\Gamma}(]a,b[)$. ∎

Lemma 2.7. *Let* $\rho \in R^+$, *and* $\eta \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,b[)$ *is a nonnegative function. Then* $S(\rho,\eta)$ *is a compact subset of the space* $\widetilde{C}^{m-1}_{1,\Gamma}(]a,\ b[)$.

Proof. Condition (2.17) yields the inequality $|z^{(2m)}(t)| \leq \eta(t)$. Thus there exists such a function $\eta_1 \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,b[)$ that

$$(2.24) \qquad z^{(2m)}(t) = \eta_1(t), \quad \text{for} \quad a<t<b$$

$$(2.25) \qquad |\eta_1(t)| \le \eta(t) \quad \text{for} \quad a < t < b$$

From Theorem 1.1, it follows that problem (2.24), (2.18) has a unique solution $z \in C^{2m-1,m}(]a,b[)$, i.e., there exists $r > 0$ such that the inequality (2.21) holds.

For any $z \in S(\rho,\eta)$, from equality (2.20) with $l = 2m$, by (2.16), (2.24) and (2.25)we get

$$(2.26) \qquad |z^{(i-1)}(t)| \le \gamma_i(t) \quad \text{for} \quad a < t < b, \quad (i = 1, ..., 2m),$$

where

$$\gamma_i(t) = \rho_i + \frac{1}{(2m-i)!}\Big|\int_c^t (t-s)^{n-i}\eta(s)ds\Big| \quad (i = 1, ..., 2m).$$

Now, let $z_k \in S(\rho,\eta)$ $(k \in N)$. By virtue of the Arzelà-Ascoli lemma and conditions (2.17), (2.26), the sequence $\{z_k\}_{k=1}^{+\infty}$ contains a subsequence $\{z_{k_\ell}\}_{\ell=1}^{+\infty}$ such that $\{z_{k_\ell}^{(i-1)}\}_{\ell=1}^{+\infty}$ $(i = 1, ..., 2m)$ are uniformly convergent on $]a,b[$. Thus, without loss of generality, we can assume that $\{z_k^{(i-1)}\}_{k=1}^{+\infty}$ $(i = 1, ..., 2m-1)$ are uniformly convergent on $]a,\ b[$. Let $\lim\limits_{k\to+\infty} z_k(t) = z_0(t)$, then $z_0 \in \widetilde{C}_{loc}^{2m-1}(]a,\ b[)$ and

$$(2.27) \qquad \lim_{k\to+\infty} z_k^{(i-1)}(t) = z_0^{(i-1)}(t) \quad (i = 1, ..., 2m) \quad \text{uniformly on} \quad]a,\ b[.$$

From (2.27), in view of the inclusions $z_k \in S(\rho,\eta)$, immediately it follows that

$$(2.28) \qquad \left|z_0^{(i-1)}\left(\tfrac{a+b}{2}\right)\right| \le \rho \qquad (i = 1, ..., 2m),$$

$$(2.29) \qquad z_0^{(i-1)}(a) = 0 \ (j = 1, \cdots, m), \quad z_0^{(i-1)}(b) = 0 \ (j = 1, ..., m).$$

$$(2.30) \qquad |z_0^{(2m-1)}(t) - z_0^{(2m-1)}(s)| \le \int\limits_s^t \eta(\xi)d\xi \qquad \text{for } a < s \le t < b.$$

From (2.28)-(2.30), it is clear that $z_0 \in S(\rho,\eta)$. To finish the proof, we must show that

$$(2.31) \qquad \lim_{k\to+\infty} ||z_k(t) - z_0(t)||_{\widetilde{C}_{1,\Gamma}^{m-1}} = 0,$$

$$(2.32) \qquad S(\rho,\eta) \subset \widetilde{C}_{1,\Gamma}^{m-1}(]a,b[).$$

First, note that, from (2.27), by the conditions $z_k \in S(\rho,\eta)$ and (2.29), we have

$$\lim_{k\to+\infty} ||z_k - z_0||_C = 0,$$

from which, by (2.22), we get

$$(2.33) \qquad \lim_{k\to+\infty} ||\Gamma(z_k - z_0)||_C = 0.$$

Let, $x_k = z_0 - z_k$, and $a_1 \in]a,b[,\ b_1 \in]a_1,b[$. Then, it is clear that $x_k \in S(\rho',\eta')$, where $\rho' = 2\rho$, $\eta' = 2\eta$. Thus, for any x_k, there exists $\eta_k \in \widetilde{L}_{2m-2,2m-2}^2(]a,b[)$ such that

$$(2.34) \qquad x_k^{(2m)}(t) = \eta_k(t),$$

$$(2.35) \qquad x_k^{(i-1)}(a) = 0 \ \ (i = 1, ..., m), \qquad x_k^{(i-1)}(b) = 0 \ \ (i = 1, ..., m),$$

where

$$|\eta_k(t)| \le 2\eta(t) \quad \text{for } a<t<b \quad (k\in N). \tag{2.36}$$

On the other hand, from (2.19) with $z=x_k$, in view of (2.35), we get

$$\begin{aligned} |x_k^{(i-1)}(t)| &\le \Big(\int_a^t |x_k^{(m)}(s)|^2 ds\Big)^{1/2}(t-a)^{m-i+1/2} \quad \text{for} \quad a<t<a_1,\\ |x_k^{(i-1)}(t)| &\le \Big(\int_t^b |x_k^{(m)}(s)|^2 ds\Big)^{1/2}(b-t)^{m-i+1/2} \quad \text{for} \quad b_1<t<b, \end{aligned} \tag{2.37}$$

for $i=1,\dots,m$.

