SOME PROPERTIES OF AUTONILPOTENT GROUPS

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Abstract. In this paper, we introduce the notion of autonilpotent groups and study the basic properties of this new notion. Among other results, it is shown that autonilpotency property is stronger than the usual nilpotency of groups. We also classify all finite abelian groups, which are autonilpotent.

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1. Introduction

Let G be a group and $Aut(G)$ denote the automorphisms group of G. For any $g \in G$ and $\alpha \in Aut(G)$,

$$[g, \alpha] = g^{-1}g^{\alpha}$$

is the *autocommutator* of g and α. Inductively, for all $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n \in$ Aut(G),

$$[g, \alpha_1, \alpha_2, \ldots \alpha_n] = [[g, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n]$$

is the autocommutator of $g, \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n \in$ of weight n+1 ($n \geq 1$).

The following definition is vital in our investigations.

Definition 1.1.

(a) Let G be a group and A denotes the automorphism group of G. Then

$$K_1(G) = K(G) = [G, A] = \langle [g, \alpha] \mid g \in G, \alpha \in A \rangle$$

is the autocommutator subgroup of G and inductively,

$$K_n(G) = K(G) = [\ G, \underbrace{A, ..., A}_{n-times}\ ] = \langle [g, \alpha_1, ..., \alpha_n] \mid g \in G, \alpha_1, ..., \alpha_n \in A \rangle.$$

is the n th-autocommutator subgroup of G. Now, we obtain the following series of characteristic subgroups

$$G = K_0(G) \supseteq K_1(G) = K(G) \supseteq K_2(G) \supseteq ... \supseteq K_n(G) \supseteq ... ,$$

which we call the lower autocentral series of the group G.

In particular, if we restrict ourselves to inner automorphisms group of G, $Inn(G)$, then the above series will be the usual lower central series of G.

(b) We call the set of elements

$$L(G) = \{g \in G \mid [g, \alpha] = 1 \text{ or } g^\alpha = g, \ \forall \alpha \in A\}$$

the autocentre of G. Clearly, it is a characteristic subgroup of G (see [1] for more information) and, if $A = Inn(G)$, then $L(G) = Z(G)$ is the centre of G.

Now, we define the upper autocentral series of G in the following way:

$$\langle 1 \rangle = L_0(G) \subseteq L_1(G) = L(G) \subseteq L_2(G) \subseteq ... \subseteq L_n(G) \subseteq ...$$

where

$$\frac{L_n(G)}{L_{n-1}G)} = L\left(\frac{G}{L_{n-1}(G)}\right),$$

or, equivalently,

$$L_n(G) = \pi_G^{-1}\left(L\left(\frac{G}{L_{n-1}(G)}\right)\right),$$

for all $n \geq 2$, in which $\pi_G : G \to \dfrac{G}{L_{n-1}(G)}$ is a homomorphism.

In particular, if we take the group of inner automorphisms we obtain the usual upper central series of G. A group G is said to be autonilpotent group of class at most n if $L_n(G) = G$, for some natural number $n \in N$.

In [1], [2], some properties of autocommutator subgroup of a finite group are studied. In [3], we showed that every finite abelian group is the nth autocentral subgroup of some finite abelian group, for any natural number n.

The main purpose of this paper is to determine all finite abelian groups, which are autonilpotent.

The following lemma follows easily from the definition.

Lemma 1.2. *Let G be a group and $x \in L_n(G)$, for some $n \geq 1$, then for all $\alpha_1, ..., \alpha_n \in Aut(G)$,*

$$[x, \alpha_1, ..., \alpha_n] = 1.$$

Corollary 1.3. *If G is an autonilpotent group of class n, then $K_n(G) = \langle 1 \rangle$.*

Remark 1.4. For any group G and each natural number n,

$$K_n(G) \geq \gamma_n(G),$$

and

$$L_n(G) \leq Z_n(G).$$

Note that the above inequalities will be attained, when the group G is taken to be the symmetric group S_3, since $Aut(S_3) = Inn(S_3)$.

One observes that autonilpotent groups are nilpotent, but the converse is not true in general.

Example 1.5. One can easily check that

$$L(Z_2) = Z_2; \quad L(Z_3) = \langle 1 \rangle; \quad L_2(Z_4) = Z_4; \quad L(Z_6) = \{e, x^3\}; \quad L_2(Z_6) = Z_6.$$

Hence the cyclic groups of orders 2 and 4 are autonilpotent and the ones of orders 3 and 6 are not, while they are nilpotent in the usual sense.