Now, let w_{2m} be the operator defined in Lemma 2.2 and Θ_1,Θ_2 are functions defined by (2.12) with $\lambda=\eta_k$. Then, conditions (2.27) yields

$$\lim_{k\to+\infty} w_{2m}(x_k)(a_1)=0, \qquad \lim_{k\to+\infty} w_{2m}(x_k)(b_1)=0 \ \ (k\in N), \tag{2.38}$$

and, from the definition of the norm $||\cdot||_{\widetilde{L}^2_{\alpha,\beta}}$, (2.36) and (2.38), it follows that, for any $\varepsilon>0$, we can choose $a_1\in]a,\,\min\{a+1,\,b\}[$, $b_1\in]\max\{b-1,b\},b[$ and $k_0\in N$, such that

$$\begin{aligned} \Theta_1(x_k,a_1,2\eta) &\le \frac{\varepsilon}{6}(b-b_1)^{m-1/2} \quad (k\ge k_0),\\ \Theta_2(x_k,b_1,2\eta) &\le \frac{\varepsilon}{6}(a_1-a)^{m-1/2} \quad (k\ge k_0). \end{aligned} \tag{2.39}$$

By using Lemma 2.5 for x_k, in view of (2.37) and (2.39), we get

$$\int_a^{a_1} |x_k^{(m)}(s)|^2 ds \le \frac{\varepsilon}{6} \qquad \int_{b_1}^b |x_k^{(m)}(s)|^2 ds \le \frac{\varepsilon}{6} \ \ (k\ge k_0), \tag{2.40}$$

$$\frac{|x_k^{(i-1)}(t)|}{\alpha_i(t)} \le \frac{\varepsilon}{2m} \qquad \text{for} \quad t\in]a,\ a_1]\cup[b_1,\ b[, \ \ (1\le i\le m, \ \ k\ge k_0). \tag{2.41}$$

Also, in view of (2.27) without loss of generality we can assume that

$$\frac{|x_k^{(i-1)}(t)|}{\alpha_i(t)} \le \frac{\varepsilon}{2m} \quad \text{for} \quad a_1\le t\le b_1, \quad (1\le i\le n-1, \ k\ge k_0), \tag{2.42}$$

$$\int_{a_1}^{b_1} |x_k^{(m)}(s)|^2 ds \le \frac{\varepsilon}{6} \quad (k\ge k_0). \tag{2.43}$$

From (2.33), (2.40)-(2.43), equality (2.31) immediately follows.

Let, now $z\in S(\rho,\,\eta)$ and $z_k=\delta_k z$, where $\lim\limits_{k\to+\infty}\delta_k=0$. Then, by (2.27), it is clear that $z_0\equiv 0$ and then, from (2.31) it follows $z\in \widetilde{C}^{m-1}_{1,\Gamma}(]a,\,b[)$, i.e., inclusion (2.32) holds. ∎

Lemma 2.8. *Let* $\tau_j \in M(]a,b[)$, $\alpha \geq 0$, $\beta \geq 0$ *and exists* $\delta \in]0, b-a[$ *such that*

$$|\tau_j(t) - t| \leq k_1(t-a)^\beta \quad for \quad a < t \leq a + \delta. \tag{2.44}$$

Then

$$\Big|\int\limits_t^{\tau(t)} (s-a)^\alpha ds\Big| \leq \begin{cases} k_1[1 + k_1\delta^{\beta-1}]^\alpha (t-a)^{\alpha+\beta} & for \;\; \beta \geq 1 \\ k_1[\delta^{1-\beta} + k_1]^\alpha (t-a)^{\alpha\beta+\beta} & for \;\; 0 \leq \beta < 1 \end{cases},$$

for $a < t \leq a + \delta$.

Proof. First note that

$$\Big|\int\limits_t^{\tau(t)} (s-a)^\alpha ds\Big| \leq (\max\{\tau(t), t\} - a)^\alpha |\tau(t) - t| \quad \text{for} \quad a \leq t \leq a + \delta,$$

and $\max\{\tau(t), t\} \leq t + |\tau(t) - t|$ for $a \leq t \leq a + \delta$. Then, in view of condition (2.44), we get

$$\Big|\int\limits_t^{\tau(t)} (s-a)^\alpha ds\Big| \leq k_1[(t-a) + k_1(t-a)^\beta]^\alpha (t-a)^\beta \quad \text{for} \quad a \leq t \leq a + \delta.$$

The last inequality yields the validity of our lemma. ∎

Analogously, one can prove

Lemma 2.9. *Let* $\tau_j \in M(]a,b[)$, $\alpha \geq 0$, $\beta \geq 0$ *and exists* $\delta \in]0, b-a[$ *such that*

$$|\tau_j(t) - t| \leq k_1(b-t)^\beta \quad for \quad b - \delta \leq t < b. \tag{2.45}$$

Then

$$\Big|\int\limits_t^{\tau(t)} (b-t)^\alpha ds\Big| \leq \begin{cases} k_1[1 + k_1\delta^{\beta-1}]^\alpha (b-t)^{\alpha+\beta} & for \;\; \beta \geq 1 \\ k_1[\delta^{1-\beta} + k_1]^\alpha (b-t)^{\alpha\beta+\beta} & for \;\; 0 \leq \beta < 1 \end{cases},$$

for $b - \delta \leq t < b$.

2.3. Lemmas on the solutions of auxiliary problems.

Lemma 2.10. *For any function* $x \in C_1^{m-1}(]a,\, b[)$ $(x \in \widetilde{C}_1^{m-1}(]a,\, b[))$ *the equality*

$$||\Gamma(x)||_{C_2^m} = ||x||_{C_{1,\Gamma}^{m-1}}, \qquad (||\Gamma(x)||_{\widetilde{C}_2^m} = ||x||_{\widetilde{C}_{1,\Gamma}^{m-1}})$$

holds

Proof. From the definition of the norms $||\cdot||_{\widetilde{C}_2^m}$ and $||\cdot||_{\widetilde{C}_{1,\Gamma}^{m-1}}$ we have

$$\begin{aligned}||\Gamma(x)||_{\widetilde{C}_2^m} &= ||\Gamma(x)||_C + \sum_{i=1}^m \sup\Big\{\frac{|\Gamma^{(i)}(x)(t)|}{\alpha_i(t)} \ : \ a<t<b\Big\} \\ &\quad + \Big(\int_a^b |\Gamma^{(m+1)}(x)(s)|^2 ds\Big)^{1/2} \\ &= ||\Gamma(x)||_C + \sum_{i=1}^m \sup\Big\{\frac{|x^{(i-1)}(t)|}{\alpha_i(t)} \ : \ a<t<b\Big\} \\ &\quad + \Big(\int_a^b |x^{(m)}(s)|^2 ds\Big)^{1/2} = ||x||_{\widetilde{C}_{1,\Gamma}^{m-1}}. \qquad \blacksquare\end{aligned}$$

Now, define the continuous operators $P_\Gamma : C_{1,\Gamma}^{m-1}(]a,b[)\times C_{1,\Gamma}^{m-1}(]a,b[) \to L_n(]a,b[)$ as

$$P_\Gamma(x,y)(t) = P(\Gamma(x),\Gamma(y))(t) \qquad \text{for} \qquad a<t<b,$$

i.e.,

$$P_\Gamma(x,y)(t)=\sum_{j=1}^m p_{j,\Gamma}(x)(t)\Gamma^{(j)}(y)(\tau_j(t))+p_{0,\Gamma}(x)(t)\int_a^b G(\tau_0(t),s)\Gamma(y)(s)ds \tag{2.46}$$

where the operators $p_{j,\Gamma} : C_1^{m-1}(]a,\, b[) \to L_n(]a,b[)$ $(j=0,\dots,m)$ are defined by the equalities

$$p_{j,\Gamma}(x)(t) = p_j(\Gamma(x))(t),$$

and $F_{p,\Gamma} : C_{1,\Gamma}^{m-1}(]a,b[) \to \widetilde{L}^2_{2m-2,2m-2}(]a,\, b[)$ as

$$F_{p,\Gamma}(x)(t) = F_{p,\Gamma}(\Gamma(x))(t) = |F(\Gamma(x))(t) - P_\Gamma(x,x)(t)|. \tag{2.47}$$

Lemma 2.11. *If the operator $P : C_2^m(]a,b[)\times C_2^m(]a,b[) \to L_n(]a,\, b[)$ is consistent with the numbers γ_0,γ, and the set $A_{\gamma_0,\Gamma} \subset \widetilde{C}_{1,\Gamma}^{m-1}(]a,b[)$ is defined by the equality*

$$A_{\gamma_0,\Gamma} = \{x \in \widetilde{C}_{1,\Gamma}^{m-1}(]a,\, b[) : ||x||_{\widetilde{C}_{1,\Gamma}^{m-1}} \le \gamma_0\},$$

then:

(i) *for any $x_0 \in A_{\gamma_0,\Gamma}$ and almost all $t \in]a,\, b[$, the inequality*

$$\begin{aligned}\sum_{j=1}^m |p_{j,\Gamma}(x_0)(t)x_0^{(j-1)}(\tau_j(t))| + |p_{0,\Gamma}(x_0)(t)\int_a^b G(\tau_0(t),s)x_0(s)ds| \\ \le \eta(t,\, ||x_0||_{\widetilde{C}_{1,\Gamma}^{m-1}})||x_0||_{\widetilde{C}_{1,\Gamma}^{m-1}}.\end{aligned} \tag{2.48}$$

holds, where $\delta \in D_n(]a,\, b[\times R^+)$.

(ii) *for any* $x_0 \in A_{\gamma_0,\Gamma}$ *and* $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,\, b[)$, *the equation*

$$z^{(2m)}(t)=\sum_{j=1}^{m} p_{j,\Gamma}(x_0)(t) z^{(j-1)}(\tau_j(t))+p_{0,\Gamma}(x_0)(t)\int_a^b G(\tau_0(t), s) z(s)ds + q(t) \tag{2.49}$$

under the boundary conditions (2.18), *has the unique solution* $z \in \widetilde{C}^{2m-1,m}(]a,\, b[)$, *and*

$$||z||_{\widetilde{C}^{m-1}_{1,\Gamma}} \le \gamma ||q||_{\widetilde{L}^2_{2m-2,\,2m-2}} \tag{2.50}$$

Proof. In view of Lemma 2.10, for any $x_0 \in A_{\gamma_0,\Gamma}$ we have $y = \Gamma(x_0) \in A_{\gamma_0}$, where A_{γ_0} is defined by (1.18), and then, from inequality (1.19), we get

$$\begin{aligned}
&\sum_{j=1}^{m} |p_{j,\Gamma}(x_0)(t) x_0^{(j-1)}(\tau_j(t))| + |p_{0,\Gamma}(x_0)(t) \int_a^b G(\tau_0(t), s) x_0(s) ds| \\
&\quad = \sum_{j=1}^{m} |p_j(\Gamma(x_0))(t) \Gamma^{(j)}(x_0)(\tau_j(t))| + |p_0(\Gamma(x_0))(t) \Gamma(x_0)(t)| \\
&\quad = \sum_{j=0}^{m} |p_j(y)(t) y^{(j)}(\tau_j(t))| \le \eta(t,\, ||y||_{\widetilde{C}^m_2}) ||y||_{\widetilde{C}^m_2} \\
&\quad = \eta(t,\, ||\Gamma(x_0)||_{\widetilde{C}^m_2}) ||\Gamma(x_0)||_{\widetilde{C}^m_2} = \eta(t,\, ||x_0||_{\widetilde{C}^{m-1}_{1,\Gamma}}) ||x_0||_{\widetilde{C}^{m-1}_{1,\Gamma}}.
\end{aligned}$$

Now, let $x_0 \in A_{\gamma_0,\Gamma}$ and $q \in \widetilde{L}^2_{2m-2,2m-2}(]a, b[)$, the function $y \in \widetilde{C}^{2m,m+1}(]a, b[)$ is a solution of problem (1.20), (1.2), for $x = \Gamma(x_0)$. Let also $z = y'$, then $z \in \widetilde{C}^{2m-1,m}(]a, b[)$ and in view of (1.2) we have the representation $y = \Gamma(z)$, and from (1.20) and (1.21) it follows

$$\begin{aligned}
z^{(2m)}(t) &= \sum_{j=0}^{m} p_j(\Gamma(x_0))(t) \Gamma^{(j)}(z)(\tau_j(t)) + q(t) \\
&= \sum_{j=1}^{m} p_{j,\Gamma}(x_0)(t) z^{(j-1)}(\tau_j(t)) + p_{0,\Gamma}(x_0)(t) \int_a^b G(\tau_0(t), s) z(s) ds + q(t)
\end{aligned}$$

and

$$||z||_{\widetilde{C}^{m-1}_{1,\Gamma}} = ||\Gamma(z)||_{\widetilde{C}^m_2} = ||y||_{\widetilde{C}^m_2} \le \gamma ||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}.$$