It is known that the symmetric group S_3 is not nilpotent and it is easily checked that $L(S_3) = \langle 1 \rangle$, hence it is not autonilpotent as well.

The following property for autonilpotent groups is similar to the one in the usual nilpotent groups.

Proposition 1.6. *If G is a non-trivial autonilpotent group, then*

$$L(G) \neq \langle 1 \rangle.$$

Proof. By the assumption there exists $n \geq 1$ such that $L_n(G) = G$. Now, if $L(G) = \langle 1 \rangle$ then by the definition $L_i(G)$ must be trivial for $i \geq 1$, which gives a contradiction.

2. Some properties of autonilpotent groups

In this section, it is shown that some of the known results on nilpotent groups can be carried over to autonilpotent groups.

In the view of Example 1.5 in the previous section, we show that all cyclic groups of order 2^n, $n > 1$, are autonilpotent, while it is not the case for arbitrary cyclic groups.

Remark 2.1. The cyclic group Z_p, of odd prime order p is not autonilpotent, while it is nilpotent, since $Aut(Z_p) = U_{p-1}$ is a cyclic group of order $p - 1$ and so it can not fix any element of Z_p.

It will be shown that this is held for all cyclic groups of order p^n, when $p \neq 2$ (see Remark 2.8, below). On the other hand, if p = 2 the following result shows that the cyclic group of order 2n is autonilpotent.

Theorem 2.2. *The cyclic group, Z_{2^n}, of order 2^n $(n > 1)$, is an autonilpotent group.*

Proof. Let $Z_{2^n} = \langle x \mid x^{2^n} = 1\rangle$ be the cyclic group of order 2^n $(n > 1)$. Clearly, if $r = 2t + 1$ is an odd number the map $\alpha : x \mapsto x^r$ is an automorphism. So

$$r2^{n-1} = (2t + 1)2^{n-1} \equiv 2^{n-1} \pmod{2^n}.$$

Hence $(x^{2^{n-1}})^\alpha = x^{r2^{n-1}} = x^{2^{n-1}}$. Now, if for each $\alpha \in Aut(Z_{2^n})$ and $s \in N$,

$$(x^s)^\alpha = x^s,$$

then $x^{rs} = x^s$ and so $x^{r(s-1)} = 1$, which implies that $2^n \mid r(s - 1)$, i.e. $s = 2^{n-1}$. Thus $L(Z_{2^n}) = \{e, x^{2^{n-1}}\}$ and hence $L\left(\frac{Z_{2^n}}{L(Z_{2^n})}\right)$ is a cyclic group of order 2^{n-1}. Continuing in this way, we get $L_n(Z_{2^n}) = Z_{2^n}$, after n-steps, which proves the claim.

In the following lemma, we determine the autonilpotency property of abelian 2-groups, except cyclic 2-groups.

Lemma 2.3. *If $Z_{2^m} = \langle x \mid x^{2^m} = 1\rangle$ is a cyclic group of order 2^m, then*

$$L(Z_{2^m} \times Z_{2^m}) = \langle 1\rangle.$$

Proof. (i) Let $Z_{2^m} \times Z_{2^m} = \langle x\rangle \times \langle y\rangle$, in which x and y are both of orders 2^m. Clearly, the following map is an automorphism of $Z_{2^m} \times Z_{2^m}$.

$$\alpha_j : \begin{cases} x \rightarrow x^i y^j \\ y \rightarrow x^{i_1} y^{j_1} \end{cases}$$

where $0 \leq i, i_1, j, j_1 < 2^m$ and are all odd, but not $i = i_1$ and $j = j_1$.

Now, if $x^r y^s \in L(Z_{2^m} \times Z_{2^m})$ then it must be fixed under all the automorphisms such as α is given above. So, $\alpha(x^r y^s) = x^r y^s$, and it follows that $x^{r_i} y^{r_j} x^{si_1} y^{sj_1} = x^r y^s$. This gives $x^{ri-r+si_1} y^{rj-s+sj_1} = 1$ or $x^{r(i-1)+si_1} y^{rj+s(j_1-1)} = 1$. Hence 2^m divides $r(i-1)+si_1$ and $rj+s(j_1-1)$. Now, if $i = i_1 = 1$ or $j = j_1 = 1$ then 2^m must divide r and s. Hence $r = s = 0$ and so $x^r y^s = e$ which implies that $L(Z_{2^m} \times Z_{2^m}) = \langle 1\rangle$ and thus the group $Z_{2^m} \times Z_{2^m}$ can not be autonilpotent.