Thus, relations (2.48), (2.48), and (2.48) are valid. ∎

Lemma 2.12. *Let* (1.22) *and all the conditions of Theorem* 1.3 *hold. Then*

$$\widetilde{F}_{p,\Gamma}(t,\, \rho) \equiv \sup\{F_{p,\Gamma}(x)(t) : ||x||_{C^{m-1}_{1,\Gamma}} \le \rho\} \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,\, b[), \tag{2.51}$$

$$||\widetilde{F}_{p,\Gamma}(\cdot,\, \min\{2\rho_0,\, \gamma_0\})||_{\widetilde{L}^2_{2m-2,\,2m-2}} \le \frac{\gamma_0}{\gamma}, \tag{2.52}$$

and, for any $\lambda \in]0,1[$*, an arbitrary solution* $z \in A_{\gamma_0,\Gamma}$ *of the equation*

$$z^{(2m)}(t) = (1-\lambda)P_\Gamma(z,\, z)(t) + \lambda F(\Gamma(z))(t) \tag{2.53}$$

under condition (2.18)*, admits the estimate*

$$||z||_{C^{m-1}_{1,\Gamma}} \le \rho_0, \tag{2.54}$$

Proof. In view of Lemma 2.10, and (2.47), by the notation $y \equiv \Gamma(x) \in C^m_2(]a,\, b[)$ and (1.22), we get

$$\begin{aligned}\sup\{F_{p,\Gamma}(x)(t) : ||x||_{C^{m-1}_{1,\Gamma}} \le \rho\} &= \sup\{F_p(\Gamma(x))(t) : ||x||_{C^{m-1}_{1,\Gamma}} \le \rho\}\\ &= \sup\{F_p(\Gamma(x))(t) : ||\Gamma(x)||_{C^m_2} \le \rho\}\\ &= \sup\{F_p(y)(t) : ||y||_{C^m_2} \le \rho\} \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,\, b[).\end{aligned}$$

Thus (2.51) holds.

From the last relation with $\rho = \min\{2\rho_0,\, \gamma_0\}$, in view of (1.23), we get (2.52).

Let now $z \in A_{\gamma_0,\Gamma}$ is a solution of problem (2.53), (2.18), and $x \equiv \Gamma(z)$. Then

$$x^{(2m+1)}(t) = \Gamma^{(2m+1)}(z)(t) = z^{(2m)}(t) = (1-\lambda)P_\Gamma(z,\, z)(t) + \lambda F(\Gamma(z))(t)$$

$$= (1-\lambda)P(\Gamma(z),\, \Gamma(z))(t) + \lambda F(\Gamma(z))(t) = (1-\lambda)P(x,\, x)(t) + \lambda F(x)(t),$$

i.e., x is a solution of problem (1.24), (1.2) end then the estimate (1.25) holds, from which by Lemma 2.10 we get (2.54). ■

Now, let the operator $q : C^{m-1}_{1,\Gamma}(]a,b[) \to \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$ be continuous and for any $x \in \widetilde{C}^{m-1}_{1,\Gamma}(]a,b[) \subset C^{m-1}_{1,\Gamma}(]a,b[)$ consider the nonhomogeneous equation

$$z^{(2m)}(t) = \sum_{j=1}^{m} p_{j,\Gamma}(x_0)(t) z^{(j-1)}(\tau_j(t)) + p_{0,\Gamma}(x_0)(t)\int_a^b G(\tau_0(t),s)z(s)ds + q(x)(t) \tag{2.55}$$

and the corresponding homogeneous equation

$$z^{(2m)}(t) = \sum_{j=1}^{m} p_{j,\Gamma}(x_0)(t) z^{(j-1)}(\tau_j(t)) + p_{0,\Gamma}(x_0)(t)\int_a^b G(\tau_0(t),s)z(s)ds, \tag{2.56}$$

and let E^n be a set of the solutions of problem (2.55), (2.18). Assume that the operator $P : C^m_2(]a,b[) \times C^m_2(]a,b[) \to L_n(]a,b[)$ be γ_0, γ consistent with the boundary condition (1.2), and operator $q : C^m_2(]a,b[) \to \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$, be continuous. Then, by Lemma 2.11, problem (2.55), (2.18) has the unique solution $z \in \widetilde{C}^{2m-1,m}(]a,\, b[)$. But, in view of Lemma 2.6, it is clear that $z \in \widetilde{C}^{m-1}_{1,\Gamma}(]a,\, b[)$. Thus, $E^n \cap \widetilde{C}^{m-1}_{1,\Gamma}(]a,\, b[) \neq \emptyset$, and there exists the operator $U : \widetilde{C}^{m-1}_{1,\Gamma}(]a,\, b[) \to E^n \cap \widetilde{C}^{m-1}_{1,\Gamma}(]a,\, b[)$ defined by the equality

$$U(x)(t) = z(t).$$

Lemma 2.13. $U : \widetilde{C}^{m-1}_{1,\Gamma}(]a,b[) \to E^n \cap \widetilde{C}^{m-1}_{1,\Gamma}(]a,b[)$ *is a continuous operator.*

Proof. Let $x_k \in \widetilde{C}^{m-1}_{1,\Gamma}(]a,\, b[)$ and $z_k(t) = U(x_k)(t)$ $(k = 1, 2)$, $y = y_2 - y_1$, and the operator P is defined by (1.17). Then

$$z^{(2m)}(t) = P_\Gamma(x_2,\, z)(t) + q_0(x_1,\, x_2)(t)$$

where $q_0(x_1,\, x_2)(t) = P_\Gamma(x_2,\, y_1)(t) - P_\Gamma(x_1,\, y_1)(t) + q(x_2)(t) - q(x_1)(t)$. Hence, by item *ii.* of Lemma 2.11 we have

$$||U(x_2) - U(x_1)||_{\widetilde{C}^{m-1}_{1,\Gamma}} \leq \gamma||q_{0,\Gamma}(x_1,\, x_2)||_{\widetilde{L}^2_{2m-2,\, 2m-2}}.$$

Since the operators P_Γ and q are continuous, this estimate implies the continuity of the operator U. ■

3. Proofs

Proof of Remark 1.1. Let z be a solution of problem (2.49), (2.18) with $p_{j,\Gamma}(x_0) = p_j$ $(j = 0, ..., m)$, then $u = \Gamma(z)$ is a solution of problem (1.8), (1.2), and from inequalities (2.19) it follows the estimate

$$|z^{(i-1)}(t)| \leq \frac{[(b-t)(t-a)]^{m-i+1/2}}{(m-i)!(2m-2i+1)^{1/2}}\Big(\frac{2}{b-a}\Big)^{m-i+1/2}||z^{(m)}||_{L^2},$$

for $a \leq t \leq b$, and

$$||z||_C \leq \frac{(b-a)^{m-1/2}}{(m-1)!(2m-1)^{1/2}2^{m-1/2}}||z^{(m)}||_{L^2}.$$

From the last estimates, by the definition of the norm $||\cdot||_{\widetilde{C}^{m-1}_{1,\Gamma}}$ and (2.22), we have

$$||z||_{\widetilde{C}^{m-1}_{1,\Gamma}} \leq \int_a^b \frac{|\varphi(s)-\varphi(a)| + |\varphi(s)-\varphi(b)|}{|\varphi(b)-\varphi(a)|}ds\frac{(b-a)^{m-1/2}}{(m-1)!(2m-1)^{1/2}2^{m-1/2}}||z^{(m)}||_{L^2}$$

$$+\Big(1+\sum_{j=1}^{m}\frac{1}{(m-i)!(2m-2i+1)^{1/2}}\Big(\frac{2}{b-a}\Big)^{m-i+1/2}\Big)||z^{(m)}||_{L^2},$$

from which, by Lemma 2.10, estimate (1.15) and the fact that $u' = z$, (1.16) immediately follows. ■

Proof of Theorem 1.3. Let δ and λ are the functions and numbers appearing in Definition 1.1, and the operator $\widetilde{F}_{p,\Gamma}(t,\rho)$ be defined by (2.51). We set

$$\eta(t) = \delta(t,\ \gamma_0)\gamma_0 + \widetilde{F}_{p,\Gamma}(t,\, \min\{2\rho_0, \gamma_0\}), \tag{3.57}$$

$$\chi(s) = \begin{cases} 1 & \text{for} \quad 0 \le s \le \rho_0 \\ 2 - s/\rho_0 & \text{for} \quad \rho_0 < s < 2\rho_0 \ , \\ 0 & \text{for} \quad s \ge 2\rho_0 \end{cases} \tag{3.58}$$

$$q(x)(t) = \chi(||x||_{\widetilde{C}^{m-1}_{1,\Gamma}}) F_{p,\Gamma}(x)(t). \tag{3.59}$$

From (1.22) it is clear that the nonnegative functions $\widetilde{F}_{p,\Gamma}$, η, admits the inclusion

$$\widetilde{F}_{p,\Gamma}(\cdot,\ \min\{2\rho_0, \gamma_0\}),\ \eta \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,\ b[), \tag{3.60}$$

and, for every $x \in A_{\gamma_0,\Gamma} \subset \widetilde{C}^{m-1}_{1,\Gamma}(]a, b[)$ and almost all $t \in]a, b[$, the inequality

$$|q(x)(t)| \le \widetilde{F}_{p,\Gamma}(t,\ \min\{2\rho_0, \gamma_0\}) \qquad \text{for } a < t < b \tag{3.61}$$

holds.

Let $U : A_{\gamma_0,\Gamma} \to E^n \cap \widetilde{C}^{m-1}_{1,\Gamma}(]a,\ b[)$ is a operator appeared in Lemma 2.13, from which it follows that U is a continuous operator. From (i) and (ii) of Definition 1.1, i.e., by Lemma 2.11, from (2.48) and (2.50) it is clear that, for each $x \in A_{\gamma_0,\Gamma}$, the conditions

$$||z||_{\widetilde{C}^{m-1}_{1,\Gamma}} \le \gamma_0, \quad |z^{(n-1)}(t) - z^{(n-1)}(s)| \le \int_s^t \eta(\xi) d\xi \qquad \text{for} \qquad a < t < b$$

hold. Thus, in view of Definition 2.3, the operator U maps the ball $A_{\gamma_0,\Gamma}$ into its own subset $S(\rho_1,\ \eta)$. From Lemma 2.2, it follows that $S(\rho_1,\ \eta)$ is the compact subset of the ball $A_{\gamma_0,\Gamma} \subset \widetilde{C}^{m-1}_{1,\Gamma}(]a,\ b[)$, i.e., the operator u maps the ball $A_{\gamma_0,\Gamma}$ into its own compact subset. Therefore, owing to Schauders's principle, there exists $z \in S(\rho_1,\ \eta) \subset A_{\gamma_0,\Gamma}$, such that

$$z(t) = U(z)(t) \quad \text{for} \quad a < t < b.$$

Thus, by (2.55) and notation (3.59), the function $z\,(z \in A_{\gamma_0})$ is a solution of problem (2.53), (2.18), where

$$\lambda = \chi(||z||_{\widetilde{C}^{m-1}_{1,\Gamma}}). \tag{3.62}$$

If $\gamma_0 = \rho_0$, then, in view of condition $z \in A_{\gamma_0,\Gamma}$, by (3.58) we have that $\lambda = 1$, and then, in view of (2.53) and (3.59), the function z is a solution of the equation

$$z^{(2m)}(t) = F(\Gamma(z))(t) \tag{3.63}$$

under boundary conditions (2.18). Thus the function $u = \Gamma(z)$ (where $u' = z$) is a solution of problem (1.1), (1.2).