Lemma 2.4. *If $G = Z_{2^n} \times Z_{2^m}$, where $n > m > 1$, then G cannot be autonilpotent.*

Proof. Let

$$G = Z_{2^n} \times Z_{2^m} = \langle x \mid x^{2^n} = 1\rangle \times \langle y \mid y^{2^n} = 1\rangle.$$

Clearly, if i and j are odd numbers such that $i < 2^n$ and $j < 2^m$, then $o(x^i) = 2^n$ and $o(y^j x^{2^{n-m}}) = 2^m$. Hence the following map

$$\alpha : \begin{cases} x \rightarrow x^i \\ y \rightarrow y^j x^k, \quad k = 0, \text{ or } 2^{n-m}, \ j \neq 0, \end{cases}$$

is an automorphism of the group G.

Now, if the element $x^r y^s$ of G is in $L(G)$, then for each i, j and k with the above conditions, we have $\alpha(x^r y^s) = x^r y^s$ and hence $x^{ir}x^{sk}y^{sj} = x^r y^s$. So $x^{r(i-1)+sk}y^{s(j-1)} = 1$ or $y^{s(j-1)} = 1$ and $x^{r(i-1)+sk} = 1$. These follow that $2^m|s(j-1)$ and $2^n|r(i-1)+sk$. By the choice of i and j as above, we may assume $i = 2^n - 1$ and $j = 3$. Then we obtain $s = 2^m k'$ and $r = 2^n - 1$, and so $x^r y^s = x^{2^{n-1}}$. It follows that

$$L_1(Z_{2^n} \times Z_{2^m}) = \langle x^{2^{n-1}} \rangle = Z_2,$$

and, by the definition, we have

$$\frac{L_2(Z_{2^n} \times Z_{2^m})}{Z_2} = L\left(\frac{Z_{2^n} \times Z_{2^m}}{Z_2}\right) = L\left(Z_{2^{n-1}} \times Z_{2^m}\right) = \langle x^{2^{n-1}} \rangle \cong Z_2.$$

Continuing the above procedure and using Lemma 2.3, we obtain

$$L_{n-m+1}(Z_{2^n} \times Z_{2^m}) = L\left(\frac{Z_{2^n} \times Z_{2^m}}{Z_{2^{n-m}}}\right) = L(Z_{2^m} \times Z_{2^m}) = \langle 1 \rangle,$$

and hence $Z_{2^n} \times Z_{2^m}$ is not autonilpotent.

Theorem 2.5. *Let G be a finite abelian group such that there exists a natural number n with n and $n-1$ are coprime to $|G|$. Then $K(G) = G$.*

Proof. The property $(n, |G|) = 1$ implies that $G^n = G$ and so the map $\alpha : G \to G$ given by $x \to x^n$ is an automorphism. Therefore, for all $x \in G$, we have

$$[x^n, \alpha] = x^{-n}\alpha(x^n) = x^{-n}x^{n^2} = x^{(n^2-n)} = (x^n)^{n-1} = (x^{n-1})^n \in G^n$$

. This implies that $G \leq K(G) \leq G$ and hence $K(G) = G$.

Corollary 2.6. *If G is a finite abelian group of odd order, then $K(G) = G$.*

Lemma 2.7. *Let $G = Z_{p^k} \times ... \times Z_{p^k}$ be the direct product of $m (m > 1)$ copies of cyclic groups of order a prime p. Then $K(G) = G$.*

Proof. If the prime p is odd, then the result follows by the above corollary. Now, let $p = 2$ and $G = \langle a_1 \mid a_1^{2^k} = 1 \rangle \times \ldots \times \langle a_m \mid a_m^{2^k} = 1 \rangle$. Clearly, the following maps are all automorphism:

$$\alpha_j : \begin{cases} a_1 \to a_1 a_j, & j \neq 1, \\ a_i \to a_i, & i \geq 2, \end{cases}$$

and

$$\phi : \begin{cases} a_1 \to a_1, & \\ a_2 \to a_1 a_2, & \\ a_i \to a_i, & i \geq 3. \end{cases}$$

Then we have

$$[a_1, \alpha_j] = a_1^{-1}\alpha_j(a_1) = a_1^{-1}a_1 a_j = a_j \in K(G)$$

and

$$[a_2, \phi] = a_2^{-1}\phi(a_2) = a_2^{-1}a_1a_2 = a_1 \in K(G).$$

Hence $G \subseteq K(G)$ and so $K(G) = G$.