Let us show now, that z admits estimate (2.50) in the case when $\rho_0 < \gamma_0$. Assume the contrary. Then either

$$\rho_0 < ||z||_{\widetilde{C}^{m-1}_{1}} < 2\rho_0, \tag{3.64}$$

or

$$\|z\|_{\widetilde{C}_1^{m-1}} \geq 2\rho_0. \tag{3.65}$$

If condition (3.64) holds, then by virtue of (3.58) and (3.62), z is a solution of equation (2.53) with $\lambda \in]0,1[$, and then (2.54) holds. But this contradicts (3.64).

Assume now that (3.65) is fulfilled. Then, by virtue of (3.58) and (3.62), we have that $\lambda = 0$. Therefore, $x \in A_{\gamma_0,\Gamma}$ is a solution of problem (2.56), (2.18). Thus, from (ii) of Lemma 2.11, it is obvious that $z \equiv 0$, because problem (2.49), (2.18) has only a trivial solution. But this contradicts condition (3.65), i.e., estimate (2.54) is valid. From estimate (2.54) and (3.58), we have that $\lambda = 1$, and then, in view of (2.55) and (3.59), the function z is a solution of problem (3.63), (2.18), which admits to the estimate (2.50). Thus, the function $u = \Gamma(z)$ (where $u' = z$) is a solution of problem (1.1), (1.2). ∎

Proof of Corollary 1.2. First, note that, in view of condition (1.28), there exists such a $\gamma_0 > 2\rho_0$ that condition (1.23) holds, and, in view of Definition 1.2, the operator P is γ_0, γ consistent.

On the other hand, from (1.28), it follows the existence of the number ρ_0, such that

$$\gamma\|\eta(\cdot,\,\rho)\|_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} < \rho \quad \text{for} \qquad \rho > \rho_0. \tag{3.66}$$

Let x be a solution of problem (1.24), (1.2) for some $\lambda \in]0,\,1[$. Then $y = x$ is also a solution of problem (1.20), (1.2), where $q(t) = \lambda\Big(F(x)(t) - P(x,\,x)(t)\Big)$. Now, let $\rho = \|x\|_{\widetilde{C}_1^{m-1}}$ and assume that

$$\rho > \rho_0. \tag{3.67}$$

holds. Then, in view of the $\gamma-$consistency of the operator p with boundary conditions (1.2), inequality (1.21) holds and thus, by condition (1.26), we have

$$\rho = \|x\|_{\widetilde{C}_1^{m-1}} \leq \gamma\|q(x)\|_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} \leq \gamma\|\eta(\cdot,\,\rho)\|_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}.$$

But the last inequality contradicts (3.66). Thus, assumption (3.67) is not valid and $\rho \leq \rho_0$. Therefore, for any $\lambda \in]0,1[$, an arbitrary solution of problem (1.24), (1.2) admits the estimate (1.25). Therefore, all the conditions of Theorem 1.3 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. ∎

Proof of Theorem 1.4. Let r_n be the constant defined in Remark 1.1. First, let us prove that the operator P is γ_0, r_n consistent with boundary conditions (1.2). From the conditions of our theorem, it is obvious that the item (i) of Definition 1.1 is satisfied. Let now x be an arbitrary fixed function from the set A_{γ_0} and let $p_j(t) \equiv p_j(x)(t)$. Thus in view of (1.31), (1.32) all the assumptions of Theorem 1.1 are satisfied, and then for any $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,\,b[)$ the problem (1.20), (1.2) has unique solution y. Also in view of Remark 1.1 there exists the constant $r_n > 0$, (which depends only on the numbers l_{kj}, $\bar{l}_{kj}, \gamma_{kj}$ $(k = 1, 2;\ j = 1, \cdots, m)$, and a, b, t^*, n) such that estimate (1.21) holds with $\gamma = r_n$. I.e., the operator P is γ_0, r_n consistent with boundary conditions (1.2). Therefore all the assumptions of Corollary 1.1 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. ∎

Proof of Theorem 1.5. Let r_n be the constant defined in Remark 1.1. First prove that operator P is r_n consistent with boundary conditions (1.2). From the conditions of our theorem it is obvious that the item (i) of Definition 1.1 is satisfied. Let now γ_0 be an arbitrary nonnegative number, x be arbitrary fixed function from the space A_{γ_0} and let $p_j(t) \equiv p_j(x)(t)$. Then, in view of (1.34), (1.35), all the assumptions of Theorem 1.1 are satisfied and then, for any $q \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a, b[)$, problem (1.20), (1.2) has a unique solution y. Also, in view of Remark 1.1, there exists the constant $r_n > 0$, (which depends only on the numbers $l_{kj}, \bar{l}_{kj}, \gamma_{kj}$ $(k = 1, 2;\ j = 1, ..., m)$, and $a, b, t^*, n,$) such that estimate (1.21) holds with $\gamma = r_n$. I.e., the operator P is γ_0, r_n consistent with boundary conditions (1.2) for arbitrary $\gamma_0 > 0$. Thus, by Definition 1.1, the operator P is r_n consistent with boundary conditions (1.2). Therefore, all the assumptions of Corollary 1.2 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. ∎

Proof of Remark 1.2. By Schwartz's inequality, the definition of the norm $||y||_{\widetilde{C}_1^{m-1}}$ and inequalities (1.36), (2.2) for any $x, y \in A_{\gamma_0}$ and $z = y - x$, we have

$$\begin{aligned} |p_j(y)(t)z^{(j-1)}(\tau_j(t))| &= |p_j(y)(t)z^{(j-1)}(t)| + |p_j(y)(t)|\Big|\int\limits_t^{\tau_j(t)} z^{(j)}(\psi)d\psi\Big| \\ &\le ||z||_{\widetilde{C}_1^{m-1}}|p_j(y)(t)|\alpha_j(t)\Big(1 + \frac{1}{\alpha_j(t)}\Big(\int\limits_t^{\tau_j(t)} (\psi - a)^{2m-2j}d\psi\Big)^{1/2}\Big) \end{aligned} \tag{3.68}$$

for $a < t < b$. On the other hand, from conditions (1.37), by Lemmas 2.8 and 2.9, it is cleat that

$$\alpha_j^{-1}(t)\Big(\int\limits_s^{\tau_j(s)} (\xi - a)^{2m-2j}d\xi\Big)^{1/2} \le \frac{\sqrt{\kappa(1+\kappa)}}{\varepsilon^{m-j+1/2}} \quad \text{for } s \in]a,\, a+\varepsilon] \cup [b-\varepsilon, b[,$$

$$\begin{aligned} \alpha_j^{-1}(t)\Big(\int\limits_s^{\tau_j(s)} (\xi - a)^{2m-2j}d\xi\Big)^{1/2} &\le \varepsilon^{-2m+2j-1}\Big(\int\limits_a^b (\xi - a)^{2m-2j}d\xi\Big)^{1/2} \\ &= \frac{(b-a)^{m-j+1/2}}{\sqrt{2m-2j+1}\varepsilon^{2m-2j+1}} \quad \text{for } s \in]a+\varepsilon, b-\varepsilon[. \end{aligned}$$

Then, if we put

$$\kappa_0 = \min\Big\{\frac{\sqrt{\kappa(1+\kappa)}}{\varepsilon^{m-j+1/2}},\ \frac{(b-a)^{m-j+1/2}}{\sqrt{2m-2j+1}\varepsilon^{2m-2j+1}}\Big\}, \tag{3.69}$$

from (3.68), by the last estimates, we get the inequality

$$
\begin{aligned}
(3.70)\qquad |p_j(y)(t)z^{(j-1)}(\tau_j(t))| &\le ||z||_{\widetilde{C}_1^{m-1}}(1+\kappa_0)|p_j(y)(t)|\alpha_j(t)\\
&\le ||z||_{\widetilde{C}_1^{m-1}}(1+\kappa_0)\delta_j(t,\,||z||_{\widetilde{C}_1^{m-1}})
\end{aligned}
$$

for $a<t<b$. Analogously, we get that

$$|(p_j(y)(t)-p_j(x)(t))x^{(j-1)}(\tau_j(t))| \le ||x||_{\widetilde{C}_1^{m-1}}(1+\kappa_0)|p_j(y)(t)-p_j(x)(t)|\alpha_j(t)$$

for $a<t<b$. From (3.70) and the last inequality, it is obvious that the operator P defined by equality (1.17) continuously acting from A_{γ_0} to the space $L_n(]a,b[)$, and the item (ii) of Definition 1.1 holds, with $\delta(t,\rho)=(1+\kappa_0)\sum\limits_{j=1}^{m}\delta_j(t,\rho)$. ■

Proof of Corollary 1.3. From conditions (1.40) and (1.37), by Remark 1.2, we obtain that the operator P defined by equality (1.17) with $p_j(x)(t)=p_j(t)$, continuously acting from A_{γ_0} to the space $L_n(]a,b[)$, for any $\gamma_0>0$, i.e., continuously acting from $\widetilde{C}_1^{m-1}(]a,b[)$ to the space $L_n(]a,\,b[)$. Therefore, it is clear that all the conditions of Theorem 1.5 would be satisfied with

$$F(x)(t)=f(t,x(\tau_1(t)),x'(\tau_2(t)),\cdots,x^{(m-1)}(\tau_m(t))),\quad \delta(t,\rho)=(1+\kappa_0)\sum_{j=1}^{m}|p_j(t)|,$$

where the constant κ_0 is defined by equality (3.69). Thus problem (1.38), (1.2) is solvable. ■

Proof of Corollary 1.4. Let the operators $F,p_1:C^{m-1}(]a,\,b[)\to L_{loc}(]a,\,b[)$, and the function $\eta:]a,\,b[\times R^+\to R^+$ be defined by equalities

$$F(x)(t)=-\frac{\lambda|x(t)|^k}{[(t-a)(b-t)]^{2+k/2}}x(\tau(t))+q(x)(t),\quad p_1(x)(t)=-\frac{\lambda|x(t)|^k}{[(t-a)(b-t)]^{2+k/2}}.$$

Then, it is easy to verify that in view of (1.43)-(1.45), conditions (1.11), (1.12), (1.26), (1.31)–(1.40) are satisfied with

$$
\begin{aligned}
&\delta(t,\,\rho)=\frac{\rho^k\lambda}{[(t-a)(b-t)]^2},\quad l_{01}=\frac{4\gamma_0^k\lambda}{(b-a)^2},\quad \bar{l}_{01}=\frac{16\gamma_0^k\lambda}{(b-a)^2},\\
(3.71)\qquad &r_2=\Big(1+\sqrt{\frac{2}{b-a}}\Big)\frac{2(1+b-a)(b-a)^2}{(b-a)^2-16\lambda\gamma_0^k(1+[2(b-a)]^{1/4})},\\
&B_0=B_1=\frac{16\lambda\gamma_0^k}{(b-a)^2}(1+[2(b-a)]^{1/4}),\quad t^*=(a+b)/2,\quad \gamma_{01}=\gamma_{11}=\frac{1}{4}.
\end{aligned}
$$

Thus all the condition of Theorem 1.4 are satisfied, from which follows solvability of problem (1.41), (1.2). ■

Proof of Corollary 1.5. Let the operators $F,p_1:C^{m-1}(]a,b[)\to L_{loc}(]a,b[)$, and the function $\eta:]a,b[\times R^+\to R^+$ be defined by equalities

$$F(x)(t)=-\frac{\lambda|\sin x^k(t)|}{[(t-a)(b-t)]^2}x(\tau(t))+q(x)(t),\quad p_1(x)(t)=-\frac{\lambda|\sin x^k(t)|}{[(t-a)(b-t)]^2}.$$

Then it is easy to verify that in view of (1.28), (1.43), and (1.46), all the conditions of Theorem 1.5 follow, where δ, l_{01}, $\bar{l}_{01}$, r_2, B_0, B_1, t^*, γ_{01}, γ_{11}, are defined by (3.71) with $\rho = 1$, $\gamma_0 = 1$, from which the solvability of problem (1.41), (1.2) follows. ∎

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References

[1] Agarwal, R.P., *Focal boundary value problems for differential and difference equations,* Kluwer Academic Publishers, Dordrecht, 1998.