Clearly, Corollary 1.3 shows that the group $G = Z_{p^k} \times ... \times Z_{p^k}$ is not autonilpotent.

Remark 2.8. Note that using similar methods as in Lemmas 2.6, 2.4 and 2.3, one can show that if

$$G = \underbrace{Z_{2^n} \times \ldots \times Z_{2^n}}_{r-times} \times Z_{2^{m_1}} \times \ldots \times Z_{2^{m_k}}$$

where $n > m_1 \geq ... \geq m_k$ and $r > 1$. Then $K(G) = G$ and hence, by Corollary 1.3, the group G can not be autonilpotent.

The following example shows that there are nilpotent groups, which are not autonilpotent.

Example 2.9. Consider $D_8 = \langle a, b | a^4 = b^2 = 1, bab = a^{-1} \rangle$, the dihedral group of order 8. Clearly, the group of automorphisms of D_8 is of order 8 and one may check that

$$L(D_8) = \{e, a^2\} \cong Z_2$$

and

$$\frac{L_2(D_8)}{L(D_8)} = L\left(\frac{D_8}{L(D_8)}\right) = L(Z_2 \times Z_2) = \langle 1 \rangle.$$

Hence $L_2(D_8) = L(D_8)$, which implies that D_8 is not an autonilpotent group, while it is nilpotent.

The following result gives the basic step in proving our main goal which says that; the abelian autonilpotent groups, are the only cyclic groups of order 2^n, for $n \geq 1$.

Theorem 2.10. *If the group $G = H \times K$ is the direct product of its characteristic subgroups H and K, then for all $n \geq 1$,*

$$L_n(H \times K) = L_n(H) \times L_n(K).$$

Proof. We proceed by induction on n. If $n = 1$, then by Theorem 1.1 [1], $K_1(H \times K) = K_1(H) \times K_1(K)$ and hence $L_1(H \times K) = L_1(H) \times L_1(K)$.

Assume the result holds for $n - 1$ and consider the canonical epimorphisms

$$\pi_H : H \longrightarrow \frac{H}{L_{n-1}(H)},$$

$$\pi_K : K \longrightarrow \frac{K}{L_{n-1}(K)}.$$

Then using induction hypothesis, we may define

$$\begin{aligned}\pi &= \pi_H \times \pi_K : H \times K \longrightarrow \frac{H}{L_{n-1}(H)} \times \frac{K}{L_{n-1}(K)} \overset{\psi}{\cong} \frac{H \times K}{L_{n-1}(H) \times L_{n-1}(K)} \\ &= \frac{H \times K}{L_{n-1}(H \times K)},\end{aligned}$$

So, we obtain an epimorphism

$$\varphi = \psi o \pi : H \times K \longrightarrow \frac{H \times K}{L_{n-1}(H \times K)} \ .$$

Hence, we have

$$\begin{aligned}L_n(H \times K) &= \varphi^{-1}\left(L\left(\frac{H \times K}{L_{n-1}(H \times K)}\right)\right) \\ &= \pi^{-1}\psi^{-1}\left(L\left(\frac{H \times K}{L_{n-1}(H) \times L_{n-1}(K)}\right)\right) \\ &= \pi^{-1}\left(L\left(\frac{H}{L_{n-1}(H)}\right) \times L\left(\frac{K}{L_{n-1}(K)}\right)\right) \\ &= L_n(H) \times L_n(K),\end{aligned}$$

which gives the result.

The following corollaries are immediate consequences of the above theorem.

Corollary 2.11. *The above theorem holds, when the subgroups H and K are of co-prime orders.*

Corollary 2.12. *If $G = H \times K$, is the direct product of its characteristic subgroups such that H or K is not autonilpotent, then so is not G.*

Theorem 2.13. *A finite abelian group is autonilpotent if and only if it is a cyclic 2-group.*

Proof. The necessity condition follows from Theorem 2.2. Now, for the reverse conclusion, we assume that G is not a cyclic 2-group. So it is either abelian 2-group or G has a direct summand Z_{p^t}, where p is an odd prime number and $t \geq 1$. In the first case, Lemmas 2.3, 2.4 and Remark 2.8 imply that the group G is not autonilpotent. In the second case, Corollary 1.3 shows that Z_{p^t} can not be autonilpotent. Thus, Corollary 2.12 gives the result.

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