[2] Agarwal, R.P., O'Regan, D., *Singular differential and integral equations with applications,* Kluwer Academic Publishers, Dordrecht, 2003.

[3] Agarwal, R.P., Kiguradze, I., *Two-point boundary value problems for higher-order linear differential equations with strong singularities,* Bound. Value Probl., 2006.

[4] Bravyi, E., *A note on the Fredholm property of boundary value problems for linear functional differential equations,* Mem. Diff. Equ. Math. Phys., 20 (2000), 133-135,

[5] Escudo, C., Hakl, R., Peral, I., Torres, P., *Existence and nonexistence results for a singular boundary value problems arising in the theory of epitaxial growth.,* Math. Methods in the Appl. Science, (2011), (submited).

[6] Escudo, C., Hakl, R., Peral, I., Torres, P., *On radial stationary solutions to a model of nonequilibrium growth,* Math. Methods in the Appl. Science (2011), (submited).

[7] Kiguradze, I.T., *On a singular multi-point boundary value problem,* Ann. Mat. Pura Appl., IV. Ser., 86 (1970), 367-399.

[8] Kiguradze, I.T., *Some singular boundary value problems for ordinary differential equations,* (Russian) Tbilisi University Press, Tbilisi, 1975.

[9] Kiguradze, I.T., Chanturia, T.A., *Asymptotic properties of solutions of nonautonomous ordinary differential equations,* Mathematics and Its Applications. Soviet Series, 89. Kluwer Academic Publishers group, Dortrecht, 1993.

[10] Kiguradze, I., Tskhovrebadze, G., *On two-point boundary value problems for systems of higher order ordinary differential equations with singularities,* Georgian Math. J., 1 (1) (1994), 31-45.

[11] Kiguradze, I., Půža, B., Stavroulakis, I.P., *On singular boundary value problems for functional differential equations of higher order*, Georgian Math. J., 8 (4) (2001), 791-814.

[12] Kiguradze, I., *Some optimal conditions for the solvability of two-point singular boundary value problems,* Funct. Differ. Equ., 10 (1-2) (2003), 259-281.

[13] Kiguradze, T., *On conditions for linear singular boundary value problems to be well posed,* (English. Russian original) Differ. Equ., 46 (2) (2010), 187-194; translation from Differ. Uravn., 46 (2) (2010), 183-190.

[14] Kiguradze, I., *On two-point boundary value problems for higher order singular ordinary differential equations,* Mem. Differential Equ. Math. Phys., 31 (2004), 101-107.

[15] Kiguradze, I., *The Dirichlet and focal boundary value problems for higher order quasi-halflinear singular differential equations,* Mem. Differential Equ. Math. Phys., 54 (2011), 126-133.

[16] Kiguradze, I., *On Extremal Solutions of Two-Point Boundary Value Problems for Second Order Nonlinear Singular Differential Equations,* Bull. Georgian National Acad. Sci., 5 (2) (2011), 31-35.

[17] Lomtatidze, A., *On one boundary value problem for linear ordynary differential equations of second order with singularities,* Differential'nye Uravneniya, 222 (3) (1986), 416-426.

[18] Mukhigulashvili, S., *Two-point boundary value problems for second order functional–differential equations,* Mem. Differential Equ. Math. Phys., 20 (2000), 1-112.

[19] Mukhigulashvili, S., Partsvania, N., *On two-point boundary value problems for higher order functional differential equations with strong singularities,* Mem. Differential Equ. Math. Phys., 54 (2011), 134-138.

[20] Mukhigulashvili, S., Partsvania, N., *Two-point boundary value problems for strongly singular higher-order linear differential equations with deviating arguments,* E. J. Qualitative Theory of Diff. Equ., 38 (2012), 1-34.

[21] Mukhigulashvili, S., *On one estimates for solutions of two-point boundary value problems for strongly singular higher-order linear differential equations,* Mem. Differential Equ. Math. Phys. (accepted).

[22] Mukhigulashvili, S., *The Dirichlet boundary value problem for strongly singular higher-order nonlinear functional-differential equations,* Czechoslovak Mathematical Journal, (submited).

[23] Mukhigulashvili, S., *Nonlocal boundary value problems for strongly singular higher-order linear functional-differential equations,* E. J. Qualitative Theory of Diff. Equ., (submited).

[24] Partsvania, N., *On solvability and well-posedness of two-point weighted singular boundary value problems,* Mem. Differential Equ. Math. Phys. **54** (2011), 139-146.

[25] Půža, N., *On a singular two-point boundary value problem for the nonlinear mth-order differential equation with deviating arguments,* Georgian Mathematical J. **4** (1997), No. 6, 557-566. MR 1478554

[26] Půža, B., Rabbimov, A., *On a weighted boundary value problem for a system of singular functional-differential equations,* Mem. Differential Equ. Math. Phys., 21 (2000), 125-130.

[27] Rachunková, I., Staněk, S., Tvrdý, M., *Singularities and Laplacians in boundary value problems for nonlinear ordinary differential equations,* Handbook of differential equations: ordinary differential equations. Vol. III, 607-722, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2006.

[28] Rachunková, I., Staněk, S., Tvrdý, M., *Solvability of nonlinear singular prob- lems for ordinary differential equations,* Contemporary Mathematics and Its Applications, 5. Hindawi Publishing Corporation. New York 2008.

[29] Schwabik, Š, Tvrdý, M., Vejvoda, O., *Differential and integral equations, boundary value problems and adjoints,* D. Reidel Publishing Co., Dordrecht-Boston, mass-London, 1979.

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